UNIVERSITY OF GAZİANTEP
GRADUATE SCHOOL OF
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GREEN'S FUNCTION FOR REGULAR AND SINGULAR BOUNDARY-VALUE PROBLEM

M. Sc. THESIS
IN
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BY
SHADY HAJ HASAN
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SHADY HAJ HASAN

Green's Function for Regular and Singular Boundary-Value Problems

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Supervisor
Assoc. Prof. Dr. Abdullah KABLAN

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ABSTRACT

GREEN'S FUNCTION FOR REGULAR AND SINGULAR BOUNDARY-VALUE PROBLEMS

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This thesis seeks to find the Green's functions for a singular boundary-value problems, by first transforming the singular problems into a regular one, and then find the Green's function of the regular problem, which directly yields the Green's function of the original singular problem. In this thesis, a five steps procedure will be used as a general method to get the singular Green's function. Lastly, we illustrate our method by constructing the Green's function of a simple singular Legendre problem.

Key Words: Singular boundary-value problems, Green's function, Legendre equation.

ÖZET

DÜZENLİ VE TEKİL SINIR-DEĞER PROBLEMLERİ İÇİN GREEN FONKSİYONU

HAJ HASAN, SHADY

Yüksek Lisans Tezi, Matematik Bölümü

Tez Danışmanı: Doç. Dr. Abdullah KABLAN

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35 sayfa

Bu tez, tekil sınır-değer probleminin Green fonksiyonunu, problemi düzenli sınır-değer problemine dönüştürüp, daha sonra bu düzenli sınır-değer probleminin Green fonksiyonunu bularak tekil sınır-değer problemin Green fonksiyonunu elde etme üzerinedir. Bu tezde tekil Green fonksiyonunu bulmak için beş aşamalı bir metot kullanılmıştır. Son olarak da, metodun bir uygulaması olarak basit tekil Legendre probleminin Green fonksiyonu bulunmuştur.

Anahtar Kelimeler: Tekil sınır-değer problemi, Green fonksiyonu, Legendre problemi.

To my parents,

ABDULKARIM, RAFEA, and my lovely wife SAFA Without whom none of my success would be possible…..

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TABLE OF CONTENTS

LIST OF SYMBOLS/ABBREVIATIONS

BVP	Boundary-value Problem
IVP	Initial-value Problem
I	Real Interval
LC	Limit-circle end point
LCNO	Limit-circle-non-oscillatory end point
$L(I,\mathbb{C})$	The set of the Lebesgue integrable complex valued functions
$L_{loc}(I,\mathbb{C})$	The Lebesgue integrable complex valued functions' set
$AC(I)$	The absolutely continuous complex valued functions over I
$M_{m\times n}(\mathbb{C})$	The set of $m\times n$ matrices with complex entries
$M_n(\mathbb{C})$	The set of squared matrices with complex entries of the order n
E^*	he conjugate of $E\in M_n(\mathbb{C})$
$\Phi(.,.,.)$	The primary fundamental matrix of the regular problem
T(.,.,.)	The primary fundamental matrix of the adjoint problem
G	The Green's function of the regular system
L	The Green's function of the adjoint system
$x^{[i]}$	The qausi-derivative of x

CHAPTER 1

INTRODUCTION

The name of the Green's functions goes back to 1828, when the British mathematician George Green published a paper about solving Poisson's equation $\nabla 2u = f$ of an electric potential u which was defined inside a bounded volume with specific boundary conditions on the volume's surface, and he introduced a function now is identified as what Riemann later coined the “Green’s function”.

Green's function has proved to have a wide range of applications with regard to BVPs, reader can consult the basic theory of Green's function from [3] and [10]. It has been used to solve BVP and IVP, Wave Equation, Kirchhoff Diffusion Equation, Diffraction Theory, Helmholtz Equation and etc. [2].

In this thesis, we will find the solution of regular Boundary-value problems (BVPs), and then use the result found to construct the solution of a singular BVP, which has been studied of first and second kind by E. A. Coddington and N. Levinson in [1]. It has also been studied with limit-circle (LC) and limit-circle-non-oscillatory (LCNO) end points in [10].

In chapter two, we will start with some basic notations and theorems that will be used in the rest of chapters of this thesis.

The Green's function of the regular system of BVPs

$$\left.\begin{array}{l} X' = (Q - \alpha W)X + R, \\ A_1 X(a_1) + A_2 X(a_2) = 0, \quad A_1, A_2 \in M_n(\mathbb{C}) \end{array}\right\},$$

and its adjoint problem will be obtained in Chapter three, we will also introduce the quasi-differential equation and find the Green's function of it to be the base of our work in the next chapter when we deal with the singular BVPs.

In Chapter four, we will find the Green's function of a singular BVP using a procedure of regularizing the singular problem (find an equivalent regular BVP) then construct the Green's function of the new regular problem, which yields to the Green's function of the original singular BVP.

In the last chapter, we will give a conclusion by finding the Green's function of a simple singular Legendre equation using a five steps procedure built on the result of the four pervious chapters.

CHAPTER 2

PRELIMINARY DEFINITION AND THEOREM

In this chapter we will start with a review of the basic theories and make some definitions to use it in the rest of this thesis's chapters. Let $L(I,\mathbb{C})$ be the set of the Lebesgue integrable complex valued functions on the open, bounded or unbounded interval $I \subseteq \mathbb{R}$, $L_{loc}(I,\mathbb{C})$ be the Lebesgue integrable complex valued functions' set on every subinterval of I which are compact and let $AC(I)$ be the absolutely continuous complex valued functions over I.

We will also denote by $M_{m\times n}(\mathbb{C})$ the set of $m\times n$ matrices with complex entries, and if $m=n$ then we write $M_{n\times n}(\mathbb{C}) = M_n(\mathbb{C})$. If the entries of the $n\times n$ matrix were from an arbitrary set S then the matrix M will be denoted by $M_n(S)$. For any $E\in M_n(\mathbb{C})$, E^* is the conjugate of E.

The proofs of theorems and lemmas will not be given in this chapter, they can be found in [10] and we will give them without proofs.

Definition 2.1 (Solution): If I is any bounded, unbounded, open or closed, interval. Let $Q: I \to M_n(\mathbb{C}),\ R: I \to M_{n,m}(\mathbb{C})$, then the function $X: I \to M_{n,m}(\mathbb{C})$ which is absolutely continuous on every compact subinterval of I and satisfy the equation

$$X' = QY + R \quad on \quad I$$

is called the solution of the equation.

Theorem 2.2 (Existence and uniqueness): Let $m, n \in \mathbb{N}$. If

$$Q \in M_n(L(I, \mathbb{C})), \tag{2.1}$$

$$R \in M_{n,m}(L(I, \mathbb{C})). \tag{2.2}$$

then the IVP

$$X' = QX + R, \tag{2.3}$$

$$X(v) = C, \quad v \in I, \quad C \in M_{n,m}(\mathbb{C}) \tag{2.4}$$

has a solution on I which is unique. And if C, Q, R are real, then the unique solution is real valued.

Let $Q \in M_n(L(I))$. Then from the previous theorem we know that there is exactly one matrix solution of the equation

$$X' = QX \quad on \quad I, \tag{2.5}$$

for each point $v \in I$, and if that solution is X then it satisfies $X(v) = I_n$, where I_n is the identity matrix of the order n.

Theorem 2.3 (Rank Invariance): Let $Q \in M_n(L_{loc}(I, \mathbb{C}))$, where $I = (a_1, a_2)$. If X is an $n \times m$ matrix solution of the equation

$$X' = QX \ \ on \ \ I, \tag{2.6}$$

then

$$rank(X(x)) = rank(X(v)), \ \ v, x \in I. \tag{2.7}$$

Furthermore, if $m = n$, then for any $v, x \in I$, we have

$$(\det X)' = (traceQ)(\det X) \tag{2.8}$$

or, in the integral form

$$(\det X)(x) = (\det X)(v) \exp\left(\int_v^x traceQ(t)dt \right). \tag{2.9}$$

Definition 2.4 (Primary fundamental matrix): Let $\Phi(.,v)$ be the fundamental matrix of the equation (2.5) for every fixed $v \in I$ satisfying

$$\Phi(v,v) = I_n.$$

Here for any fixed $v \in I, \Phi(.,v)$ is from $M_n(AC_{loc}(I))$, and if I is a compact and $Q \in M_n(L(I,\mathbb{C}))$, then v could be an endpoint of the interval I and $\Phi(.,v)$ is from $M_n(AC(I))$ (see [10]). $\Phi(x,v)$ is invertible for every $x, v \in I$ By Theorem 2.3, and we can note that

$$\Phi(x,v) = X(x)X^{-1}(v) \tag{2.10}$$

where X is any fundamental matrix of (2.6).

We call Φ the primary fundamental matrix of (2.6) and we write

$$\Phi = \Phi(Q) = (\Phi_{rs})_{r,s=1}^{n} \ , \ \Phi(Q)(x,v) = \Phi(x,v,Q). \tag{2.11}$$

Furthermore ΦC is a solution of the equation (2.6), for any constant $n \times m$ matrix C see [10].

Proposition 2.5: Suppose that Φ is the fundamental matrix of $Q \in M_n(L_{loc}(I))$, then Φ satisfies these five identities:

1. $\Phi(t,t) = I,$
2. $\Phi(x,s)\Phi(s,t) = \Phi(x,t), \ \ t, x, s \in I,$
3. $\Phi(x,t)^{-1} = \Phi(t,x),$
4. $\frac{\partial}{\partial t}\Phi(x,t) = -\Phi(x,t)Q(t),$
5. $\frac{\partial}{\partial x}\Phi(x,t) = Q(x)\Phi(x,t),$

The first two identities are called Chapman-Kolmogorov identities.

Proof: From (2.10), we have $\Phi(t,t) = X(t)X^{-1}(t) = I$ for 1, and to prove 2 we have

$$\Phi(x,s)\Phi(s,t) = X(x)X^{-1}(s)X(s)X^{-1}(t) = X(x)X^{-1}(t) = \Phi(x,t).$$

The other identities are easy to establish. We have

$$\Phi(x,t)^{-1} = (X(x)X^{-1}(t))^{-1} = (X^{-1}(t))^{-1}X^{-1}(x) = X(t)X^{-1}(x) = \Phi(t,x)$$

which proves 3. Also,

$$\frac{\partial}{\partial t}\Phi(x,t) = \frac{\partial}{\partial t}(X(x)X^{-1}(t)) = X(x)\left(\frac{\partial}{\partial t}X^{-1}(t)\right)$$

and since X is a fundamental matrix solution, X' exist and X is nonsingular, and by differentiating $X(t)X^{-1}(t) = I$ we have

$$\begin{aligned}
&\frac{\partial}{\partial t}(X(t)X^{-1}(t)) = 0 \\
&\Rightarrow \left(\frac{\partial}{\partial t}X(t)\right)X^{-1}(t) + X(t)\left(\frac{\partial}{\partial t}X^{-1}(t)\right) = 0 \\
&\Rightarrow X(t)\left(\frac{\partial}{\partial t}X^{-1}(t)\right) = -\left(\frac{\partial}{\partial t}X(t)\right)X^{-1}(t) \\
&\Rightarrow \left(\frac{\partial}{\partial t}X^{-1}(t)\right) = -X^{-1}(t)\left(\frac{\partial}{\partial t}X(t)\right)X^{-1}(t)
\end{aligned}$$

substituting this in the last relation, gives

$$\frac{\partial}{\partial t}\Phi(x,t) = -X(x)X^{-1}(t)\left(\frac{\partial}{\partial t}X(t)\right)X^{-1}(t) = -\Phi(x,t)\left(\frac{\partial}{\partial t}X(t)\right)X^{-1}(t).$$

And again since X is a fundamental matrix solution, it follows that $\partial X(t)/\partial t = Q(t)X(t),$ and thus

$$\frac{\partial}{\partial t}\Phi(x,t) = -\Phi(x,t)Q(t)X(t)X^{-1}(t) = -\Phi(x,t)Q(t),$$

gives 4. Finally,

$$\begin{aligned}
\frac{\partial}{\partial x}\Phi(x,t) = \frac{\partial}{\partial x}(X(x)X^{-1}(t)) &= \left(\frac{\partial}{\partial t}X(x)\right)X^{-1}(t) \\
&= Q(x)X(x)X^{-1}(t) \\
&= Q(x)\Phi(x,t).
\end{aligned}$$

gives 5.

Lemma 2.6: Let $Q \in M_n(L_{loc}(I))$ and $\Phi(x,t)$ be the primary fundamental matrix in respect of Q. If $traceQ(x) = 0,\ x \in I$. Then

$$\det(\Phi(x,t)) = 1,\ \ t, x \in I$$

Theorem 2.7 (Variation of Parameters Formula): Suppose that I is any interval, and $Q \in M_n(L_{loc}(I,\mathbb{C}))$, and let $\Phi(.,.,Q)$ be the primary fundamental matrix of the equation(2.6). Let $R \in M_{n,m}(L_{loc}(I,\mathbb{C}))$, $v \in I$ and $C \in M_{n,m}(\mathbb{C})$. Then

$$X(x) = \Phi(x,v,Q)C + \int_v^x \Phi(x,t,Q)R(t)dt, \quad x \in I \tag{2.12}$$

is a solution of the equation (2.3), (2.4). We can note that if I is compact and $Q \in M_n(L(I))$, $R \in M_{n,m}(L(I))$, then $X \in M_{n,m}(AC(I))$, and v can be an endpoint or an interior point of the interval I.

The next definition is introduced because of its relationship to the adjoint BVPs in the sense of Lagrange. Also we will see an important case of Lagrange Hermitian which generates symmetric differential operators when $Q^+ = Q$.

Definition 2.8 (Lagrange adjoint matrix): For $Q \in L_{loc}(I)$, we define Lagrange adjoint matrix as

$$Q^+ := -K^{-1}Q^*K. \tag{2.13}$$

where K is defined as

$$K := ((-1)^i \delta_{i,n+1-j})_{1\le i,j\le n}, \tag{2.14}$$

δ refer to the Delta Kronecker, and

$$K^{-1} = (-1)^{n+1}K = K^*. \tag{2.15}$$

we note that $Q^+ \in L_{loc}(I)$ and has the properties

$$\left.\begin{array}{l} 1.(Q^+)^+ = Q, \\ 2.(Q+P)^+ = Q^+ + P^+, \\ 3.(QP)^+ = -P^+Q^+, \\ 4.(cQ)^+ = \overline{c}Q^+,\ c \in \mathbb{C}. \end{array}\right\} \tag{2.16}$$

Lemma 2.9 (adjointness lemma): If $Q \in M_n(L_{loc}(I))$. Then $Q^+ \in M_n(L_{loc}(I))$. Suppose that $\Phi(x,t)$ and $T(x,t)$ are the fundamental matrices in respect of Q, Q^+, then

$$\Phi(x,t) = K^{-1}T^*(t,x)K, \quad T(x,t) = K^{-1}\Phi^*(t,x)K, \quad t,x \in J. \tag{2.17}$$

Moreover, if $Q \in L(I)$, then $Q^+ \in L(I)$ and (2.5:2), (2.17) hold for $-\infty = a_1 \le t,\ x \le a_2 = +\infty$.

Proof: If t is a fixed point in I, and let

$$X(x) = K^{-1*}\Phi^*(x,t,Q)K^*\Phi(x,t,Q^+),\ x \in I.$$

Then $X(t) = I$ and $X'(x) = 0$ for all $x \in I$. Thus, $X(x) = I$ for all $x \in I$, so $\Phi(x,t,Q^+) = K^{-1}(\Phi^{-1})^*(x,t,Q)K$ and by using the properties of $\Phi(x,t,Q)$ from (2.5:3) we find $\Phi(x,t,Q^+) = K^{-1}\Phi^*(t,x,Q)K.$ which complete the proof of the first part.

To proof the second part, we can use the result of the first part and (2.15) to find the second equality. The last part of the lemma could be found by taking the limits as t and x approach a_2 and a_1, which exist by [10].

CHAPTER 3

GREEN'S FUNCTIONS FOR A REGULAR SYSTEM BVPs

3.1 Constructing Green's function for regular systems

Let us consider the system of BVPs:

$$\left.\begin{array}{l} X' = (Q - \alpha W)X, \\ A_1 X(a_1) + A_2 X(a_2) = 0, \quad A_1, A_2 \in M_n(\mathbb{C}). \end{array}\right\} \tag{3.1}$$

We start by constructing the Green matrix function for the system (3.1) above.

Theorem 3.1: Let Q, W be from $M_n(L(I))$, and let α be a complex number and $\Phi(x,t,\alpha)$ be the primary fundamental matrix of the problem (3.1) then the following statements are equivalent:

(1) The problem in (3.1) which is homogeneous system has a non-trivial solution if and only if

$$\det[A_1 + A_2\Phi(a_2, a_1, \alpha)] = 0. \tag{3.2}$$

(2) If $\det[A_1 + A_2\Phi(a_2, a_1, \alpha)] \neq 0$, then for every $R \in L(I)$, the inhomogeneous system

$$X' = (Q - \alpha W)X + R, \quad A_1 X(a_1) + A_2 X(a_2) = 0 \tag{3.3}$$

has a unique solution X given by

$$X(x) = \int_{a_1}^{a_2} G(x,t,\alpha) R(t)\, dt, \quad a_1 \le x \le a_2. \tag{3.4}$$

We call $G(t,s,\lambda)$ the Green matrix of the BVP (3.1) which is related to each of x,t,α,Q,W,A_1, and A_2, and given by

$$G(x,t,\alpha)=\begin{cases}\Phi(x,a_1,\alpha)V\Phi(a_1,t,\alpha), & a_1\le x<t\le a_2,\\ \Phi(x,a_1,\alpha)V\Phi(a_1,t,\alpha)+\Phi(x,t,\alpha), & a_1\le t<x\le a_2,\\ \Phi(x,a_1,\alpha)V\Phi(a_1,t,\alpha)+\frac{1}{2}\Phi(x,t,\alpha), & a_1\le t=x\le a_2,\end{cases}\qquad(3.5)$$

where

$$V=-[A_1+A_2\Phi(a_2,a_1,\alpha)]^{-1}A_2\Phi(a_2,a_1,\alpha).\qquad(3.6)$$

Proof: Let X be the solution of

$$X'=(Q-\alpha W)X+R\,on\,I\qquad(3.7)$$

we determine a solution X of (3.7) for $C\in M_{n,m}(\mathbb{C})$ by the initial condition

$X(a_1,\alpha)=C.$

Since $\Phi(x,t,\alpha)$ is the primary fundamental matrix of the equation, then

$X(x,\alpha)=\Phi(x,a_1,\alpha)C,\quad a_1\le x\le a_2,$

and by the variation of parameters formula (2.12) we have

$$X(x)=\Phi(x,a_1,\alpha)C+\int_{a_1}^{x}\Phi(x,t,\alpha)R(t)dt,\quad a_1\le x\le a_2.\qquad(3.8)$$

Especially at the point a_2 we have

$X(a_2,\alpha)=\Phi(a_2,a_1,\alpha)C+\int_{a_1}^{a_2}\Phi(a_2,t,\alpha)R(t)dt.$

Let $D=[A_1+A_2\Phi(a_2,a_1,\alpha)]$ then the conditions in (3.1) will be

$$\begin{aligned} A_1X(a_1)+A_2X(a_2) &= A_1C+A_2\Phi(a_2,a_1,\alpha)C+A_2\int_{a_1}^{a_2}\Phi(a_2,t,\alpha)R(t)dt \\ &=[A_1+A_2\Phi(a_2,a_1,\alpha)]C+A_2\int_{a_1}^{a_2}\Phi(a_2,t,\alpha)R(t)dt \\ &=DC+A_2\int_{a_1}^{a_2}\Phi(a_2,t,\alpha)R(t)dt. \end{aligned} \tag{3.9}$$

From (3.9) we see that, when $R=0$ on I there is a non-trivial solution X satisfies the conditions $A_1X(a_1)+A_2X(a_2)=0$ if and only if $\det D=0$ i.e. $\det[A_1+A_2\Phi(a_2,a_1,\alpha)]=0$. Also we can see that there is a unique solution X of the nonhomogeneous problem satisfying the boundary conditions $A_1X(a_1)+A_2X(a_2)=0$ for every $R\in L(I)$, if and only if $D(\alpha)\neq 0$, and the following equality is confirmed

$$D(\alpha)C+A_2\int_{a_1}^{a_2}\Phi(a_2,t,\alpha)R(t)dt=0,$$

which yields

$$C=D^{-1}(\alpha)(-A_2)\int_{a_1}^{a_2}\Phi(a_2,t,\alpha)R(t)dt,$$

and from (3.8) we got

$$\begin{aligned} X(x) &= \Phi(x,a_1,\alpha)D^{-1}(\alpha)(-A_2\int_{a_1}^{a_2}\Phi(a_2,t,\alpha)R(t)dt)+\int_{a_1}^{x}\Phi(x,t,\alpha)R(t)dt \\ &=\int_{a_1}^{a_2}\Phi(x,a_1,\alpha)D^{-1}(\alpha)(-A_2\Phi(a_2,t,\alpha))R(t)dt+\int_{a_1}^{x}\Phi(x,t,\alpha)R(t)dt \\ &=\int_{a_1}^{x}\Phi(x,a_1,\alpha)D^{-1}(\alpha)(-A_2\Phi(a_2,t,\alpha))R(t)dt+ \\ &\quad \int_{x}^{a_2}\Phi(x,a_1,\alpha)D^{-1}(\alpha)(-A_2\Phi(a_2,t,\alpha))R(t)dt)+\int_{a_1}^{x}\Phi(x,t,\alpha)R(t)dt, \end{aligned}$$

since $\Phi(a_2,t,\alpha)=\Phi(a_2,a_1,\alpha)\Phi(a_1,t,\alpha)$ we got a modified form of Green's function given in [10].

$$\begin{aligned} X(x) &= \int_{a_1}^{x}\Phi(x,a_1,\alpha)D^{-1}(\alpha)(-A_2\Phi(a_2,a_1,\alpha)\Phi(a_1,t,\alpha))+\Phi(x,t,\alpha)R(t)dt+ \\ &\quad \int_{x}^{a_2}\Phi(x,a_1,\alpha)D^{-1}(\alpha)(-A_2\Phi(a_2,a_1,\alpha)\Phi(a_1,t,\alpha))R(t)dt, \end{aligned}$$

which yield to

$$X(x)=\int_{a_1}^{a_2} G(x,t,\alpha)R(t)dt,$$

where $G(x,t,\alpha)$ is given by(3.5). □

For more special cases of the construction reader can see [8,9,11].

3.2 The Green's function of the adjoint problem

Let the adjoint problem be

$$H'=(Q-\alpha W)^{+}H,\quad A_3H(a_1)+A_4H(a_2)=0,\quad A_3,A_4\in M_n(\mathbb{C}),\tag{3.10}$$

then it's Green's function L will be given by

$$L(x,t,\bar{\alpha})=\begin{cases}\mathrm{T}(x,a_1,\bar{\alpha})U\mathrm{T}(a_1,t,\bar{\alpha}), & a_1\le x<t\le a_2,\\ \mathrm{T}(x,a_1,\bar{\alpha})U\mathrm{T}(a_1,t,\bar{\alpha})+\mathrm{T}(x,t,\bar{\alpha}), & a_1\le t<x\le a_2,\\ \mathrm{T}(x,a_1,\bar{\alpha})U\mathrm{T}(a_1,t,\bar{\alpha})+\dfrac{1}{2}\mathrm{T}(x,t,\bar{\alpha}), & a_1\le t=x\le a_2,\end{cases}\tag{3.11}$$

where $L(x,t,\bar{\alpha})=L(x,t,\bar{\alpha},Q^{+},W^{+},A_3,A_4)$ and

$$U=-[A_3+A_4\mathrm{T}(a_2,a_1,\bar{\alpha})]^{-1}A_4\mathrm{T}(a_2,a_1,\alpha).\tag{3.12}$$

Lemma 3.2: Assume that $\alpha\in\mathbb{C}$ and $Q,W\in M_n(L(I))$. Let the primary fundamental matrices of $(Q-\alpha W),(Q-\alpha W)^{+}$ be $\Phi(x,t,\alpha)$ and $\mathrm{T}(x,t,\bar{\alpha})$, respectively. If $G(x,t,\alpha)$ and $L(x,t,\bar{\alpha})$ are the Green's matrices of the BVP (3.1) and the adjoint BVP (3.10), respectively and

$$\det[A_1+A_2\Phi(a_2,a_1,\alpha)]\neq 0\neq\det[A_3+A_4\mathrm{T}(a_2,a_1,\bar{\alpha})].\tag{3.13}$$

Then the both of Green's matrices $G(x,t,\alpha)$ and $L(x,t,\alpha)$ exist and

$$G(x,t,\alpha)+K^{-1}L^{*}(t,x,\bar{\alpha})K=\Phi(x,a_1,\alpha)\Omega\Phi(a_1,t,\alpha),\quad a_1\le x,t\le a_2,\tag{3.14}$$

where

$$\Omega=V+K^{-1}U^{*}K+I.\tag{3.15}$$

Proof: It follows from (3.5) and (3.11) for $a_1 \le x < t \le a_2$ that

$$\begin{aligned} G(x,t,\alpha) + K^{-1}L^{*}(t,x,\alpha)K &= \Phi(x,a_1,\alpha)V\Phi(a_1,t,\alpha) + \\ &\quad K^{-1}[\mathrm{T}(t,a_1,\overline{\alpha})U\mathrm{T}(a_1,x,\overline{\alpha}) + \mathrm{T}(t,x,\alpha)]^{*}K \\ &= \Phi(x,a_1,\alpha)V\Phi(a_1,t,\alpha) + \\ &\quad K^{-1}[\mathrm{T}(t,a_1,\overline{\alpha})U\mathrm{T}(a_1,x,\overline{\alpha})]^{*}K + K^{-1}\mathrm{T}^{*}(t,x,\alpha)K \\ &= \Phi(x,a_1,\alpha)V\Phi(a_1,t,\alpha) + \\ &\quad K^{-1}\mathrm{T}^{*}(a_1,x,\overline{\alpha})U^{*}\mathrm{T}^{*}(t,a_1,\overline{\alpha})K + K^{-1}\mathrm{T}^{*}(t,x,\alpha)K \end{aligned}$$

now using the fact in (2.17) and (2.5:2) we have

$$\begin{aligned} G(x,t,\alpha) + K^{-1}L^{*}(t,x,\alpha)K &= \Phi(x,a_1,\alpha)V\Phi(a_1,t,\alpha) + \\ &\quad \Phi(x,a_1,\alpha)K^{-1}U^{*}K\Phi(a_1,t,\alpha) + \Phi(x,a_1,\alpha)\Phi(a_1,t,\alpha) \\ &= \Phi(x,a_1,\alpha)[V + K^{-1}U^{*}K + I]\Phi(a_1,t,\alpha) \\ &= \Phi(x,a_1,\alpha)\Omega\Phi(a_1,t,\alpha). \end{aligned}$$

Similarly, we can find that (3.14) holds if $a_1 \le t < x \le a_2$ and $a_1 \le t = x \le a_2$ also, which complete the proof.

Furthermore, if the notation of the pervious lemma hold. Then

$$G(x,t,\alpha) = -K^{-1}L^{*}(t,x,\alpha)K, \quad a_1 \le t, x \le a_2 \tag{3.16}$$

if and only if

$$V = -[K^{-1}U^{*}K + I]. \tag{3.17}$$

And

$$G(x,t,\alpha) = +K^{-1}L^{*}(t,x,\alpha)K, \quad a_1 \le t, x \le a_2, \tag{3.18}$$

if and only if

$$V = +[K^{-1}U^{*}K + I]. \tag{3.19}$$

Theorem 3.3: Let the notations of lemma 3.2 hold. Then the Green's matrices of the BVP (3.3) and its adjoint problem exist, and (3.16) holds if and only if

$$A_1 K A_3^{*} = A_2 K A_4^{*}. \tag{3.20}$$

Proof: When (3.16) hold then $\Omega = 0$ which means

$$\begin{aligned} -I &= V + K^{-1}U^{*}K \\ &= -[A_1 + A_2\Phi(a_2,a_1,\alpha)]^{-1}A_2\Phi(a_2,a_1,\alpha) \\ &\quad -K^{-1}\mathrm{T}^{*}(a_2,a_1,\bar{\alpha})A_4^{*}[A_3 + A_4\mathrm{T}(a_2,a_1,\bar{\alpha})]^{-1*}K. \end{aligned} \tag{3.21}$$

Multiplying (3.21) from the right by $K^{-1}[A_3 + A_4\mathrm{T}(a_2,a_1,\bar{\alpha})]^{*}$ and from the left by $-[A_1 + A_2\Phi(a_2,a_1,\alpha)]$ and simplifying we got the equality

$$\begin{aligned} A_1K^{-1}A_3^{*} &= A_2\Phi(a_2,a_1,\alpha)K^{-1}\mathrm{T}^{*}(a_2,a_1,\bar{\alpha})KK^{-1}A_4^{*} \\ &= A_2\Phi(a_2,a_1,\alpha)\Phi(a_1,a_2,\alpha)K^{-1}A_4^{*} \\ &= A_2K^{-1}A_4^{*}, \end{aligned}$$

since $K^{-1} = (-1)^{n+1}K$, then

$$A_1KA_3^{*} = A_2KA_4^{*}.$$

And this completes the proof.

Theorem 3.4: If the notation of the lemma 3.2 hold, and if $U = V$, then for every α satisfying (3.13) there exist t and x, where $a_1 \le t, x \le a_2$, such that

$$G(x,t,\alpha) \ne +K^{-1}L^{*}(t,x,\alpha)K. \tag{3.22}$$

Notably, (3.22) holds if $Q = Q^{+}$, $W = W^{+}$, $A_1 = A_3$, $A_2 = A_4$, and $\alpha = \bar{\alpha}$ satisfies (3.13).

Now, we will construct the Green's function for the quasi-differential equation which will be the base we depend on when we deal with the singular BVPs, but first let us start by defining the quasi-derivatives of a function x.

As in [6], let

$$H_n(I) = \left\{ \begin{array}{r} Q = (q_{ij}) \in M_n(L_{loc}(I)), q_{i,i+1} \ne 0 \; for\, 1 \le i \le n-1, \\ q_{ij} = 0 \; for\, 2 \le i+1 < j \le n \end{array} \right\}. \tag{3.23}$$

Let $Q \in H_n(L_{loc}(I))$. We define the quasi-derivative as follow:

$$P_0 = \{x : I \to \mathbb{C}, x\, is\, masurable\}$$

we define $x^{[0]} = x$ for $x \in P_0$.

And inductively, we define $P_i = \{x \in P_{i-1} : x^{[i-1]} \in AC_{loc}(I)\}Q$ for $i = 1,\ldots,n,$ and

$$x^{[i]} = q_{i,i+1}^{-1}\left\{\left(x^{[i-1]}\right)' - \sum_{j=1}^{i} q_{ij} x^{[j-1]}\right\} \text{ for } x \in P_i \tag{3.24}$$

where $q_{n,n+1} = 1$. Now we set the quasi-differential expression

$$Mx = M_q x = i^n x^{[n]} \text{ for } x \in P_n.$$

generated by Q, we call $x_q^{[i]}$ the i-th quasi-derivative of x with respect to Q , and P_n the expression domain of M_q, when it is clear we will remove the subscript Q of $x^{[i]}$ and M .

Note that the quasi-differential expression also includes the classical one Lx when we define H_n as follows:

$$H_n(I) = \left\{\begin{array}{r} Q = (q_{ij}) \in M_n(L_{loc}(I)), q_{i,i+1} = 1\, for\, 1 \le i \le n-1, \\ q_{ij} = 0\, for\, 2 \le i+1 < j \le n \\ q_{ii} = 0\, for\, 1 \le i \le n \end{array}\right\}$$

and also they are more general than the expressions in [7]. For more discussion of M_x and its relationship with the classical expressions, the reader can see [4,6,7,11].

Next, we will write the equation $x^{[n]} = r$ as a system.

Lemma 3.4: Let $Q \in H_n(I), r \in L_{loc}(I)$, then the equation

$$x^{[n]} = r \tag{3.25}$$

is equivalent to the system

$$X' = QX + R, \tag{3.26}$$

where

$$X = \begin{bmatrix} x \\ x^{[1]} \\ \vdots \\ x^{[n-1]} \end{bmatrix}, \quad R = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ r \end{bmatrix}. \tag{3.27}$$

And if x is a solution of the scalar equation (3.25) then for X and R defined as (3.27) , (3.26) holds and X is a solution of the first-order system (3.26). And vice versa if X is a vector solution of (3.26) then the upper component of X is a solution of (3.25) .

Proof: First we will write X in a new form as

$$X = \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{bmatrix}$$

where $x_0, x_1, ..., x_{n-1} \in AC_{loc}(I)$, and let $x_0 = x$. In the next steps we will show by induction that $x \in P_i$ and $x_i = x^{[i]}$ for $i = 0,1,...,n-1$.

For $i = 0$, it is clear that $x_0 = x^{[0]} = x$ by assumption. Now, suppose that it holds for $j \le i-1$. i.e. $x^{[i-1]} = x_{i-1} \in AC_{loc}(I)$ which means that $x \in P_i$. The r^th^ component of (3.26) gives

$$\left(x^{[i-1]}\right)' = x'_{i-1} = \sum_{j=1}^{i+1} q_{ij} x_{j-1} = \sum_{j=1}^{i} q_{ij} x^{[j-1]} + q_{i,i+1} x_i \tag{3.28}$$

which yields $x_i = x^{[i]}$.

From $x^{[i-1]} = x_{i-1} \in AC_{loc}(I)$ we derive that $x \in P_n$. And the last component of (3.26) gives

$$\left(x^{[n-1]}\right)' = x'_{n-1} = \sum_{j=1}^{n} q_{nj} x_{j-1} + r = \sum_{j=1}^{n} q_{nj} x^{[j-1]} + r \tag{3.29}$$

which means that $x^{[n]} = r$.

Let $x \in P_n$ then, by the definition, $x^{[i]} \in AC_{loc}(I)$ for $i = 0,1,...,n-1$. So X which is given by (3.27) belongs to $\left(AC_{loc}(I)\right)^n$. And the quasi-derivative's definition gives (3.26), from (3.28) and (3.29).

In the remainder of this section, we will suppose that $Q = (q_{ij}) \in H_n(I)$ and satisfies

$$q_{ij} \in L(I),\ 1 \le i \le j,\ j = 1,2,...,n;\ q_{j,j+1}^{-1} \in L(I),\ j = 1,2,...,n-1. \tag{3.30}$$

Let W be $n \times n$ matrix in the form

$$W = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ w & 0 & \cdots & 0 \end{bmatrix} \text{ for } w \in L(I). \tag{3.31}$$

Let us consider for $\alpha \in \mathbb{C}$ and $r \in L(I)$ the equation

$$Mx = \alpha w x + r \tag{3.32}$$

with the equivalent system

$$X' = (Q + \alpha W)X + R. \tag{3.33}$$

Here X and R are as defined in (3.27). We consider the BVP from (3.32) with the conditions

$$A_1 X(a_1) + A_2 X(a_2) = 0. \tag{3.34}$$

Since the quasi-derivative is absolutely continuous on every compact subinterval of I so it exists and is continuous at each point of I and this makes the problem well defined [10].

Theorem 3.5: Suppose that $\Phi(x,t,\alpha)$ is the primary fundamental matrix of $Q - \alpha W$, with $\alpha \in \mathbb{C}$, W given by (3.31), and $Q = (q_{ij}) \in H_n(I)$ satisfies (3.30) then the following statements are equivalent.

(1) The homogeneous system of BVPs

$$X' = (Q - \alpha W)X,\ A_1 X(a_1) + A_2 X(a_2) = 0,\ A_1, A_2 \in M_n(\mathbb{C}), \tag{3.35}$$

has a non-trivial solution if and only if

$$\det[A_1 + A_2\Phi(a_2, a_1, \alpha)] = 0. \tag{3.36}$$

(2) The homogeneous scalar equation

$$M_x = \alpha w x,\ \ A_1 X(a_1) + A_2 X(a_2) = 0 \tag{3.37}$$

has a non-trivial solution if and only if

$$\det[A_1 + A_2\Phi(a_2, a_1, \alpha)] = 0. \tag{3.38}$$

(3) If $\det[A_1 + A_2\Phi(a_2, a_1, \alpha)] \neq 0,$ then the inhomogeneous system problem

$$X' = (Q - \alpha W)X + R,\ \ A_1 X(a_1) + A_2 X(a_2) = 0,\ \ R \in L(I), \tag{3.39}$$

has a unique solution X given by

$$X(x) = \int_{a_1}^{a_2} G(x, t, \alpha) R(t)\,dt,\ \ a_1 \le x \le a_2, \tag{3.40}$$

G in the last relation is a Green's matrix and is given by (3.5) and (3.6).

(4) If $\det[A_1 + A_2\Phi(a_2, a_1, \alpha)] \neq 0,$ then the inhomogeneous scalar problem

$$M_x = \alpha w x + r,\ \ A_1 X(a_1) + A_2 X(a_2) = 0, \tag{3.41}$$

has a unique solution x given by

$$x(x) = \int_{a_1}^{a_2} G_{1n}(x, t, \alpha) r(t)\,dt,\ \ a_1 \le x \le a_2. \tag{3.42}$$

where G_{1n} is the first component of the n$^{\text{th}}$ column of the Green's matrix defined by $G(x, t, \alpha) = G_{ij}(x, t, \alpha),\ 1 \le i, j \le n.$

And G_{1n} is continuous and is unique on the square $[a_1, a_2] \times [a_1, a_2]$.

Let us now consider the system of BVPs

$$H' = (Q - \alpha W)^{+} H + R = (Q^{+} - \overline{\alpha} W^{+})H + R = (Q^{+} - (-1)^{n}\overline{\alpha W})H + R \tag{3.43}$$

and the equivalent equation to it

$$M^{+}h=(-1)^{n}\bar{\alpha}\bar{w}h+r,\ M^{+}=M_{Q^{+}}, \tag{3.44}$$

with the boundary condition

$$A_3H(a_1)+A_4H(a_2)=0,\ A_3,A_4\in M_n(\mathbb{C}). \tag{3.45}$$

Theorem 3.6: Suppose that $Q=(q_{ij})\in H_n(I)$ is satisfying (3.30) and W to be given by (3.31) where $w\in L(I)$. Let $\alpha\in\mathbb{C}, A_1,A_2,A_3,A_4\in M_n(\mathbb{C})$, and let K be given by (2.14), then $Q^{+}\in H_n(I)$ and satisfies (3.30). If $\Phi(x,t,\alpha)$ and $\mathrm{T}(x,t,\bar{\alpha})$ are the primary fundamental matrices of $(Q-\alpha W)$ and $(Q-\alpha W)^{+}$, respectively, and

$$\det[A_1+A_2\Phi(a_2,a_1,\alpha)]\neq 0\neq\det[A_3+A_4\mathrm{T}(a_2,a_1,\bar{\alpha})]. \tag{3.46}$$

Then the following three statements are equivalent:

(1) $G(x,t,\alpha)=-K^{-1}L^{*}(t,x,\bar{\alpha})K,\ a_1\leq t,x\leq a_2;$ (3.47)

(2) $G_{1n}(x,t,\alpha)=(-1)^{n}\bar{L}_{1n}(t,x,\bar{\alpha}),\ a_1\leq t,x\leq a_2;$ (3.48)

(3) $A_1KA_3^{*}=A_2KA_4^{*},$ (3.49)

where $G(x,t,\alpha)$ and $L(x,t,\alpha)$ are the Green matrices of the problems (3.35) and (3.43) respectively.

Proof: From the non-singularity of Φ and the equality (3.14) it follows that (1) holds if and only if $\Omega=0$. We have (1) and (3) equivalent by the theorem 3.3. It is clear that (1) implies to (2) and to show the reverse we must show that $\Omega=0$ which follows from (3.14) and since $\Phi_{1j}(x,a_1,\alpha), j=1,2,...,n$ and $\Phi_{jn}(a_1,t,\alpha), j=1,2,...,n$ are linear independent as functions of x and t, respectively.

Let $C(t)=\Omega\Phi(a_1,t,\alpha)$ for a fixed t, then by the theorem (3.14) we have

$$\sum_{j=1}^{n}\Phi_{1j}(x,a_1,\alpha)C_{jn}(t)=0,\quad a_1\leq x\leq a_2.$$

Since $\Phi_{1j}(x,a_1,\alpha), j=1,2,\ldots,n$ are linear independent then, we have

$$C_{jn}(t)=\sum_{j=1}^{n}\Omega_{jk}\Phi_{kn}(a_1,t,\alpha)=0, \quad a_1 \le t \le a_2,$$

and since $\Phi_{kn}(a_1,t,\alpha), k=1,2,\ldots,n$ are linear independent, we observe that $\Omega_{jk}=0, k=1,2,\ldots n$ which holds for every j, and means that $\Omega_{jk}=0, j,k=1,2,\ldots,n$, which completes the proof.

Theorem 3.7: Let the notations of the theorem 3.6 hold, and suppose that $Q=Q^{+}, W=W^{+}, A_1=A_3, A_2=A_4$ and $\alpha\in\mathbb{R}$, then there is x and t, $a_1\le t,x\le a_2$, so that

$$G_{1n}(x,t,\alpha)\neq(-1)^{n+1}\bar{G}_{1n}(t,x,\alpha). \tag{3.50}$$

Proof: Since we have $Q=Q^{+}, W=W^{+}, A_1=A_3, A_2=A_4$ and $\alpha\in\mathbb{R}$, then it follows that $(Q-\alpha W)=(Q-\alpha W)^{+}$ and so (3.50) holds as a result of the theorem (3.4).

CHAPTER 4

GREEN'S FUNCTIONS FOR SINGULAR BVPS

In this chapter, we will discuss the construction of the Green's function of the singular problems on $I=(a_1,a_2)$ with singular or regular (LC) limit-circle end points, which may be finite or infinite with no oscillatory on the solutions or the coefficients, but first let us show that the singular scalar equations can be transformed into regular problems. Where the components of Q and W are assumed to be in $L_{loc}(I)$ with no necessity to be in $L(I)$.

Let $Q\in H_n(I), W$ be given by (3.31) and let $\alpha\in\mathbb{C}$. Then $Q^+\in H_n(I)$. Consider the scalar equations

$$Mx=\alpha wx \quad on\, I, \tag{4.1}$$

$$M^+h=(-1)^n\overline{\alpha}\overline{w}x \quad on\, I, \tag{4.2}$$

where the system formulation of them is given by

$$X'=(Q-\alpha W)X \quad \text{on}\, I, \tag{4.3}$$

$$H'=(Q-\alpha W)^+H \quad \text{on}\, I. \tag{4.4}$$

We will denote the primary fundamental matrices of $(Q-\alpha W)$, $(Q-\alpha W)^+$ by Φ and T. respectively, for a fixed $c\in I$ and an arbitrary, but fixed after choosing $s\in\mathbb{R}$, we have

$$\left.\begin{array}{ll} V=\Phi(.,c,s), & U=\mathrm{T}(.,c,s)=(u_{ij}), \\ V=(v_{ij})=(v_i^{[j-1]})^T, & V^{-1}=(V_{ij}). \end{array}\right\} \tag{4.5}$$

Theorem 4.1: Suppose that the trace of Q is $trace(Q)=0$, and that for $\alpha = s$ all solutions of the problems (4.1) and (4.2) are from $L^2(I,|w|)$, now for any $\alpha \in \mathbb{C}$ and any solution $X = X(.,\alpha)$ of (4.3) define $Y = Y(.,\alpha)$ by

$$Y(x) = V^{-1}(x)X(x), \quad for\, x \in I. \tag{4.6}$$

Then

1. For

$$P = \begin{bmatrix} v_1 V_{1n} & v_2 V_{1n} & \cdots & v_n V_{1n} \\ v_1 V_{2n} & v_2 V_{2n} & \cdots & v_n V_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ v_1 V_{nn} & v_2 V_{nn} & \cdots & v_n V_{nn} \end{bmatrix}, \tag{4.7}$$

we have

$$Y' = (s-\alpha)wPY \quad on\, I, \tag{4.8}$$

2. $wP \in L^1(I)$,
3. The limits

$$Y(a_1) = \lim_{t \to a_1^+} Y(x), \qquad Y(a_2) = \lim_{t \to a_2^-} Y(x) \tag{4.9}$$

exist and are finite.

Proof:

1. By taking the first normal derivative of Y, we have

$$\begin{aligned} Y' &= -V^{-1}V'V^{-1}X + V^{-1}X' \\ &= V^{-1}[-(Q-sW)VV^{-1}X + (Q-\alpha W)X] \\ &= (s-\alpha)(V^{-1}WV)Y \quad on\, I. \end{aligned}$$

Here we replaced V', X' by $(Q-sW)V$ and $(Q-\alpha W)X$ respectively. Comparing the results above with (4.8), we observe that

$$V^{-1}WV = wP, \tag{4.10}$$

and this proves (1).

2. To prove the second statement, we will use the adjointness lemma, the relation $\Phi(c,x)=\Phi^{-1}(x,c)$, and the assumption that $trace(Q)=0$, which means that $\det\Phi(c,x)=1$ for all $x\in I$, to obtain that

$$\mathrm{T}(x,c)=K^{-1}\Phi^{*}(c,x)K=K^{-1}\Phi^{-1*}(x,c)K=K^{-1}(V^{-1})^{*}K, \tag{4.11}$$

which yields

$$\mathrm{T}_{1,n+1-j}(x,c)=\pm\overline{V}_{j,n},\quad j=1,2,\dots,n. \tag{4.12}$$

Using the assumption that every solution of the equations (4.1) and (4.2) are from $L^2(I,|w|)$ with the Schwarz inequality we can find that

$$\left(\int_I |wv_jV_{kn}|\right)^2\le\int_J |w||v_j|^2\int_I |w||V_{kn}|^2<\infty,\quad j,k=1,2,\dots,n, \tag{4.13}$$

which completes the proof of (2).

3. The proof of this section of the theorem follows from (2) as in [10].

Corollary 4.2: If the notations of the theorem 4.1 hold then any solution of the equations (4.1) and (4.2) are from $L^2(I,|w|)$.

Theorem 4.3: Let $w\in L_{loc}(I)$. Assume that $Q\in H_n(I)$, $trace(Q)=0$. Let M and M^{+} be the nth-order differential expressions in respect of Q and Q^{+}, respectively. If V,U are determined by (4.5), Y by (4.6) and let P be given by (4.7). Let $A_1,A_2,A_3,A_4\in M_n(\mathbb{C})$. We consider the following BVPs:

$$Mx=\alpha wx\ \text{ on } I,\quad \alpha\in\mathbb{C},\ A_1Y(a_1)+A_2Y(a_2)=0,\ \text{ with } Y=V^{-1}X, \tag{4.14}$$

$$M^{+}h=(-1)^n\overline{\alpha}\overline{w}x\ \text{ on } I,\ \overline{\alpha}\in\mathbb{C},\ A_3\Theta(a_1)+A_4\Theta(a_2)=0,\text{with } \Theta=U^{-1}H, \tag{4.15}$$

and

$$X'=(Q-\alpha W)X+R\qquad \text{on } I,\quad A_1Y(a_1)+A_2Y(a_2)=0, \tag{4.16}$$

$$H' = (Q-\alpha W)^{+} + G \qquad \text{on } I, \quad A_3\Theta(a_1) + A_4\Theta(a_2) = 0, \tag{4.17}$$

$$Y' = (s-\alpha)wPY + V^{-1}R \qquad \text{on } I, \quad A_1Y(a_1) + A_2Y(a_2) = 0, \tag{4.18}$$

$$\Theta' = (s-\bar{\alpha})\bar{w}P\Theta + V^{-1}G \qquad \text{on } I, \quad A_3\Theta(a_1) + A_4\Theta(a_2) = 0. \tag{4.19}$$

Assume that $s-\alpha$, $s-\bar{\alpha}$ are not eigenvalues of the BVPs (4.18) and (4.19) respectively, then the Green matrices of (4.18) and (4.19)

$$K(x,t,s-\alpha,P,W), \quad L(x,t,s-\bar{\alpha},P^{+},\bar{W})$$

exist. Define

$$\begin{aligned} G(x,t,\alpha,Q,W,A_1,A_2) &= V(x)K(x,t,s-\alpha,P,W)V^{-1}(t), \quad t,x \in I, \\ G(x,t,\bar{\alpha},Q^{+},\bar{W},A_3,A_4) &= U(x)L(x,t,s-\bar{\alpha},P^{+},\bar{W})U^{-1}(t), \quad t,x \in I, \end{aligned}$$

the P^{+} here is a matrix defined just as P with Q^{+} instead of Q then the regular problem (4.18) has a unique solution Y for each $V^{-1}R \in L(I)$ which is given by

$$Y(x) = \int_{a_1}^{a_2} K(x,t,s-\alpha,P,W)V^{-1}(t)R(t)dt, \quad a_1 < x < a_2,$$

ehich yields to

$$V^{-1}(x)X(x) = \int_{a_1}^{a_2} V^{-1}(x)G(x,t,\alpha,Q,W)V(t)V^{-1}(t)F(t)dt, \quad a_1 < x < a_2,$$

and therefore, the solution of the system form will be

$$X(x) = \int_{a_1}^{a_2} G(x,t,\alpha,Q,W)R(t)dt, \quad a_1 < x < a_2.$$

With a similar approach, we can find the solution of (4.17) which will be given in the form

$$X(x) = \int_{a_1}^{a_2} G(x,t,\bar{\alpha},Q^{+},\bar{W})G(t)dt, \quad a_1 < x < a_2.$$

Now with given above, the following statements are equivalent:

1. $A_1KA_3^{*} = A_2KA_4^{*}$;

2. $K(x,t,P,s-\alpha)=K^{-1}L^{*}(t,x,P^{+},s-\bar{\alpha})K,\ t,x\in I;$

3. $G(x,t,\alpha,Q,W)=-K^{-1}G^{*}(t,x,\bar{\alpha},Q^{+},\overline{W})K,\ t,x\in I;$

4. $K_{1n}(x,t,\alpha,Q,W)=(-1)^{n}\overline{L}_{1n}(t,x,\bar{\alpha},Q^{+},\overline{W}),\ t,x\in I.$

In particular, we can use 1-4 to prove that

$$G_{1n}(x,t,\alpha,Q,W)=(-1)^{n}G_{1n}(t,x,\bar{\alpha},Q^{+},\overline{W}),\ t,x\in I.$$

Proof: the equivalence between 1 and 2 can be seen from the theorem 3.3, and to show that 3 is equivalent to 2 we start with:

$$\begin{aligned}-K^{-1}G^{*}(t,x,\bar{\alpha},Q^{+},W)K &= -K^{-1}[U(t)L(t,x,s-\bar{\alpha},P^{+},W)U^{-1}(x)]^{*}K\\ &= -K^{-1}U^{-1*}(x)KK^{-1}L^{*}(t,x,s-\bar{\alpha},P^{+},W)KK^{-1}U^{*}(t)K\\ &= V(x)K(x,t,s-\alpha,P,W)V^{-1}(t)\\ &= G(x,t,\alpha,Q,W).\end{aligned}$$

As a last step, the equivalence of 3 and 4 can be shown similarly to result of the Theorem 3.6.

CHAPTER 5

(APPLICATION) THE GREEN'S FUNCTION OF LEGENDRE EQUATION

In this last chapter, we will give an illustration of the above results by finding the Green's function of the Legendre singular equation. For this we will take the Legendre equation in the form

$$-(px')' = \alpha x\,, \quad p(s) = 1 - s^2 \;\; on \;\; I = (-1,1), \tag{5.1}$$

which is one of the simplest singular differential equations and has a lot of applications related to it in applied and pure mathematics. In the equation above the q is zero, with a weight function w equaled to 1 and p is the simple quadratic function $1-s^2$ with two singular points at -1 and 1 which are regular singularities. The equation (5.1) and its self-adjoint operators, in spite of its simplicity, plays an important role in a lot of phenomena.

Next, we will use five steps as shown above is the pervious chapters to find the Green's function of the given Legendre equation, which are:

(1) Transform the singular second-order scalar equation of Legendre (5.1) into a first-order singular system.

(2) Construct a regular system equivalent to the singular system 'Regularizing'.

(3) Find the Green's matrix of the constructed regular system of BVPs.

(4) Build the singular Green's matrix of the singular system which is equivalent to the regular one.

(5) Determine the Green's function of the singular problem (5.1), be extracting the upper right entrie from the singular Green's matrix.

For $\alpha = 0$, we can find two linearly independent solutions of (5.1) given by

$$x_1(s) = 1, \qquad x_2(s) = -\frac{1}{2}\ln\left(\left|\frac{1-s}{s+1}\right|\right). \tag{5.2}$$

We can write the equation (5.1) like a second order system of the form

$$X' = (P - \alpha W)X \quad on(-1,1), \tag{5.3}$$

where

$$X = \begin{pmatrix} x \\ px' \end{pmatrix}, \quad P = \begin{pmatrix} 0 & 1/p \\ 0 & 0 \end{pmatrix}, \quad W = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{5.4}$$

Now let

$$V = \begin{pmatrix} x_1 & x_2 \\ px_1' & px_2' \end{pmatrix} = \begin{pmatrix} 1 & x_2 \\ 0 & 1 \end{pmatrix}. \tag{5.5}$$

And note that $\det V = 1$ for $s \in I$, let's define H as

$$H = V^{-1}X. \tag{5.6}$$

Then H' will be

$$\begin{aligned} H' &= (V^{-1})'X + V^{-1}X' \\ &= -V^{-1}V'V^{-1}X + (V^{-1})(P - \alpha W)X \\ &= -V^{-1}V'H + (V^{-1})(P - \alpha W)VH \\ &= -V^{-1}(PV)H + V^{-1}(PV)H - \alpha(V^{-1}WV)H \\ &= -\alpha(V^{-1}V)H. \end{aligned}$$

Which we can write it in the form

$$H' = -\alpha GH, \tag{5.7}$$

by letting

$$G = V^{-1}WV = \begin{pmatrix} -x_2 & -x_2^2 \\ 1 & x_2 \end{pmatrix}. \tag{5.8}$$

The equation (5.7) is called the regularized Legendre system.

In order to show the relationship between the regularized system ant the scalar singular equation (5.1), we present the following theorem.

Theorem 5.1: Let G be given by (5.8) and let $\alpha \in \mathbb{C}$ then:

(1) Every component of the matrix G is in $L^1(-1,1)$ which make the system (5.7) regular.

(2) For any $c_1, c_2 \in \mathbb{C}$ the solution of the IVP

$$H' = -\alpha G H, \qquad H(-1) = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \tag{5.9}$$

is defined uniquely on the close interval $[-1,1]$.

(3) Let X given by

$$X = \begin{pmatrix} x(s,\alpha) \\ (px)(s,\alpha) \end{pmatrix}$$

be a solution of the singular system (5.3) and H be given by

$$H = V^{-1} X = \begin{pmatrix} h_1(s,\alpha) \\ h_2(s,\alpha) \end{pmatrix},$$

then H is a solution of the regular system (5.7), and furthermore for any $s \in (-1,1)$ we get the relationships

$$x(s,\alpha) = x_1(s) h_1(s,\alpha) + x_2(s) h_2(s,\alpha) = h_1(s,\alpha) + x_2(s) h_2(s,\alpha) \tag{5.10}$$

$$(px')(s,\alpha) = (px_1')(s) h_1(s,\alpha) + (px_2')(s) h_2(s,\alpha) = h_2(s,\alpha) \tag{5.11}$$

(4) The quasi-derivative px' for any solution x of the singular Legendre equation (5.1), is continuous for every $s \in [-1,1]$, so

$$\lim_{s \to -1^+} (px')(s,\alpha) = h_2(-1,\alpha), \quad \lim_{s \to 1^{-1}} (px')(s,\alpha) = h_2(1,\alpha). \tag{5.12}$$

And therefore, the quasi-derivative is continuous on $[-1,1]$ for every $\alpha \in \mathbb{C}$.

(5) If $h_2(1,\alpha)\neq 0,$ the $x(s,\alpha)$ given by (5.10) is unbounded at 1, and similarly if $h_2(-1,\alpha)\neq 0,$ the $x(s,\alpha)$ is unbounded at -1.

(6) For a fixed $s\in[-1,1]$, $c_1,c_2\in\mathbb{C}$. And let consider

$$H=\begin{pmatrix} h_1(s,\alpha)\\ h_2(s,\alpha)\end{pmatrix}$$

as a solution of the regularized system (5.7) with the initial conditions $h_1(-1,\alpha)=c_1, h_2(-1,\alpha)=c_2$, then the function $h_i(t,\alpha), i=1,2$ is an entire function of α, the same could be made for the initial conditions $h_1(1,\alpha)=c_1, h_2(1,\alpha)=c_2$.

(7) For each $\alpha\in\mathbb{C}$ there are non-trivial solutions which are bounded in neighborhoods of 1 and -1, and they are generally different.

(8) If x is a non-trivial solution of the singular equation (5.1), then y is bounded in 1 and -1 if and only if $h_2(1,\alpha)=0$ and $h_2(-1,\alpha)=0$ respectively.

Proof: From the definition of the matrix G the first part (1) is proved, and the second part (2) comes directly from (1) and the general theory of general systems, to prove (3) we use the fact that $X=VH$ which yields to (4) and (5) too, (2) and the basic theory of the general systems gives (6), to prove (7) we find the solutions $x_1(s,\alpha), x_{-1}(s,\alpha)$ of the equation (5.1) as a power series using the Forbenius method as in [5] in the form:

$$x^1(s,\alpha)=1+\sum_{n=1}^{\infty}a_n(\alpha)(s-1)^n,\quad |s-1|<2, \tag{5.13}$$

$$x^{-1}(s,\alpha)=1+\sum_{n=1}^{\infty}b_n(\alpha)(s+1)^n,\quad |s+1|<2. \tag{5.14}$$

Here -1 and 1 are not refer to powers but the point that each solution is bounded at.

To prove the last part (8) of this theorem we start from the relation (5.10) that if $h_2(-1,\alpha)\neq 0$ then $x(s,\alpha)$ is unbounded at -1, assume $h_2(-1,\alpha)=0$, if the equivalent solution $x(s,\alpha)$ is unbounded at -1, then there will be two solutions unbounded at -1 so every non-trivial solutions is unbounded at -1 and this contradiction makes (8) and complete the proof.

Remark 5.2: From the previous theorem we note that, the equation (5.1) has a bounded solution $x^1(s,\alpha)$ at 1, and bounded solution $x^{-1}(s,\alpha)$ at -1, for every $\alpha \in \mathbb{C}$.

It is known that the Legendre polynomials P_n are bounded solutions on $(-1,1)$ for $\alpha_n = n(n+1) : n \in \mathbb{N}_0$.

Next we find the primary fundamental matrix of the regular system (5.7).

Definition 5.3: For a fixed $\alpha \in \mathbb{C}$. $\Phi(.,.,\alpha)$ is the primary fundamental matrix of the regular system (5.7), that means for every $t \in [-1,1], \Phi(.,t,\alpha)$ is the matrix unique solution of the problem

$$\Phi(t,t,\alpha) = I_2 \tag{5.15}$$

here I_2 is 2×2 identity matrix. And since the system is regular the primary fundamental matrix $\Phi(s,t,\alpha)$ is defined for every $t,s \in [-1,1]$ and is entire functions of α for every fixed t,s.

Now let $A_1, A_2 \in M_2(\mathbb{C})$, and take the two-point conditions for the BVP

$$H' = -\alpha G H, \qquad A_1 H(-1) + A_2 H(1) = 0. \tag{5.16}$$

Lemma 5.4: The equation (5.16) has a complex eigenvalue $-\alpha$, if and only if

$$\Delta(\alpha) = \det\left[A_1 + A_2 \Phi(1,-1,-\alpha)\right] = 0. \tag{5.17}$$

And furthermore, $-\alpha \in \mathbb{C}$ is an eigenvalue with multiplicity of 2 if and only if

$$A_1 + A_2 \Phi(1,-1,-\alpha) = 0. \tag{5.18}$$

Proof: Using the initial condition $H(-1) = C$ the solution is

$$H(s) = \Phi(s,-1,-\alpha)C, \quad s \in [-1,1]. \tag{5.19}$$

Then the BVP (5.16) has non-trivial solution if and only if

$$\left[A_1 + A_2\Phi(1,-1,-\alpha\right]H(-1) = 0 \tag{5.20}$$

has a non-trivial solution for $H(-1)$.

For the second part of this lemma, note that any two linearly independent solutions of (5.20) with $H(-1)$ are giving two linearly independent solutions of (5.16), and the converse also holds.

For any real eigenvalue $\alpha \in \mathbb{R}$ and any solutions x, h of the equation (5.1), the Lagrange from will be defined as

$$[x,h](s) = x(s)(ph')(s) - h(s)(px')(s).$$

Then for the given above solutions $x_1(s), x_2(s)$ of the equation (5.1), we have

$$[x_1,x_2](s) = +1, \quad [x_2,x_1](s) = -1, \quad [x,x_1](s) = -(px')(s), \quad for\, s \in \mathbb{R},$$

$$[x,x_2](s) = x(s) - x_2(s)(px')(s), \quad for\, s \in \mathbb{R}, \quad s \neq \pm 1.$$

We will now find the Green's function of the singular equation of Legendre type (5.1) which include the equation

$$Mx = -(px')' = \alpha x + b \quad on\ I = (-1,1), \quad p(s) = 1 - s^2, \quad s \in I, \tag{5.21}$$

with the two-point boundary conditions

$$A_1\begin{bmatrix}(-px')(-1) \\ (x - x_2 px')(-1)\end{bmatrix} + A_2\begin{bmatrix}(-px')(1) \\ (x - x_2 px')(1)\end{bmatrix} = \begin{bmatrix}0 \\ 0\end{bmatrix}, \tag{5.22}$$

where x_2 is given as in (5.2), and $A_1, A_2 \in M_2(\mathbb{C})$ are complex. The construction is based on the method we have already discussed above. Let us consider the system

$$H' = -\alpha G H + R, \quad A_1 H(-1) + A_2 H(1) = 0, \tag{5.23}$$

which is regular, and

$$R = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}, \quad r_i \in L^1(I,\mathbb{C}), \ i = 1,2. \tag{5.24}$$

Theorem 5.5: Consider the non-homogeneous system (5.23) with (5.24) on $I=(-1,1)$. For $-\alpha\in\mathbb{C}$ let $\Delta(-\alpha)=\left[A_1+A_2\Phi(1,-1,-\alpha)\right]$. then the following statements are equivalent.

(1) If $R=0,$ then the problem (5.23) has the trivial solution only.

(2) $\left[A_1+A_2\Phi(1,-1,-\alpha)\right]$ is non-singular.

(3) For every $R\in L^1(I)$ the non-homogeneous problem has a unique solution H given as

$$H(s,-\alpha)=\int_{-1}^{1}G(s,t,-\alpha)R(t)dt,\quad -1\le s\le 1, \tag{5.25}$$

For

$$G=\begin{cases}\Phi(s,-1,-\alpha)\Delta^{-1}(-\alpha)(-A_2)\Phi(1,t,-\alpha), & -1\le s\le t\le 1\\ \Phi(s,-1,-\alpha)\Delta^{-1}(-\alpha)(-A_2)\Phi(1,t,-\alpha)+\Phi(s,t,-\alpha), & -1\le t\le s\le 1\\ \Phi(s,-1,-\alpha)\Delta^{-1}(-\alpha)(-A_2)\Phi(1,t,-\alpha)+\dfrac{1}{2}\Phi(s,t,-\alpha), & -1\le t=s\le 1\end{cases} \tag{5.26}$$

Proof: The proof is similar to the one we studied inTtheorem 3.1.

Theorem 5.6: Suppose that $\Delta(-\alpha)$ is non-singular, and the function b in the equation (5.21) satisfies

$$b,x_2b\in L^1(I,\mathbb{C}), \tag{5.27}$$

then the Legendre problem (5.21), (5.22) has a unique solution $x(s,-\alpha)$ given as

$$x(s,-\alpha)=\int_{-1}^{1}L_{12}(s,t)b(t)dt,\quad -1<s<1. \tag{5.28}$$

Where L_{12} is the upper right component of the 2×2 matrix L given by the relation

$$L(s,t,-\alpha)=V(s)G(s,t,-\alpha)V^{-1}(t),\quad -1\le s,t\le 1. \tag{5.29}$$

Proof: Let R and B be given as

$$R = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = V^{-1}B, \quad B = \begin{pmatrix} 0 \\ -b \end{pmatrix}. \tag{5.30}$$

Then $r_1, r_2 \in L^1(I, \mathbb{C})$, and since $X(s,-\alpha) = V(s)H(s,-\alpha)$ then from (5.25) we got

$$\begin{aligned} X(s,-\alpha) &= V(s)H(s,-\alpha) \\ &= V(s)\int_{-1}^{1} G(s,t,-\alpha)R(t)dt \\ &= \int_{-1}^{1} V(s)G(s,t,-\alpha)V^{-1}(t)B(t)dt \\ &= \int_{-1}^{1} L(s,t,-\alpha)B(t)dt, \quad -1 < s < 1, \end{aligned} \tag{5.31}$$

so

$$x(s,-\alpha) = -\int_{-1}^{1} L_{12}(s,t,-\alpha)b(t)dt, \quad -1 < s < 1. \tag{5.32}$$

CHAPTER 6

CONCLUSION

In this thesis, our main objective was to construct the Green's function for a singular BVPs. We started with finding the Green's function and its adjoint form for the regular system of BVPs. Then we defined the quasi-derivatives and found the Green's function for the quasi-differential equation which was the basis for the singular system of BVPs. We have also constructed the Green's function of the singular problems on real interval with singular or regular (LC) end points, which may be finite or infinite with no oscillatory on the solutions or the coefficients, where the components of Q and W were assumed to be in $L_{loc}(I)$ with no necessity to be in $L(I)$.

As an illustration, a simple singular Legendre problem was considered. Using some given five steps, we were able to formulate the Green's function for the singular Legendre problem.

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