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INTRODUCTION TO VERTEX-COLORING PROBLEM

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
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THE DEGREE OF MASTER OF SCIENCE
IN
MATHEMATICS

BY

MOHAMMED JABBAR ABDULLAH AL-SHAFEAY

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INTRODUCTION TO VERTEX-COLORING PROBLEM

By Mohammed Jabbar Abdullh AL-SHAFEAY

February 2022

We certify that have read this thesis and that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

Advisor : Asst. Prof. Dr. Celalettin KAYA
Co-Advisor : Asst. Prof. Dr. Mohammed Shakir ZABIBA

Examining Committee Members:

Chairman : Assoc. Prof. Dr. İbrahim ÖZEN
Mathematics Department
Marmara University

Member : Assoc. Prof. Dr. Faruk KARAASLAN
Mathematics Department
Çankırı Karatekin University

Member : Asst. Prof. Dr. Celalettin KAYA
Mathematics Department
Çankırı Karatekin University

Approved for the Graduate School of Natural and Applied Sciences

Prof. Dr. İbrahim ÇİFTÇİ
Director of Graduate School

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Mohammed Jabbar Abdullh AL-SHAFEAY

ABSTRACT

INTRODUCTION TO VERTEX-COLORING PROBLEM

Mohammed Jabbar Abdullh AL-SHAFEAY
Master of Science in Mathematics
Advisor: Asst. Prof. Dr. Celalettin KAYA
Co-Advisor: Asst. Prof. Dr. Mohammed Shakir ZABIBA
February 2022

The vertex coloring problem is about using the smallest number of colors possible to color the vertices of a graph properly, and also related to ways to find this coloring. In the first chapter of this thesis, we introduced this famous problem by giving its origin and historical development. In the second chapter, we gave basic definitions and theorems about graphs. In the third chapter, we defined the chromatic number of a graph, and introduced related results. We also presented some of its applications such as finding a time line for solving some real-life problems. In the fourth chapter, we looked at how to calculate chromatic number for a graph, and its upper and lower bounds. We looked at one of the ways to find the chromatic number, which is the greedy coloring algorithm. We also found the chromatic number of the Cartesian product of two simple graphs. In the fifth and final chapter, after introducing the four-color problem, we reviewed the topic of coloring a graph on a surface. Finally, we also mentioned Hajos and Hadwiger conjectures.

2022, 32 pages

Keywords: Graphs, Vertex coloring, Chromatic number, Chromatic polynomaial, Chromatic bounds

ÖZET

KÖŞE RENKLENDİRME PROBLEMİNE GİRİŞ

Mohammed Jabbar Abdullh AL-SHAFEAY
Matematik, Yüksek Lisans
Tez Danışmanı: Dr. Öğr. Üyesi Celalettin KAYA
Eş Danışman: Dr. Öğr. Üyesi Mohammed Shakir ZABIBA
Şubat 2022

Köşe renklendirme problemi, bir grafın köşelerini uygun bir şekilde renklendirmek için mümkün olan en az sayıda rengi kullanmak ve bu renklendirmeyi bulmanın bazı yollarını göstermek ile ilgilidir. Bu tezin birinci bölümünde, bu ünlü problemi kökenini ve tarihsel gelişimini vererek tanıttık. İkinci bölümde, graflarla ilgili temel tanım ve teoremleri verdik. Üçüncü bölümde, bir grafın kromatik sayısını tanımladık ve ilgili sonuçları sunduk. Bunun, bazı gerçek hayat problemlerinin çözümü için bir zaman çizelgesi bulma gibi bazı uygulamalarını sunduk. Dördüncü bölümde, bir graf için kromatik sayının nasıl hesaplanacağını ve üst ve alt sınırlarını inceledik. Kromatik sayıyı bulmanın yollarından birisi olan açgözlü renklendirme algoritmasına değindik. Ayrıca, iki basit grafın Kartezyen çarpımının kromatik sayısını bulduk. Beşinci ve son bölümde, dört renk problemini tanıttıktan sonra, bir yüzey üzerindeki bir grafın renklendirilmesi konusunu ele aldık. Son olarak, Hajos ve Hadwiger varsayımlarından da bahsettik.

2022, 32 sayfa

Anahtar Kelimeler: Graflar, Köşe renklendirme, Kromatik sayı, Kromatik polinom, Kromatik sınırlar

PREFACE AND ACKNOWLEDGEMENTS

I begin by thanking God Almighty for what He has bestowed upon us.

The graph coloring problem is one the fundamental and very famous problems in graph theory and it is also closely related to chromatic number which is another famous notion. In this thesis, we introduced the basic notions and theorems about graph coloring, and exhibited this very rich and active research area at a basic level. In other words, we did not solve an open problem in this area or obtain an original result about vertex coloring, but we presented a survey / a tutorial source about vertex coloring and related topics for non-experts to this area. And in our thesis, we especially benefited from the textbook "Chromatic Graph Theory" written by Gary Chartrand and Ping Zhang, you can find this resource in the references. We express our profound thanks to Gary Chartrand and Ping Zhang for their excellent textbook.

I want to express my thanks and gratitude to the supervisor **Asst. Prof. Dr. Celalettin KAYA**, I am also grateful to him again for teaching me introduction Graph theory. I would like to thank the **Chairperson and Members** of the discussion committee cussing the research. I can extend my thanks to **my family** for standing with me and encouraging me throughout my studies. I am very grateful to **Dr. Mohamed Shakir ZABIBA in University of Kufa** for providing him with the necessary technical suggestions as I pursue my written. And I have not forgotten my thanks to all of my teachers in the Department of Mathematics, my gratitude for their.

Mohammed Jabbar Abdullh AL-SHAFEAY
Çankırı-2022

CONTENTS

LIST OF SYMBOLS

$G=(V,E)$	Graph G with vertex set V and edge set E
v	Vertex
e	Edge
$V(G)$	The set of vertices in a graph G
$E(G)$	The set of edges in a graph G
$\mid G \mid$ or n	The number of vertices in a graph G
$\mid\mid G \mid\mid$ or m	The number of edges in a graph G
$\Delta(G)$	Maximum degree of a graph G
$\delta(G)$	Minimum degree of a graph G
$deg(v)$	Degree of a vertex v
$\sum$	Summation
max	Maximum
ϕ	The empty graph
C_n	Cycle graph with n vertices
P_n	Path graph with n vertices
K_n	Complete graph on n vertices
$c(v)$	Coloring of a vertex v
$\chi(G)$	Chromatic number of a graph G
$\omega(G)$	Clique number of a graph G
$\alpha(G)$	Independence number of a graph G
$\ell(D)$	Length of a directed path D
$H\square K$	Cartesian product
$K_{n,m}$	Complete bipartite graph
$\overline{G}$	The complement graph of a graph G
$G[S]$	The graph induced by a subset $S\subseteq V$
D	Directed graph or digraph
$V(D)$	The vertex set of a digraph D
$A(D)$	The arc family of a digraph D
$P(G,\lambda)$	The chromatic polynomial of a graph G
$g(G)$	The density of a graph G

LIST OF FIGURES

1. INTRODUCTION

The problem with coloring graph vertices is a side effect of the initial problem, the four-color issue that is the coloring of planar graphs such as coloring a map.

In 1852 Francis Guthrie said: Is it possible to suffice with four colors to color the map of England? This problem was brought up for discussion in 1879 at the London Mathematical Society by Arthur Cayley. At the time Alfred Quimby published a paper proving that saying (Kubale 2004). But in 1890 Heywood refuted Quimby's proof of this and he gave a guess that five colors could be chosen to color the planar graph. In 1912 George Ra Ven used chromatic polynomials to prove and find the chromatic number of graphs (Jensen and Toft 1995; Van Lint et al. 2001). In 1972 the problem of colors became a computational problem and several algorithms were found and developed to prove the coloring of graphs including (Karp 1972). For example, the greedy coloring algorithm was introduced and it was considered a complete problem NP. The idea of coloring vertices appeared by choosing a point in each of the areas of the planar graph or the map where these points are called vertices and straight or curved lines, which are called the edges, are connected between each two vertices of two adjacent areas of the graph to form a system called a proper graph coloring. The goal of such a coloring of vertices is to use the fewest colors possible (called the chromatic number) (Chakraborty et al. 2018).

To color such a graph and ways to find it are the objective of this thesis. We also gave some applications in engineering science and communication as well as solved some problems in our lives. We also mentioned an applied example that we always encounter in our academic life.

2. PRELIMINARIES

This section contains a set of basic definitions. Some theorems, corollaries, and some types of graphs that are be used in the following sections.

2.1 Basic Definitions

Definition 2.1. (Saoub 2021)
The graph H is made up of two units that are the vertex set denoted by $V(H)$ and the edge set denoted by $E(H)$. As a shorthand we will use H and sometimes $H = (V,E)$.
An edge *ad* is a pair of vertices where the order does not matter.

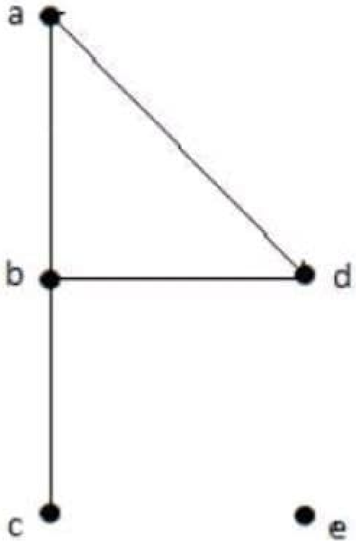


Figure 2.1 An example to illustrate the graph H (Tutorialspoint 2021)

Definition 2.2. (Diestel 2018)
The **order** of a graph H expressed as $\mid H \mid$ or n is the total of all vertices.
The **size** of a graph H is the number of edges possessed by a graph, and $\| H \|$ or m represents this number.

Example 2.3. For the graph seen in Figure 2.1:
The set $V(H) = \{e,d,a,b,c\}$ is the set of vertices of the graph H.
So $n = 5$.
The set $E(H) = \{ab,ad,bc,bd\}$ is the set of the edges of the graph H.
So $m = 4$.

Definition 2.4. (Saoub 2021)

If G is a graph, then:

1. The a, b are the **endpoints** of the edge ab.
2. If a is the endpoint of ab, we call it **incident** to edge ab.
3. If the two vertices are incident to a certain edge, we will call the vertices **adjacent** and denote it by $a \sim b$.
4. We call two edges **adjacent** if they contain one common endpoint.
5. We say two vertices are **neighbors** if they are adjacent, and the collection of all neighbors of any vertex a is designated by $N(x)$.
6. The two vertices or edges are called **independent** if they are not adjacent.
7. We name a vertex as **isolated** if this is not incident to any edge.
8. The term **loop** is used for an edge, if both of its endpoints are the same vertex.
9. We consider two edges as **multi-edges** or **parallel edges** when they have the same terminal vertices.
10. We define a graph **simple** if it has no multi-edges and no loops.

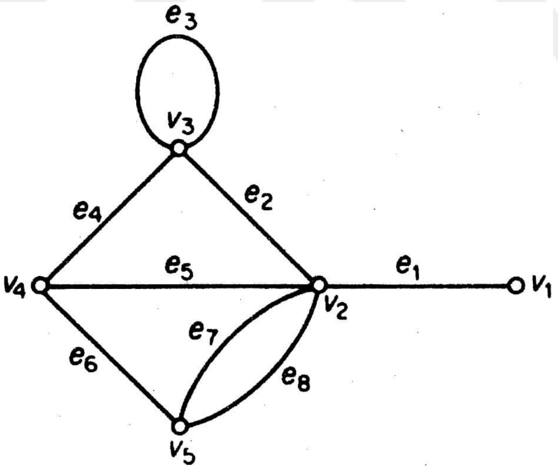


Figure 2.2 This graph shows the loop and the multi edges (Bondy and Murty 1976)

Example 2.5. For the graph given in Figure 2.2:

The vertices v_1 and v_2 are endpoints to the edge e_1.

v_1 incident to the edge e_1. There is no isolated vertex.

v_3 and v_4 adjacent vertices, and e_4, e_5 and e_6 are adjacent edges.

For example, $N(v_1) = \{v_2\}$, but $N(v_2) = \{v_1, v_3, v_4, v_5\}$.

For example, v_1 and v_3 independent vertices, and e_1 and e_4 independent edges.

The edge e_3 is a loop, the edges e_7 and e_8 multi-edges. The graphs is not simple.

Definition 2.6. (Kubale 2004)

The $deg(v)$ is the number of all edges that are incident with the vertex v in a graph H, and it is called the **degree** of the vertex v. That is, $deg(v) = |\{e \in E : v \in e\}|$.

$\Delta(H)$ is the greatest degree of all vertices in a graph H, whereas $\delta(H)$ indicates the lowest degree.

That number $g(H) = \frac{2m}{n(n-1)}$ is the **density** for a graph H.

Example 2.7. For the graph given in Figure 2.1:

$deg(e) = 0$, $deg(c) = 1$, $deg(a) = deg(d) = 2$ and $deg(b) = 3$.

Then $\Delta(H) = 3$ and $\delta(H) = 0$.

Theorem 2.8. First Theorem to Graphs (Chartrand and Zhang 2019)

Suppose G is a size m graph, then:

$$\sum_{v \in V(G)} deg(v) = 2m$$

2.2 Types of Graphs

Definition 2.9. (Gross and Yellen 2005)

The edge sets of a **null graph** is empty.

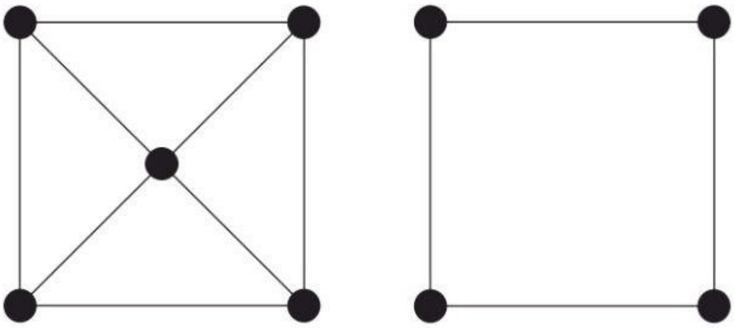
Figure 2.3 A null graph with only three vertices (Tutorialspoint 2021)

Example 2.10. For the graph seen in Figure 2.3:

a, b and c are vertices in this graph, but there are no edges between them. As a result it is a null graph.

Definition 2.11. (Diestel 2018)

We claim that H_1 **includes** H_2 if $V(H_2) \subseteq V(H_1)$ and $E(H_2) \subseteq E(H_1)$. If this is the case, H_2 is a **subgraph** of a graph H_1, and written as $H_2 \subseteq H_1$.



(a) Graph H_1 **(b)** Graph H_2

Figure 2.4 The graph H_2 is a subgraph of the graph H_1 (Diestel 2018)

Example 2.12. In Figure 2.4: The graph H_2 is a subgraph of the graph H_1.

Definition 2.13. (Diestel 2018)

The graph of order zero is called **empty** and is denoted by the letter (ϕ,ϕ) or ϕ. A **trivial graph** is a graph of order one.

Definition 2.14. (Diestel 2018)

The graph A is considered **connected** when it is not empty and also if there is a path for each of two vertices of it.

Example 2.15. In Figure 2.2:

We can move from a vertex let it be v_1 to any other vertex such as v_3 in the graph and through the following path $v_1 - v_2 - v_3$.

Definition 2.16. (Rahman 2017)

A graph consisting of a series of vertices $v_1, v_2, \ldots, v_n$ and edges $v_i v_{i+1}$ joining two of them for every $1 \leq i \leq n-1$, is called the **path graph** and the two vertices v_1 and v_n are called **endvertices** of graph.

Example 2.17. In (Figure 2.5)

A path P_6 is the path graph containing six vertices. P_n is a path graph that has n vertices.

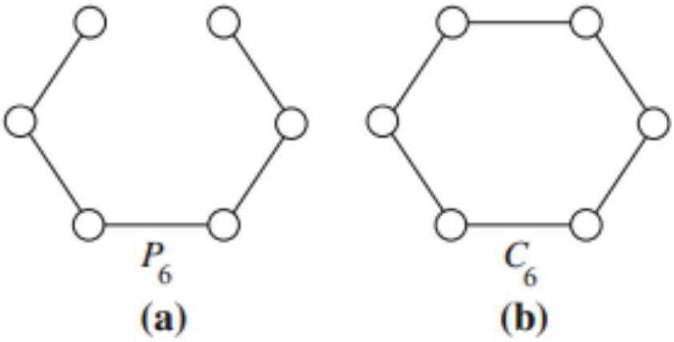
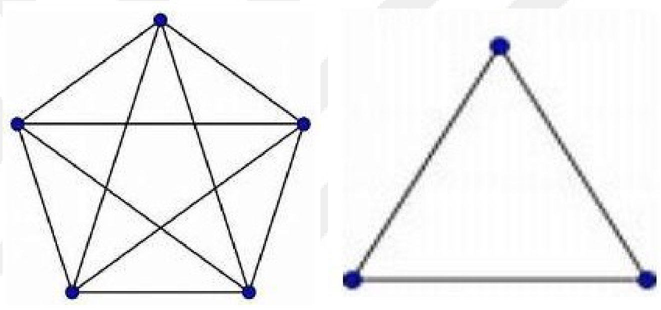


Figure 2.5 P_6 is a path graphs and C_6 is a cyclic graph (Rahman 2017)

Definition 2.18. (Diestel 2018)
A **complete graph** is one in which all of the vertices are connected by edges, and it is indicated by K_n if it has n vertices, and K_3 if it has 3 vertices and we call it **triangle**.



(a) Graph K_5 **(b)** Graph K_3

Figure 2.6 Complete graphs K_5 and triangle K_3 (Hammack et al. 2011)

Definition 2.19. (Chakraborty et al. 2018)
A **cyclic graph** is a graph that consists of a closed cycle that begins at one of the vertices and ends there.

Example 2.20. For the graph given in Figure 2.5(b):
It is a cyclic graph, and it is denoted by C_6. The symbol for a cyclic graph containing n vertices is C_n.

Definition 2.21. (Chartrand et al. 1996)
If all of its vertices of a graph have always the same degree r, it is a **regular** of degree r. These graphs are also known as r-**regular** graphs.
So the complete graph K_n is $(n-1)$ - regular and also all of cycle graphs are 2 - regular.

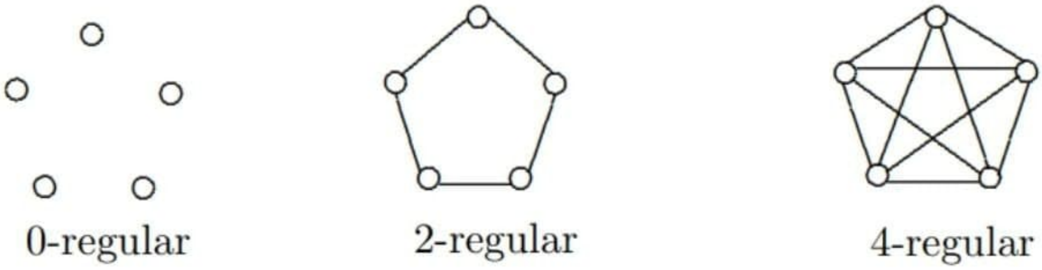


Figure 2.7 Some regular graphs of degree $r = 0$, 2, and 4 (Chartrand et al. 1996)

Example 2.22. Look at the Figure 2.7:
In 0-regular graph, there is no edge through it is vertices.
In 2-regular graph, each vertex has two edges.
In 4-regular graph,, each vertex has four edges.

Definition 2.23. (Dharwadker and Pirzada 2011)
If we have a graph B with its vertices divided into a group V_1 and a group V_2 where $V(B) = V_1 \cup V_2$, and if $ad \in E(B)$ then it must be $a \in V_1$ and $d \in V_2$ or $a \in V_2$ and $d \in V_1$ then it is called a **bipartite** binary graph and is also called **2-partite**.
If $ad \in E$ for all $a \in V_1$ and $d \in V_2$ so it is called **complete bipartite** and denoted by $K_{n,m}$ if it is $\mid V_1 \mid = n, \mid V_2 \mid = m$.

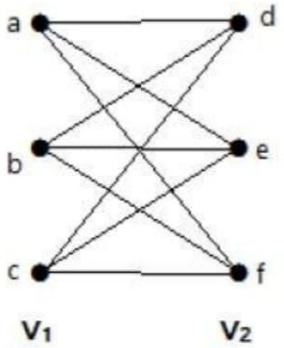


Figure 2.8 Bipartite graph B (Tutorialspoint 2021)

Example 2.24. Look at the Figure 2.8:
It is a bipartite graph such that V_1 and V_2 two partition of that graph.
In fact, it is $K_{3,3}$, that is, the complete bipartite graph such that $\mid V_1 \mid = 3, \mid V_2 \mid = 3$.

Definition 2.25. (Saoub 2021)
Let $H = (V,E)$ be a simple graph withe no multi-edges and no loops. Then the graph

$\overline{H} = (V,\overline{E})$ is its **complement** if $v_1v_2 \in \overline{E}$ if and only if $v_1v_2 \notin E$.

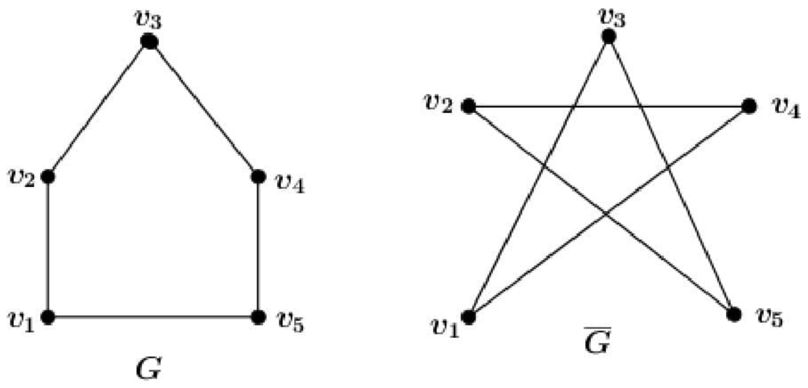


Figure 2.9 The graph G and it is complement $\overline{G}$ (Sadiquali and Sudhahar 2017)

Example 2.26. In the case of the graph depicted in Figure 2.9:
$G \subseteq K_5$ and $\overline{G} \subseteq K_5$ are complement of each other.

Definition 2.27. (Hartsfield and Ringel 2013)
A **tree** is a graph in which any two vertices are connected exactly one path.

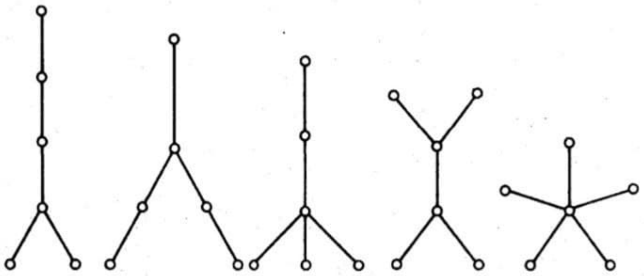


Figure 2.10 Some examples of trees (Bondy and Murty 1976)

3. COLORING OF THE VERTICES

3.1 The Chromatic Number of Graphs

Definition 3.28. (Kubale 2004)
If $H=(V,E)$ is a graph then coloring its vertices is the following function:

$$c:V\rightarrow N$$

We call c as a **coloring function**, where both vertices $a,b\in V$ are colored differently if they are adjacent. That is:

$$\{a,b\}\in E\Rightarrow c(a)\neq c(b).$$

If a graph H also has vertex-coloring which requires k colors it is termed k**-colorable** and such a coloring is called a k**-coloring**.
In this instance, a coloring function induces a split of the vertex set of the graph H such that $\{V_1,V_2,\ldots,V_k\}$ for any $V_i\cap V_j=\emptyset$ and $V_1\cup V_2\cup\ldots\cup V_k=V$.

Definition 3.29. (Kubale 2004)
Any subset V_1 of a graph's vertices is **independent** if for any

$$V_1\subseteq V\ \&\ a,b\in V_1\Rightarrow\{a,b\}\notin E$$

The number of elements of a largest independent subset is called the **independence number** of the graph G, and it is indicated by $\alpha(G)$.

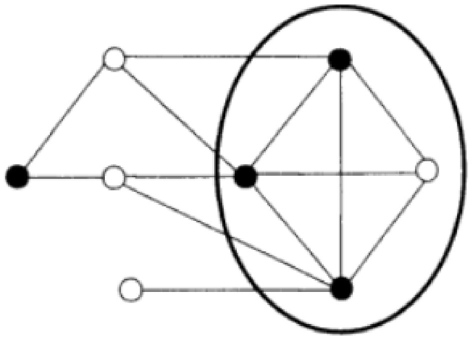


Figure 3.1 In this graph $\alpha(G)=4$ (Kubale 2004)

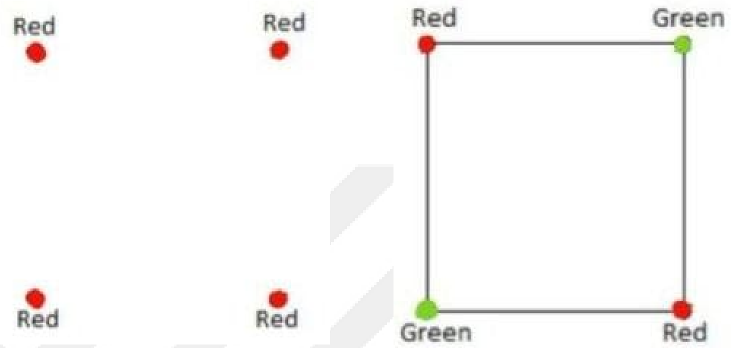
Example 3.30. For the graph given in Figure 3.1, $\alpha(G)=4$.

Definition 3.31. (Guichard 2017)

The largest value to n for which K_n is a subgraph of the graph G is called the **clique number**, and we denote it by $\omega(G)$.

Definition 3.32. (Kubale 2004)

The **chromatic number**, $\chi(A)$, for a graph A is the smallest integer k such that the graph A has one k-coloring. In this case, A is referred to be k-**chromatic**.



(a) Coloring graph A **(b)** Coloring graph B

Figure 3.2 Chromatic number for graphs A and B (Tutorialspoint 2021)

Example 3.33. In Figure 3.2:

$\chi(A) = 1$ of empty graph A and $\chi(B) = 2$ of graph B.

Remark 3.34. (Chakraborty et al. 2018; Chartrand and Zhang 2019)

1. If a graph H is k-colorable, then $\chi(H) \leq k$.
2. For each m-order graph H, $1 \leq \chi(H) \leq m$.
3. A graph H with order m has chromatic number m if and only if $H = K_m$.
4. A graph H has chromatic number $\chi(H) = 1$ if and only if H is an empty graph.
5. A graph H has chromatic number 2 if and only if it is a bipartite graph.

Example 3.35. Consider the colorings of the graph G given in Figure 3.3:

In A, 2-colors are used to color it.

In B, 3-colors are used to color it.

In A, 2-colors are used to color it.

Since the given graph is not empty, $\chi(G) \geq 2$. And we have a 2-coloring of the graph.

Therefore, $\chi(G) = 2$.

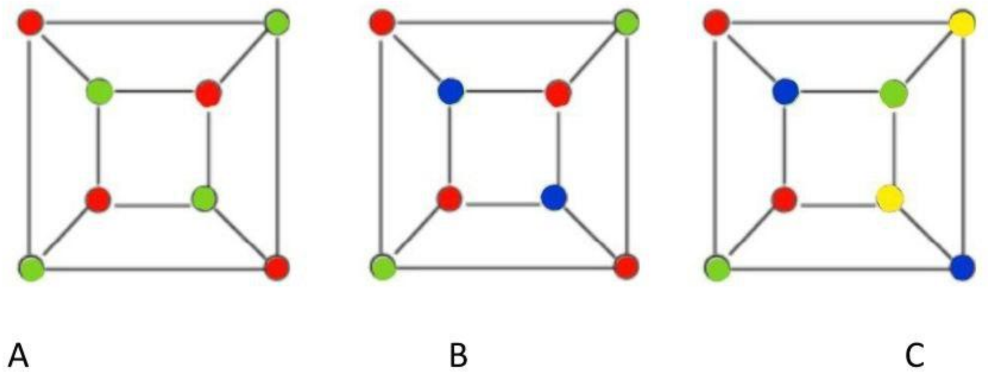


Figure 3.3 Some colorings of a graph G (Koh et al. 2015)

Theorem 3.36. (Dharwadker and Pirzada 2011)

- If $H_1,\ldots,H_n$ are components of a graph H, then $\chi(H)=\max\{\chi(H_i):1\le i\le n\}$.
- If a graph A has a subgraph B, then $\chi(B)\le\chi(A)$.
- For a graph A, the odd cycle exists if and only if $\chi(A)\ge 3$.

Theorem 3.37. (Chartrand and Zhang 2019)
If r is the order of the graph B, then

$$\frac{r}{\alpha(B)}\le\chi(B)\le r-\alpha(B)+1.$$

Theorem 3.38. (Chartrand et al. 1996)
For a graph A when each of its vertices is in at most n odd cycles, then

$$\chi(A)\le\lceil\frac{1+\sqrt{8n+9}}{2}\rceil.$$

Theorem 3.39. (Gross and Yellen 2005)
For every graph A, we have

$$\chi(A)\ge\omega(A).$$

3.2 Applications of Coloring

We use theorems concerning coloring of vertices of graphs to solve a range of problems that we face in various areas of life. We model the given problem as a graph such that the basic points of this problem are the vertices of this graph, then we determine the coloration of the vertices of the graph so that we can find an appropriate solution to the problem (Gross and Yellen 2005).

3.2.1 Making time table

The problem is setting a timetable over a period of time with specific conditions for solving a particular problem. For example, we want to define a set of tests for university students. What is the minimum time required for this purpose without conflicting exam times? We will illustrate this with an example that we have all experienced in our study life (Lewis 2015).

Example 3.40. (Chartrand and Zhang 2019)
Let $S = \{s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8\}$ be the set of eight students, and let $A = \{A_1, A_2, A_3, A_4, A_5, A_6, A_7\}$ be the set of seven courses. The students who will take the exams of these courses are listed as follows:

$A_1 = \{s_2, s_6, s_8\}$ $A_2 = \{s_1, s_4, s_7\}$ $A_3 = \{s_1, s_7, s_8\}$ $A_4 = \{s_5, s_6, s_7\}$
$A_5 = \{s_4, s_5, s_6\}$ $A_6 = \{s_2, s_3, s_4\}$ $A_7 = \{s_1, s_2, s_3\}$

How can we create an exam schedule with a minimum number of days such that all students enter their exams?

Solution:

We start by converting this problem into a graph and use the vertex coloring to solve this problem. Let $V(A) = \{A_1, A_5, A_6, A_7, A_2, A_4, A_3\}$. This set is the set of vertices of this graph, and there are edges between A_i and A_j if $A_i \cap A_j \neq \emptyset$. Now, we want to obtain a coloring of the vertices of this graph such that no two adjacent vertices have the same color as in Figure 3.4.

First, it is obvious that the independence number of this graph $\alpha(A) = 2$.

Therefore by Theorem 3.37, we have:

$$\frac{7}{2} = \frac{r}{\alpha(A)} \leq \chi(A) \leq r - \alpha(A) + 1 = 7 - 2 + 1.$$

Therefore, $\chi(A) = 4$. Thus the minimum number of colors needed is 4, and such a coloring is shown in Figure 3.4, and so the exams can be schedule as follows:

First day: A_1, A_4; Second day: A_2, A_5; Third day: A_3; Fourth day: A_6, A_7.

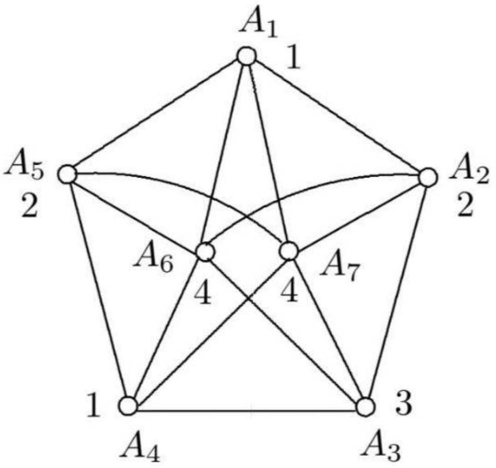
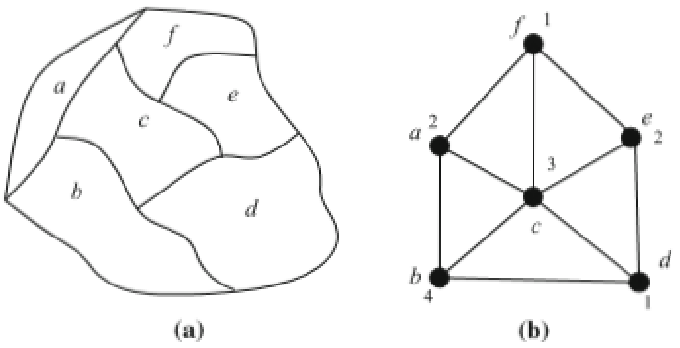


Figure 3.4 The graph *A* about an exam schedule (Chartrand and Zhang 2019)

3.2.2 Map coloring

The most well-known graph problem is about coloring maps: How to color the map of any country, state or province? How many colors for coloring each region on the map so that the adjacent regions have different colors? We put a vertex in the center of each district like the center of each city, and we connect every two adjacent vertices to get a graph. The coloration of the area on the map corresponds to color graph vertices such that adjacent graph vertices cannot have the same color. For example, Figure 3.5 shows that coloring the vertices needs four colors so we will need four colors to color the map (Rahman 2017).



(a) (b)

Figure 3.5 An example of a map coloring (Rahman 2017)

3.2.3 Chemical storage

Factories that need to operate require different and incompatible chemicals, and usually when these chemicals stored together can cause an explosion, fire or toxic gases. Therefore, storing them requires taking into account this while providing the least number warehouses required for storage. In the following example, we will show how to use the chromatic number to solve such problems (Koh et al. 2015).

Example 3.41. (Koh et al. 2015)
If in a factory we have six chemicals which are $\{G, H, L, N, T, P\}$, and some of them are incompatible with each other as follows:

Chemical	Incompatible with
P	$\{G, N\}$
G	$\{P, H, T\}$
H	$\{G, L, T\}$
L	$\{H, N\}$
N	$\{P, L\}$
T	$\{G, H\}$

Find a safe way with the fewest number of warehouses required to stock these items.

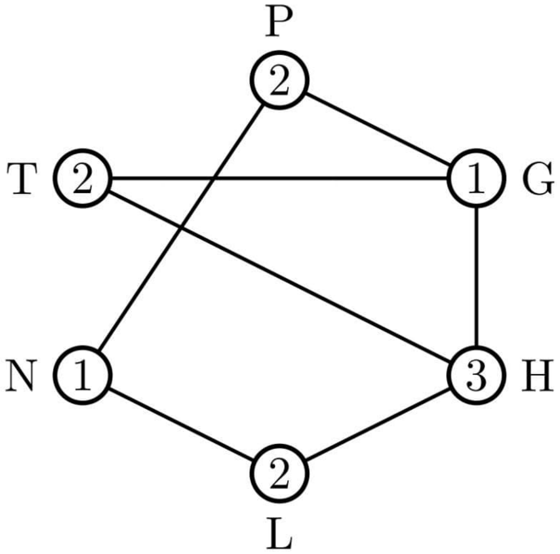


Figure 3.6 Graph C of chemical storage (Koh et al. 2015)

Solution:

We must find a graph C illustrating this problem provided that the chemicals represent the vertices of that graph C, and that there are edges between incompatible chemicals. Then coloring these vertices by the conditions and finding the chromatic number represent the least number of warehouses required to store these items. As in Figure 3.6. Then, since $\chi(C) = 3$, it is enough to allocate three warehouses for safe storage.

4. BOUNDS FOR CHROMATIC NUMBER

4.1 Upper and Lower Bounds to Chromatic Number

We want to determine upper and lower bounds of chromatic number to graphs, and find $\alpha(G)$ (independence number) and $\omega(G)$ (clique number) in the case of the graph of order n. We know that the clique number as well as the value of $\frac{n}{\alpha(G)}$ are lower bounds for chromatic number. On the other hand, we also know that n as well as $n-\alpha(G)+1$ are upper bounds for chromatic number.

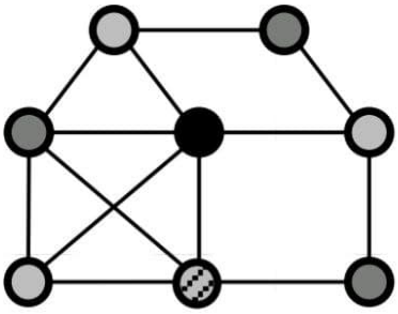


Figure 4.1 An example of a graph G (Lewis 2015)

Example 4.42. For graph given in Figure4.1:
$\alpha(G)=3$, $n=V(G)=8$, $\chi(G)=4$. Therefore by Theorem 3.37, we have:
$\frac{n}{\alpha(G)}=\frac{8}{3}=3$ is a lower bound for G.
$n-\alpha(G)+1=8-3+1=6$ is an upper bound for G.

Theorem 4.43. (Dharwadker and Pirzada 2011)
For each graph A,

$$\chi(A) \leq \triangle(A)+1$$

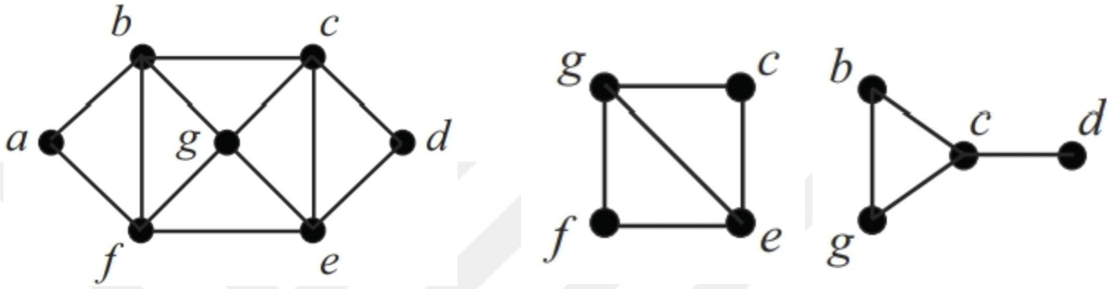
Example 4.44. Let $G=K_{1,n}$ a star graph, then $\chi(K_{1,n})=2$, but $\Delta(K_{1,n})=n$.

Theorem 4.45. Brooks's Theorem (Chartrand and Zhang 2019)
For each connected graph B which is neither an odd cycle and nor a complete graph, the following inequality is satisfied:

$$\chi(B) \leq \Delta(B)$$

Definition 4.46. (Benjamin et al. 2015)
Let S be a subset containing vertices from $V(G)$, then the subgraph H **induced by** S contains S and two vertices a, b of S are adjacent if and only if the two vertices are also adjacent in G. We call H the **induced subgraph** of G.

Example 4.47. In Figure 4.2, a garap G, and its two induced subgraphs H_1 are H_2 are depicted.



(a) Graph G **(b)** Graph H_1 **(c)** Graph H_2

Figure 4.2 Graphs H_1, H_2 are induced subgraphs of G (Werner 2017)

Theorem 4.48. (Chartrand and Zhang 2019)
If a graph B has order n and size m, then

$$\chi(B) \geq \frac{n^2}{n^2 - 2m}.$$

Theorem 4.49. (Chartrand et al. 1996)
For any graph A of order n, we have:

$$\chi(A) \leq \lfloor \frac{n + \omega(A)}{2} \rfloor.$$

Theorem 4.50. (Arif et al. 2018)
For each graph B, we have:

$$\chi(B) \leq 1 + max\{B'\},$$

where the maximum is calculated across every induced subgraphs B' of B.

4.2 Greedy Coloring Algorithm

One of the ways to color the vertices of a graph is the greedy coloring algorithm. This algorithm begins by arranging the vertices of the graph, and proceeds by coloring the vertices in a sequential manner and by allocating the lowest color to a vertex provided that the adjacent vertices have different colors.

Definition 4.51. (Herrmann and Sally 2012)

Let G be a graph of order n, and let $1, 2, \ldots, k$. The greedy coloring algorithm is carried out as follows provided that k-coloring is achievable:

1. Order the vertices of G as $1, 2, \ldots, n$.
2. Assign the first color to the first vertex.
3. Go to the second vertex, and assign the first color if it is not adjacent to the first vertex, otherwise assign the second color.
4. Repeat the previous steps for the rest of the vertices to finish coloring, and never assign the same color for any two adjacent vertices.

Example 4.52. (Koh et al. 2015)

Let color the graph H given in Figure 4.3 using the greedy algorithm. The procedure is shown in this figure:

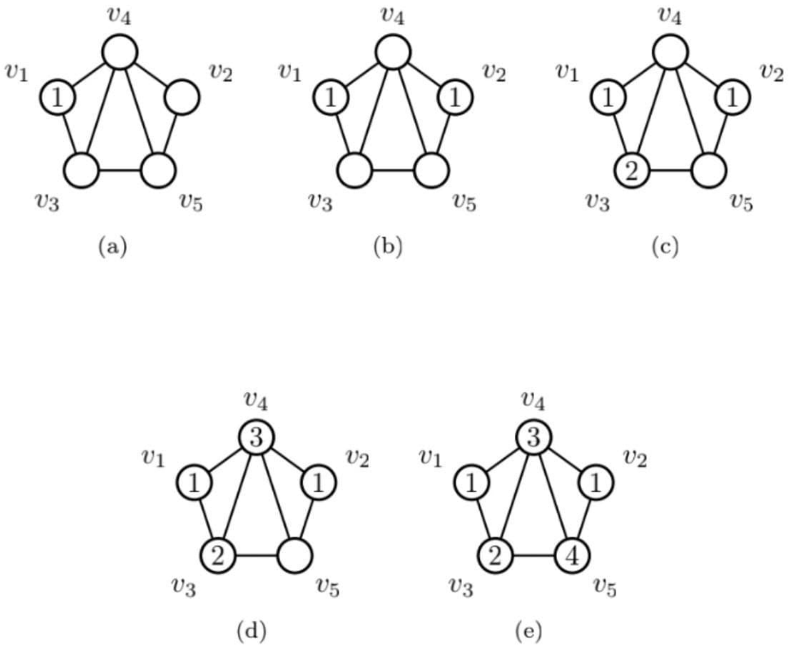


Figure 4.3 Greedy coloring algorithm for the graph H (Koh et al. 2015)

But if we take another arrangement of vertices of the graph H, and follow the algorithm to color its vertices, then this time we get another coloring as shown in Figure 4.4:

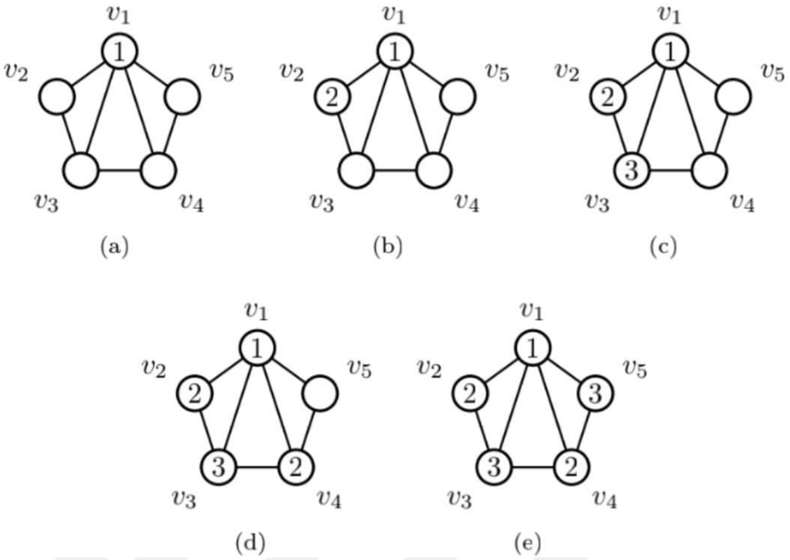


Figure 4.4 Greedy coloring algorithm for graph H (Koh et al. 2015)

Remark 4.53. (Koh et al. 2015)

We conclude from the previous example that the necessary number of colors used to color any graph using the algorithm depends directly on the arrangement of the vertices of the graph.

4.3 Upper Bounds to the Oriented Graphs

Definition 4.54. (Wilson 1979)

A **directed graph** or a **digraph** D consist of a set of vertices $V(D)$ and a set of arranged pairs of vertices which are called the **arcs** of D. The collection of arcs is denoted by $A(D)$, and an arc (a,b) is also denoted by the symbol ab.

Example 4.55. In Figure 4.5:

The Digraph $D = (V,F)$ has

$V(D) = \{u,v,w,z\}$ and

$F(D) = \{(u,v),(v,v),(v,w),(v,w),(w,v),(w,u),(z,w)\} = \{arcs\}$

Definition 4.56. (Thulasiraman and Swamy 2011)

The **length** of a directed path from w to v is the number of edges on this path.

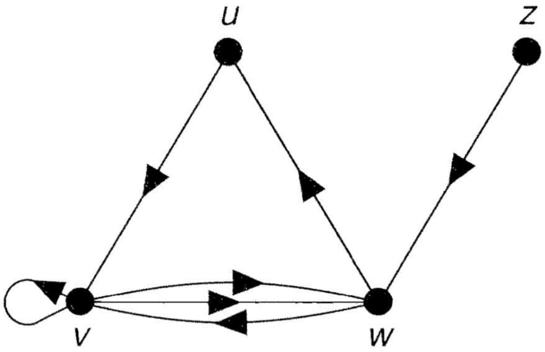


Figure 4.5 A directed graph (Wilson 1979)

Definition 4.57. (Wilson 1979)
A graph G becomes **orientable** if every edges of G is directed.

Theorem 4.58 (The Gallai – Roy – Vitaver Theorem). (Chartrand and Zhang 2019)
For each orientation D of a graph H, we have:

$$\chi(H) \leq 1 + \ell(D),$$

where $\ell(D)$ is the length of a longest path in D.

4.4 Cartesian Products for a Chromatic Number

Definition 4.59. (Avgustinovich and Fon-Der-Flaass 2000)
The **Cartesian product** G_3 of simple graphs G_1 and G_2 is denoted by:

$$G_3 = G_1 \square G_2,$$

$V(G_3) = V(G_1) \times V(G_2)$ and two vertices (v,u) in $V(G_3)$ are adjacent when either $u = b$ and $v = a$ are adjacent in G_1 or $v = a$ and $u = b$ are adjacent in G_2.

Example 4.60. Consider Figure 4.6:
G_1 is a triangle, and G_2 is a path of length three. The Cartesian product G of G_1 and G_2 is denoted in the same figure. Note that G has $V(G_1) \times V(G_2) = 3 \times 4 = 12$ vertices.

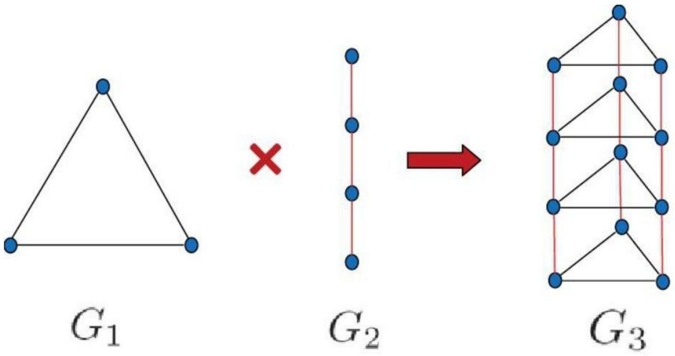


Figure 4.6 Cartesian product $G_3 = G_1 \square G_2$ (Qiao and Sipahi 2012)

Theorem 4.61. (Chartrand and Zhang 2019)

For each of the two graphs H_1, H_2, the following inequality is satisfied:

$$\chi(H_1 \square H_2) = \max\{\chi(H_1), \chi(H_2)\}$$

Example 4.62. For Figure 4.6 and by Theorem 4.61, we know that:
$\chi(H_1 \square H_2) = \max\{\chi(H_1), \chi(H_2)\} = \max\{3, 2\} = 3.$

5. COLORING GRAPHS ON SURFACES

If we have a map of several areas as in Figure 5.1, and we are instructed to coloring them in different colors in such a way that no two neighboring regions have the same hue, what is the smallest number of colors required to color this map? (Koh et al. 2015)

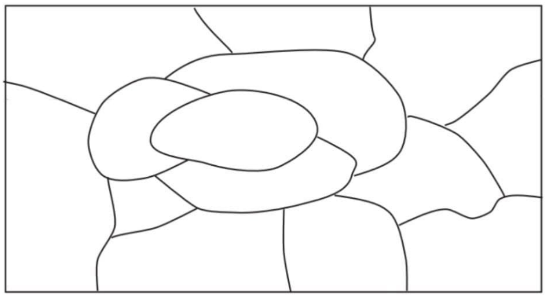


Figure 5.1 The four-color problem and map *A* (Koh et al. 2015)

Figure 5.2 is one of the colorings of the previous map, where the required condition is satisfied with four colors. The essential problem is the following: Can all planar maps be colored with only four colors or lees than four colors? We may think that yes, but it is difficult to prove it and scientists did not reach the proof for a long time. This problem was known as the problem of the four colors, and this problem first appeared in the year 1852 as a conjecture by Francis Guthrie (Koh et al. 2015).

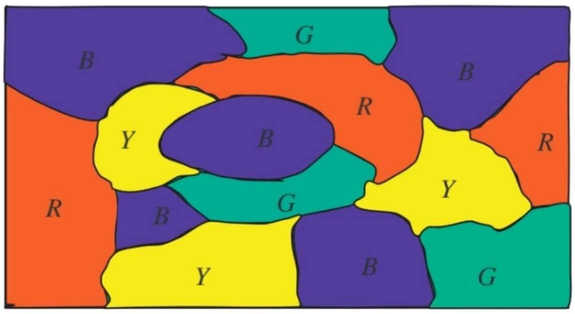


Figure 5.2 The four-color problem and bap *B* (Koh et al. 2015)

Definition 5.63. (Guichard 2017)
If the graph *P* is plotted on a specific plane provided that its edges do not cross then it is called a **planar graph**.

Example 5.64. In Figure 5.3, two examples of planar graphs are given.

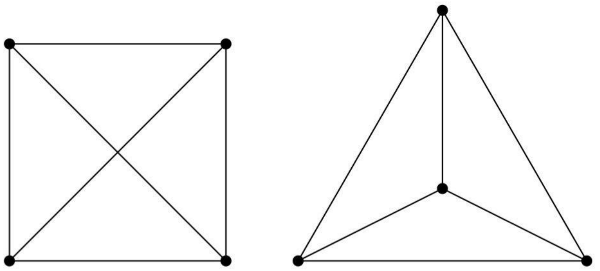
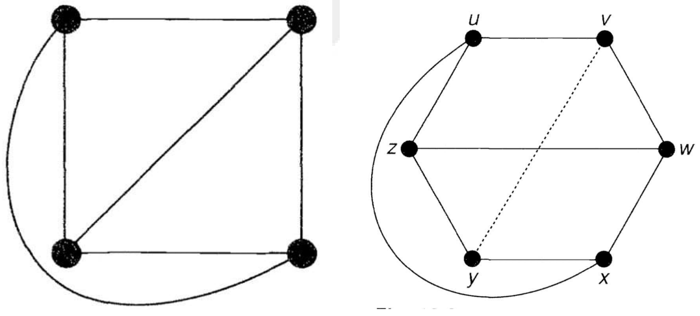


Figure 5.3 Examples of planar graphs (Dharwadker and Pirzada 2011)

Definition 5.65. (Dharwadker and Pirzada 2011)
If we have drawing of a graph that does not have two intersecting edges, such a drawing is called a **planar drawing** of this graph.

Example 5.66. In Figure 5.4, a planar drawing and a nonplanar drawing of two graphs are given.



(a) A plannar drawing of *A*   **(b)** A nonplanar drawing of *B*

Figure 5.4 A planar and a nonplanar drawings (Wilson 1979)

Theorem 5.67. (Dharwadker and Pirzada 2011)

- K_5 is nonplanar.
- $K_{3,3}$ is nonplanar.

Theorem 5.68. (Hammack et al. 2011)
Graphs that do not contain K_5 as well as $K_{3,3}$ as a subdivision if and only if it is planar.

Theorem 5.69. Four color Theorem (Chartrand and Zhang 2019)
For every planar graph, $\chi \leq 4$.

Theorem 5.70. The Five color (Chartrand and Zhang 2019)
Every map has an admissible 5-coloring.

5.1 The Chromatic Polynomial of a Graph

For example, if we want to color the vertices $\{v_1, v_2, \ldots, v_n\}$ of the graph K_n by choosing π colors, we start by coloring the vertex v_1 with one of the π colors, and then the vertex v_2 with one of the remaining $\pi - 1$ colors, until we reach the last vertex. Then the number of ways needed to color this graph is $\pi(\pi-1)(\pi-2)\ldots(\pi-n+1)$, and it is an n degree polynomial. This polynomial is denoted by $P(K_n, \pi)$ (Kocay and Krehe 2016).

Definition 5.71. (Kocay and Krehe 2016)
The polynomial representing the ways of coloring a graph G with colors less than or equal to π colors is called the **chromatic polynomial**, and it is denoted by $P(G, \pi)$.

Example 5.72. Figure5.5 illustrates the chromatic polynomial for triangle.

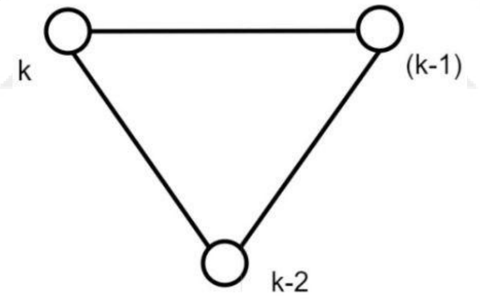


Figure 5.5 Choromatic polynomial for triangle (Zhang 2018)

Theorem 5.73. (Zhang and Dong 2017)
Let A be a triangle, and let $k \geq 3$ be the number of colors to color the vertices of this triangle, then the chromatic polynomial of A is:

$$P(A, k) = k(k-1)(k-2).$$

Example 5.74. Consider the triangle A given in Figure 5.6.
Let $\{1, 2, 3, 4\}$ be some color set. Now we proceed step by step to color the vertices of A. First, we color the vertex a, and for which there are four possible colors to color it.

Second, for the vertex b, there are three remaining possible colors to color it, other than the color picked by a. Third, for the vertex c, there are two remaining possible colors to color it, other than the colors picked by a and b.
Therefore, $P(A,4) = 4(4-1)(4-2) = 4 \times 3 \times 2 = 24$.

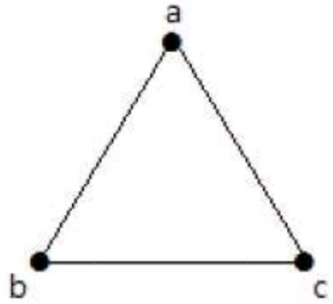


Figure 5.6 Chromatic polynomial of triangle (Tutorialspoint 2021)

Theorem 5.75. (Kocay and Krehe 2016)
Let T be a tree of order m, and let k be the number of colors to color the vertices of this tree, then the chromatic polynomial of T is:

$$P(T,k) = k(k-1)^{(m-1)}.$$

Example 5.76. Figure5.7 illustrates the chromatic polynomial for a tree.

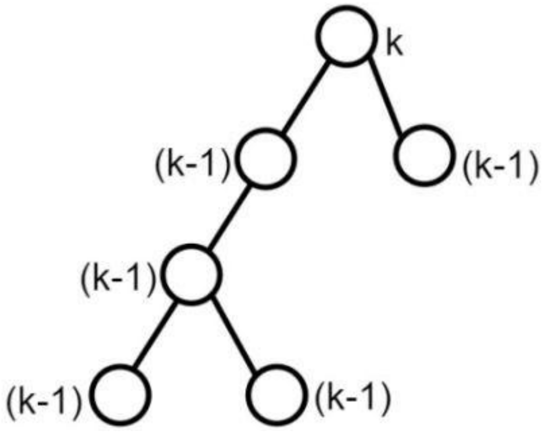


Figure 5.7 Coloring tree graphs (Zhang 2018)

Theorem 5.77. (Berman and Fryer 2014)

Let K_n be the complete graph of order n, and let $m \geq n$ be the number of colors to color the vertices of this graph, then the chromatic polynomial of K_n is:

$$P(K_n, m) = n(n-1)(n-2)\ldots(n-m+1).$$

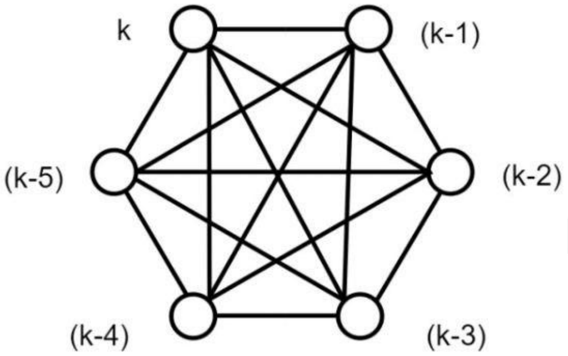


Figure 5.8 Complete Graph with using chromatic polynomials (Zhang 2018)

Example 5.78. Consider the complete graph K_6 given in Figure5.8:
Since $P(K_n, m) = n(n-1)(n-2)\ldots(n-m+1)$ by Theorem 5.77, for $m = 6$, we have:
$P(K_6, 6) = 6(6-1)(6-2)(6-3)(6-4)(6-5) = 720$.

Theorem 5.79. (Zhang and Dong 2017)

Let A be an empty graph containing m vertices, and let k be the number of colors to color the vertices of this empty graph, then the chromatic polynomial of A is k^m.

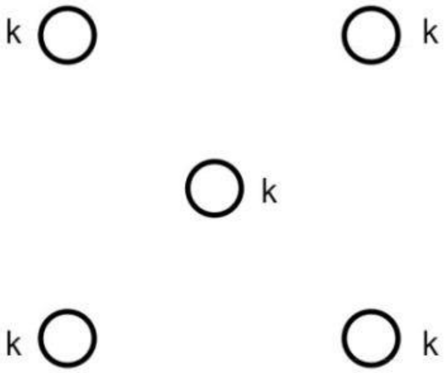


Figure 5.9 Choromatic polynomial of an empty graph (Zhang 2018)

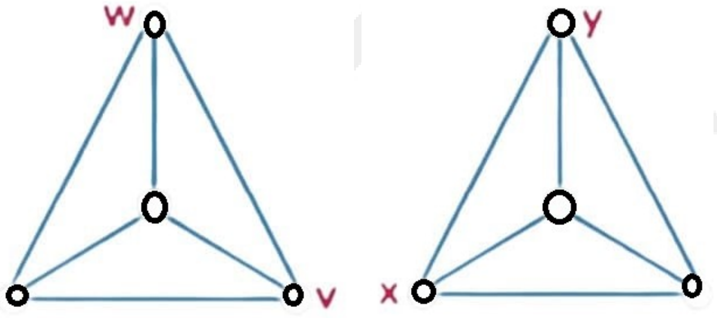
Example 5.80. For Figure 5.9: Let $\{1,2\}$ be a set of colors. Then, since $V(A) = 5$, by Theorem 5.79, the value of the chromatic polynomial of A is $2^5 = 32$.

5.2 The Conjectures of Hajos and Hadwiger

The conjectures of Hajos and Hadwiger states that if a graph G is k-chromatic, then it contains a subdivision of the complete graph K_k (Thomassen 2005).

Conjecture 5.81. Hajos Conjecture (Thulasiraman et al. 2016)
If a graph H is r-chromatic with $r \geq 2$, then the graph H has a subdivision of K_r.

Example 5.82. Consider Figure 5.10: Let H_1 and H_2 be graphs of type K_4, vw be an in H_1 and xy be an edge in H_2. Then we construct a new graph from these two graphs by merging the two vertices v and x to become a single vertex and deleting edges xy and vw. In addition connect the vertices y and w. Then we obtain the graph shown in Figure 5.11.



(a) Graph H_1 **(b)** Graph H_2

Figure 5.10 Two graphs of type K_4 to show Hajos conjecture (Miran 2016)

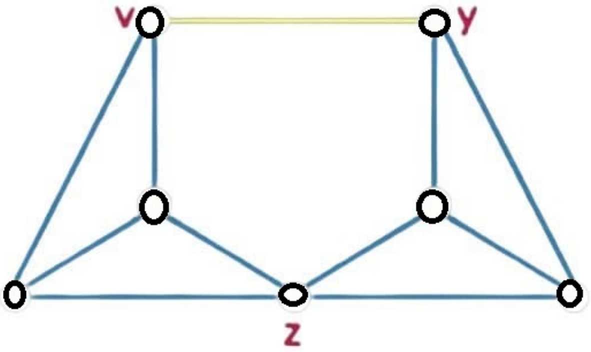


Figure 5.11 A graph related to Hajos conjecture (Miran 2016)

Definition 5.83. (Thulasiraman et al. 2016)
A graph B is said to be a **minor** of a graph A if that can be produced from A using any series of contractions, edge removals as well as vertex removals.

Conjecture 5.84. Hadwiger Conjecture (Thulasiraman et al. 2016)
Each r-chromatic graph includes K_r as a minor.

6. CONCLUSIONS AND RECOMMENDATION

The vertex-coloring problem is one of the fundamental problems in graph theory, and there are still lots of open problems related to vertex coloring, thus it is one of the active research areas of mathematics.

Throughout this thesis, we shed light on the coloring of the vertices of the different graphs, and we noticed that there are some graphs that need only one color as a chromatic number for them such as empty graphs, and some of them need two colors as a chromatic number for them such as bipartite graphs, and some of them need more than two colors, and you may need n colors as their chromatic number, like the complete graphs K_n . Therefore, for any graph G, $1 \leq \chi(G) \leq n$.

We noticed and shown in this thesis that the chromatic number and its bounds for the different graphs may differ from one graph to another. We also noticed that coloring the vertices of the graphs is a conclusion produced from coloring the maps, and it is the basis of the problem that shows that to color a map or a planar graph, we need only four colors, which was put forward in 1852 by Francis Guthrie.

The challenge in coloring graphs involves selecting the least amount of colors needed for a coloring of all different graphs is not a puzzle to be solved only, but knowing the least number of colors means that the problem can be solved with the least number of attempts or variables or methods. And a large group of applications appeared in various fields which depend on two principles or two variables or two independent conditions, and among these practical applications we have studied making time tables as an example in this thesis.

Therefore, we recommend focusing on such applications and working on finding broader areas of applications, because it is easy to solve related problems using the idea of coloring the vertices of the graphs. It should also be tried to find ways and algorithms whether manually or through the computer to solve such color problems for graphs, and because it will open a wide path for us to many applications that serve researchers.

In this thesis, since we have introduced the basic notions and theorems about graph colorings, and since we have surveyed some of the recent results about vertex coloring; on one side it is an introduction source, but on the other side it presents some of the recent results and developments in this area in a very compact form. Therefore, we hope it will be useful for students who want to learn the basics of this fascinating area.

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CURRICULUM VITAE

Personal Information

Name and Surname : Mohammed Jabbar Abdullh AL-SHAFEAY

Education

MSc	Çankırı Karatekin University	
	Graduate School of Natural and Applied Sciences	2020-2022
	Department of Mathematics Mathematics Program	
Undergraduate	Qadisiya University	
	College of Education	1990-1995
	Department of Mathematics	