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CERTAIN SMOOTHING TECHNIQUES AND THEIR APPLICATIONS TO IMAGE PROCESSING

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February 2022

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ABSTRACT

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This thesis consists of five chapters.

The first chapter is devoted to the introduction. In this chapter, the concepts of noise and denoising in image processing are introduced first, then the types and history of these terms are mentioned and some required preliminary informations are also presented.

In the second chapter, some definitions are given that is needed further.

In the third chapter, a global smoothing approximation technique is proposed to make the total variation function smooth (continuous and differentiable) to facilitate its use in denoising.

In Chapter 4, an application of the given algorithm in Chapter 3 is represented, proposed algorithm is applied to a set of test images to show its effectiveness.

The fifth and the last chapter is devoted to the conclusion and recommendations.

2022, 42 pages

Keywords: Smoothing techniques, Image processing, Optimization, Non-smooth function.

ÖZET

BAZI DÜZGÜNLEŞTİRME TEKNİKLERİ VE GÖRÜNTÜ İŞLEMEYE UYGULAMALARI

Mohammed Hashim Shakir SHAKIR

Çankırı Karatekin Üniversitesi
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Şubat 2022

Bu tez beş bölümden oluşmaktadır.

İlk bölüm giriş kısmına ayrılmıştır. Bu bölümde, ilk olarak görüntü işlemedeki gürültü ve gürültü giderme kavramları tanıtılmış, daha sonra bu terimlerin tarihçesinden, çeşitlerinden bahsedilmiş ve bazı gerekli ön bilgiler de sunulmuştur.

İkinci bölümde, ileride ihtiyaç duyulacak olan bazı tanımlar verilmiştir.

Üçüncü bölümde, gürültü gidermedeki faydasına katkıda bulunmak amacıyla, toplam değişim fonksiyonunu düzgünleştiren (sürekli ve türevli kılan) genel bir düzgünleştirme tekniği önerilmiştir.

Dördüncü bölümde, üçüncü bölümde önerilen algoritmanın bir uygulaması sunulmuştur, geçerliliğini göstermek amacıyla bu algoritma bazı örnek görüntülere uygulanmıştır.

Beşinci ve sonuncu bölüm ise sonuç ve öneriler için ayrılmıştır.

2022, 42 sayfa

Anahtar Kelimeler: Düzgünleştirme teknikleri, Görüntü işleme, Optimizasyon, Düzgün olmayan fonksiyon.

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Mohammed Hashim Shakir SHAKIR
Çankırı-2022

CONTENTS

LIST OF SYMBOLS

x_k^*	k-th local minimizer
$\mathbb{R}$	The set of real numbers
x_{int}	Start point
$\mathbb{R}^n$	n dimensional Euclidean space
f	The objective function $f : \mathbb{R}^n \to \mathbb{R}$
$\|\|.\|\|$	Euclidean norm
∇f	The gradient of f
$\mathbb{D}$	Domain of the objective function f
B_k^*	The basin of x_k^*
σ, λ, τ	Parameters
β	Smoothing parameters

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1. INTRODUCTION

The rapid and successive development in information and communication technology and its applications not only presents the world with new opportunities, but also presents new problems and changes. By the use of some means, tools and devices which are produced in order to facilitate the storage and retrieval of a huge amount of information, the data is processed in the least possible amount of space as quick as possible.

Thanks to the astonishing development in computer science and its software, photography is one of the fast-growing fields which has surpassed multiple techniques. Digital photography is the most popular one among all photography technologies. Digital photography is nothing but a camera that allows taking photos at high speed and storing them in various electronic media, by converting light rays reflected from the object to be photographed into electronic light points (pixels) using photosensitive slide types. These are then converted into digital signals within the camera.

The way of using a digital camera is not much different from a film camera. However, the image captured by the digital camera is not recorded on the film inside the camera. Instead, light is collected through a group of lenses attached to the camera into a photosensitive device called *symbolized charge coupled device*. This device is tasked with capturing light signals and there exists a chip which is tasked with receiving light signals and converting them into bitmap images. This bitmap is then transferred to another device which converts the image from analog to digital.

The purpose of using enhancement techniques for digital images is to process a specific image so that the result for a given application is smoother than the original image and is free of *noise*, which is known as undesired information that distorts the image. This noise arises from different sources, including the process of converting the image from a sine signal to a digital image. In addition, this conversion is one of the basic operations that cause noise in the image. In each step of this transformation, there are frequencies that add random values to the light of the pixel, which is one of the units that make up the image. Moreover, noise sources increase during the image acquisition (image digitization) and transmission process.

The performance of imaging sensors is affected by several factors. These are the environmental conditions during the image acquisition process, the quality of the sensors themselves, the illumination levels and temperatures of the sensors. In addition, the image transmission channels and the degree of illumination in them can expose the image to noise.

Methods for smoothing images can be examined in two main categories, namely spatial domain methods and frequency domain methods. The methods in the first category are the level of the image itself and these methods depend on the direct processing of the image elements (pixels). As for the second category, these methods are based on the Fourier transform applying of the image to be processed.

On the other hand, raw images taken by remote sensing satellites are often insufficient for accurate reconnaissance and imaging. Therefore, the use of digital processing methods is resorted to in order to highlight the spectral information in a way that enables the interpreter to derive meaning from these data more effectively. In general, digital processing methods aim to improve and highlight the spectral information used in land use applications.

Satellite images are usually stored in matrices. Each matrix element represents a pixel in the image. Some color images require a three-dimensional matrix. The first dimension of the matrix represents the intensity of the red color, the second dimension represents the intensity of the green color and the third dimension represents the intensity of the blue color. These colors are represented by RGB letters. Representing the image as a matrix makes it possible to work with the image mathematically, which means that calculations and operations can be performed on it (Al-Juaidi *et al.* 2003).

One of the general operations performed on these matrices is to create an image based on more than two space images with the same parameter, based on the correlation coefficient and standard deviation and find the *optimum index factor* (Chavez *et al.*1982). By these two parameters, channels with the lowest optimum indicator factor are obtained (Alan and Jia 2006). Optimum index factor is a statistical value that can be used to determine the optimal composition of the three bands in the satellite image for which we want to create a good color combination. In addition, the use of principal component analysis method, which is one of the specific operations on images and uses the eigenvalue of each of the space mirror bands, is under examination. Devices used in remote sensing sensors are cameras that operate at a certain wavelength or wavelength range. Detection satellites differ according to the sensor cameras inside.

An image is a function of the light intensity of a two-dimensional matrix and is denoted by $f(x,y)$. The function f must have a finite value and not equal to zero at the cordinate point, because the value at that point gives the amount of energy, called *pixels*, and is light or energy density.

In the continuation of this chapter, types of images will be introduced first, then certain noises will be examined which occur in below mentioned images.

1.1 Representation of Images

Image processing is the processing of the data that makes up the images and includes the intensity of the image illumination or the processing of a set of mathematical and logical functions to analyze and manipulate the image. Images are divided into two types: analog images and digital images.

In this thesis, we consider digital images which are derived from analog images, but they are in a discontinuous space, whereas analog images are in a continuous space. So, digital images are derived from analog images through a process called *quantization* and *sampling*, which converts data from continuous (analog) values to discrete (digital) values. It is easy to process and analyze in computing. Digital images are generally divided into several types as given below.

1.1.1 Index images

Index images contain an array of data x and a color map of the map matrix. Matrix data can consist of 8 bits or 16 bits. However, the colormap of the map image is a 3-dimensional matrix containing values ranging from 0 to 1. Each row of the map designates the components of the colors red, green and blue as a single color. It uses a direct plot of values in the image, as each color of an image point is determined by its corresponding x values in the color map. A value of 1 refers to the first row in the colormap, and values of 2 represent the second row. Thus, the colormap is stored with the indexed images and automatically loaded with the image when the indexed image is read.

1.1.2 Intensity images

Intensity images are a matrix of data representing the intensity values of the image points, which can be represented by 8-bit or 16-bit. The density image is stored with a color map in case you reapply the colors. The matrix elements of the image represent different intensity values, called the *grayscale level*. Zero density represents the black color and the intensity 1 or 256 or 65536 represents the white color, which represents gray scale end level (Figure 1.1).

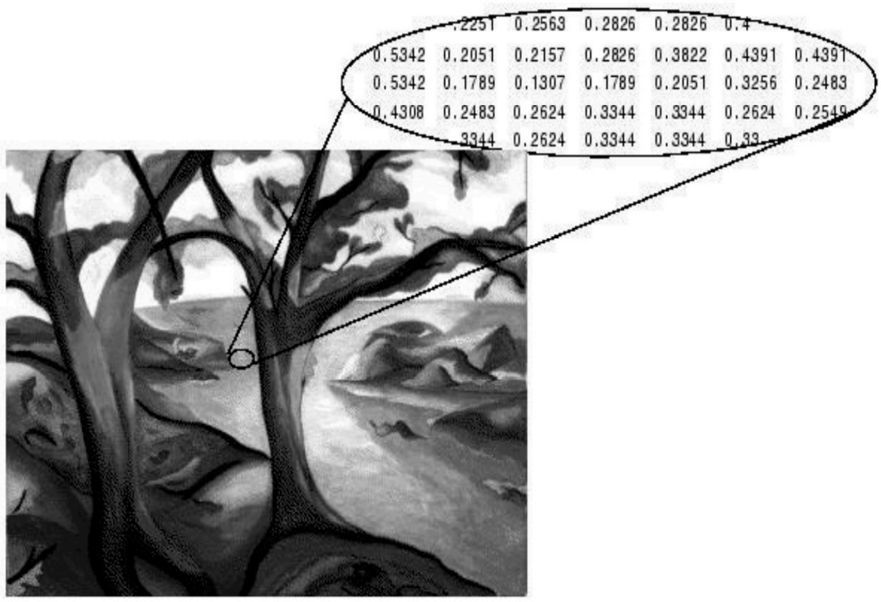

Figure 1.1 Reading an intensity image

1.1.3 Binary images

In digital binary images, each image dot has only one value, either zero or one. These values are mainly represented as on or off. The digital image is stored as a logical binary matrix consisting of zero values and unit values in (0,1) (Figure 1.2).

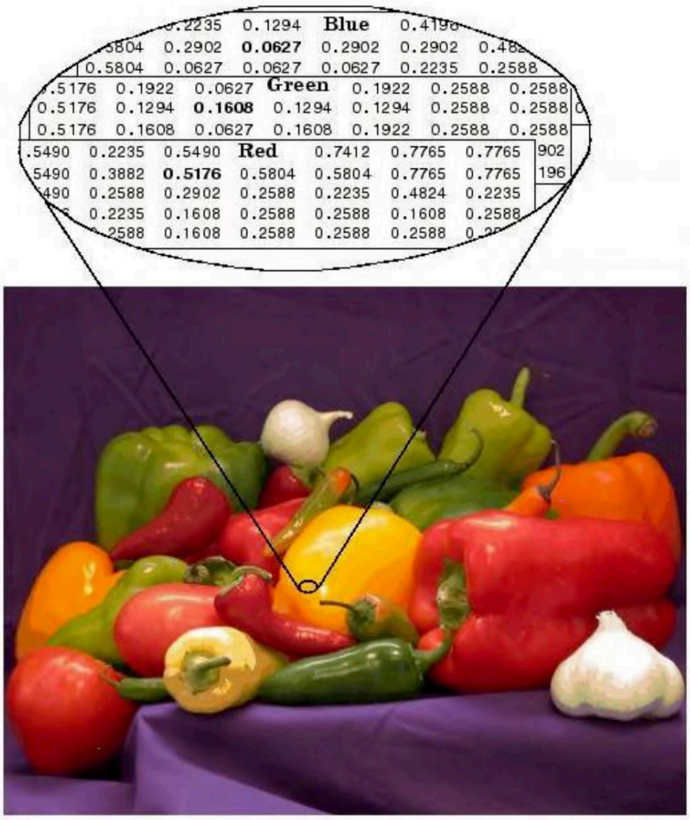

Figure 1.2 Reading a binary image

1.1.4 Red - green - blue images

Images, called red - green - blue (RGB) and sometimes true color images, are stored in the form of a three-dimensional matrix. These images are stored in the form of a three-dimensional matrix, as they describe the components of the red, green, and blue colors in each viewport and their dimensions are $M-N-3$. The color of each image point is determined by the intensities of the three colors stored at each color level at a location to give that image point (Figure 1.3).

The RGB matrix can be 8-bit to give 256 colors and 16-bit to give 65536 colors. Since the color components (red, green and blue) are 8 bits for each color, the image file can be stored as a 24-bit image. This results in the possibility to change the color range to 16 million colors.

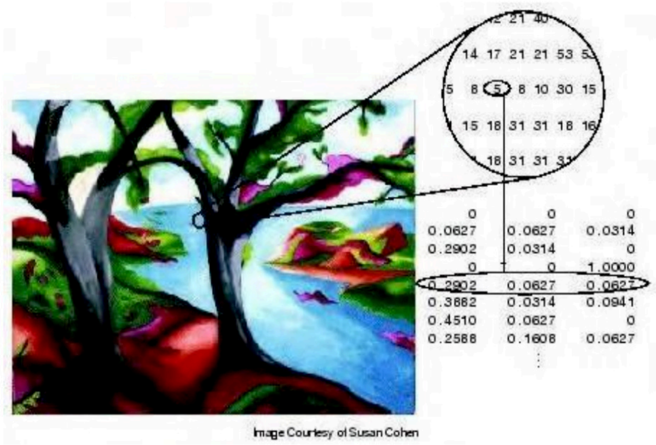

Figure 1.3 Reading a red - green - blue - image

1.2 Noise

Noise is defined as undesired information in digital images and occurs as a result of a variety of reasons, including a malfunction in the imaging equipment used and an error in the data transmission or image acquisition process.

In general, any distortion negatively affects the extraction of information from the image, so it must be removed before diagnosing the information in the image. Below is a presentation of the most important types of noise that occur in digital images:

1.2.1 Gaussian noise

Gaussian noise occurs as a result of a malfunctions in the imaging equipments, and the probability density function $p(z)$ for this type of noise is defined in Equation (1.1) and represented in Figure 1.4 as follows:

$$p(z) = \begin{cases} \frac{1}{b-a}, & \text{if } a \leq z \leq b \\ 0, & \text{otherwise} \end{cases} \tag{1.1}$$

where a, b, z represent the gray levels.

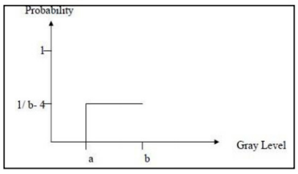

Figure 1.4 Probability density function of Gaussian noise

1.2.2 Salt and pepper-type noises

Salt and pepper-type noise is the noise caused by the error in the transmission process of the digital image data, it is dark or light in color. The probability density function $p(z)$ for this type of noise is defined in Equation (1.2) and represented in Figure 1.5 as follows:

$$p(z) = \begin{cases} A \\ B \end{cases}, \tag{1.2}$$

where A, B represent the gray levels.

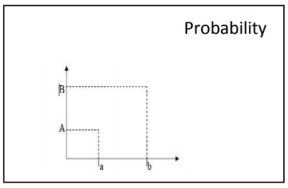

Figure 1.5 Probability density function of salt and pepper-type noise

1.2.3 Uniform noise

Uniform noise is the noise that arises from the image acquisition system, where the continuous electrical signal is converted into a digital form accepted by the computing. The probability density function $p(z)$ for this type of noise is defined in Equation (1.3) and represented in Figure 1.6 as follows:

$$p(z) = \frac{1}{\sqrt{2\pi\sigma}} \mathrm{e}^{-(z-\mu)^2/2\sigma^2}, \tag{1.3}$$

where z represents the gray level and μ represents the standard deviation.

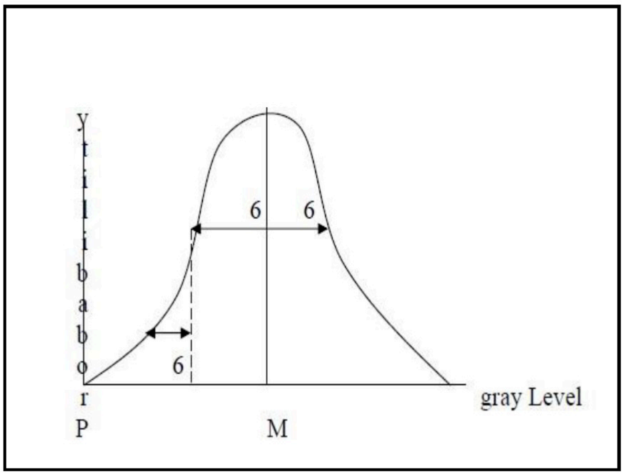

Figure 1.6 Probability density function of uniform noise

1.3 Image Processing

Image processing is the process of performing an image operation to obtain some usable information which plays important role in the fields of smart transportation, remote sensing, moving object tracking, defense surveillance, bio-medical imagery and autonomous visual control systems. In image capture and transmission, digital pictures are polluted by noise because it's not perfect in the imaging. The sound generally impacts pictures by replacing certain pixels with new pixels of the original picture. The value of new pixels is more or less than the value of the original pixel. The digital processing of images is an imaging sensor and the digitisation of a sensor signal. The sensor may be a monochrome or color television camera that generates a complete 1:30 sec picture of the issue area. In addition, the imaging sensor might be a line-scan camera which generates a single line at a time. In this example, a two-dimensional picture is generated by the item moving through the line scanner. If camera output or any other the image sensor is not already digitally, it is digitized by an analog to digital converter.

The application determines the type of the sensor and the image it generates. The following step is to process this image after digital image has been produced. The main aim of preprocessing is to enhance the image so that the chances of success of other processes can be increased. These procedures generally include contrast enhancement approaches, noise removal and isolation areas whose texture indicates an alphanumeric information chance.

Segmentation is discussed in the following stage. The segmentation divides an input into broadly defined components or objects. Independent segmentation is often one of the hardest problems in digital image processing. The robust segmentation method leads to an effective solution to an image problem on the one hand. On the other hand, weak or unpredictable segmentation algorithms ensure a failure nearly always. With regard to the recognition of characters, the main purpose of the segmentation is to extract from background characters and words. The segmentation stage output is generally raw pixel data, which is either the limit of a region or all the spots in the region itself. In any scenario, it is required to transform the data to a computer-friendly form. The initial option is whether the data should be displayed as a border or a whole region. When focused on exterior form features, such as entrances or reflections, boundary representation is suitable. Regional representation is useful if internal features, such as texture and skeleton form, are the emphasis. However, these statements coexist in some applications. This happens in applications for character recognition, which typically need algorithms based on border form, skeletons and other inner features.

Choosing a representative is just half of the solution to convert raw data into a computer-friendly form. In order to emphasize aspects of interest, a technique should also be given for characterizing the data. Description, also known as selection of the feature, addresses extracting features that provide certain quantitative information of interest or feature essential for discriminating between classes of objects. Descriptors like lakes or bays are powerful characteristics in terms of character identification, helping to distinguish between one portion of the alphabet and another. The last step is recognition and interpretation. Recognition is the process by which an item is assigned a label based on the descriptor's information (Syberfeldt and Vuoluterä 2020).

1.4 Components of Image Processing System

The general purpose image processing system consists of following basic components :

1. Image sensor: An image sensor in image processing system requires for transforming the radiant energy into an intensity image. Basically, cameras like still and television cameras, scanners such as optical scanners, X-ray scanners, magnetic resonance imaging MRI system, etc. can capture the images and are used as image sensors devices.

2. Digitizer: A digitizer produces digital images to deal with the processing system.
3. Frame buffer: Frame buffer is a storage module where the entire digital image can be stored.
4. Processor: Processor handles the arithmetic and logic operations to perform image processing.
5. Display unit: It is a unit that helps to display visual form after processing on the data stored in the digital computers. Devices like printer, TV monitor, CRTs, etc. are used as display tools.

1.5 Background and History of Image Processing

The improvement of digital newspaper images carried by underwater cable between London and New York was one of the first uses of image processing in the first category. In the early 1920s, the introduction of the Bartlane cable image transmission technology decreased the amount of time needed to transfer a picture across the Atlantic to less than 3 hours for over one week.

Cable transmission images are coded and re-created at the receiving end by specialized printers (Woźniak and Połap 2020). At the end of 1921, the printing method was discontinued based on a photographic technique created from tapes drilling at the telegraph-receiving terminal. In 1929, that capacity rose to 15 levels. In the next 35 years, improvements in processing technologies have progressed. However, the Jet Propulsion Laboratory started utilizing computer techniques to improve the photos of a space-sample in 1964, when a PC was relayed through Ranger 7 to rectify many kinds of image distortion inherent in the on-board camera. The area of image processing has been growing steadily since 1964 till now.

Digital image processing techniques may be utilized to tackle many challenges in addition to applications in the space program and in medical imaging. In the medical field, the contrast between the computer methods or the color intensity of the code makes X rays and other biological pictures simpler to comprehend. To investigate pollution trends from aerial and satellite images, geographers employ the same or comparable approaches.

Image enhancement and restoration methods are used to handle unrecoverable items or experimental findings that are too costly to reproduce the picture. In the field of archeology, imaging techniques have successfully recovered fluid photos that had only been shot to document rare items which had been lost or destroyed. The pictures of experiments in areas like high-energy plasmas and electron microscopy are improving frequently in physics and related disciplines. In commercial applications, medical, radar, sound processing, robotics, astronomy, biology, nuclear medicine, enforcement, defence, and industrial applications, effective implementations of image processing ideas may be also foamed.

The representation of the picture encompasses tasks from purchases, digitalizes and displays up to the following processing of mathematical images. Proper representation is a necessary prerequisite for the development, filtering and restoration of the processing technology, analysis, reconstruction of projections, image communication, image processing and techniques, which invokes concepts from different fields, such as physical optics, digital signal processing, assessment theory, information theory, visual perception, stochastic processes, and other aspects.

In recent years, the area of digital image processing has constantly changed. The interest in picture morphology, neural networks, image processing, compression of image data, image recognition and image analysis system based on knowledge was much improved. Our covering of picture data compression was fully achieved with a more modern viewpoint on this issue, including the image compression standard.

1.6 Uses of Image Processing

Nowadays, most of the image analysis is performed by employing the techniques of digital image processing. Improvement of quality of an image is one of the main applications of image processing (Qureshi *et al.* 2019). Some of the other application areas are given below:

1. The recognition of logo and icon of any organization and number plate of a vehicle, address area written on an envelope, optical character recognition and document processing etc.
2. Industrial areas like exploration of oil and natural gas, seismography, robotics etc.
3. Medical applications like ECG, automated radiology and pathology, X-ray image analysis etc.

4. Remote sensing makes use of the concept of image processing to deal with survey of natural resources, to estimate the information regarding agriculture, forestry, urban planning, pollution control, monitoring traffic along roads, airfields etc.
5. Biometric identification like handwritten signature, fingerprint, human face recognition etc.
6. Meteorology analysis like weather forecasting, climate change detection with the help of satellite and other remote sensing data etc.
7. Image transmission, video-conferencing, video-editing etc.
8. Military applications like detection of missile, target identification etc.

1.7 Total Variation

Camille Jordan initially proposed the notion of total variation for functions with a single real variable in his article in 1881. To establish a convergence theorem for discontinuous periodic functions whose variation is limited, he employed the novel notion. However, for a variety of reasons, extending the idea to include functions with more than one variable is not straightforward.

The total variation of any function is outlined as the overall deviations between the adjacent values of the function. The total variation of any real-valued function f on an interval $[a,b] \subset \mathbb{R}$ is obtained as

$$TV(x) = \int \| D(x) \| \, dx = \sum_i \sqrt{(D_i^h x)^2 + (D_i^v x)^2},$$

where D_i^h and D_i^v denote first-order horizontal and vertical finite-difference of x at i-th pixels. The total variation of any signal calculates deviations between adjacent sample points and hence, we define this as total variation using finite differences. It is discovered that noisy signals have high total variation, i.e., the integral of the absolute gradient of these signals are high.

1.8 Total Variation Denoising

Total variation is used for removing noise from the input data. As mentioned before, noise is an undesired piece of information in the picture. The noise in the picture is additive or multiple. In image recovery, we assume that noise is unrelated with regard to picture content and is independent of spatial coordinates (Singh and Shree 2018, Roonizi and Jutten 2021). Recall some well known noises that can be seen as follows.

- Gaussian noise is also called as sensor noise or electronic circuit noise occurs in sensor and electronic circuit because of high temperature or poor illumination.
- Erlang noise and exponential noise occurs in laser image.
- Impulse noise occurs due to faulty switching while image acquisition.
- Uniform noise occurs during quantization process hence it is also called as quantization noise.

Signal processing technique in total variation denoising also known as total variation regularization has applications in noise reduction. Signals with excessive and perhaps false detail have high total variation, meaning that the integral of the signal's absolute gradient is large, according to this theory's foundation. Reduced fluctuation in the signal, as long as it's near to the original signal, eliminates unnecessary detail while maintaining critical features such as edges, according to this concept. Rudin, Osher, and Fatemi pioneered the concept in 1994 (Rudin and Osher 1994).

As a result, it is superior to other noise reduction techniques such as linear or median filtering which decrease noise but also smooth out edges to varying degrees. When it comes to maintaining edges while smoothing out noise in flat regions, total variation denoising excels, even at low signal-to-noise levels.

The fundamental thought in smoothing tecniques is to approximate non-smooth functions by using smooth functions. This process is controlled by appropriate parameters, which are said to be *smoothing parameters*. The smoothing approaches are also used for smoothing of the piecewise smooth model functions in data modeling. In a smoothing study, the most required step is to convert the objective function into a continuously differentiable function before the optimization. During this step, it is focused on the appropriate alterations which helps minimizing to be simple and easy. For this reason, in the third chapter of this thesis, we use a new method to smooth the absolute value function. In Chapter 4, we use the technique that mentioned in Chapter 3 to make total variation function continuous and differentiable. At last, we apply the proposed algorithm on some common pictures to illustrate the impact of our method.

2. PRELIMINARIES

In this chapter, we recall some definitions that we need throughout the thesis. Note that, we denote $\|y\|$ by the Euclidean norm of the function y in $\mathbb{R}^2$ and define as

$$\|y\| = \sqrt{\sum_{k=1}^{n} y_k^2}, \quad k = 1, 2 \ldots n.$$

On the other hand, we use the following $L^1[a,b]$-norm defined by

$$\|F\|_{L^1[a,b]} = \int_a^b |F(s)| ds,$$

where F is a continuous function on the interval $[a,b]$ (Sahiner *et al.* 2018).

Definition 2.1. Assume a continuous function $g : \mathbb{R}^n \to \mathbb{R}$. Then, the other function $\breve{g} : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$ is said to be a smoothing function of g, if $\breve{g}(., \beta)$ is continuously differentiable in $\mathbb{R}^n$ for any constant β, and for any $x \in \mathbb{R}^n$,

$$\lim_{z \to x, \beta \to 0} \breve{g}(z, \beta) = g(x).$$

(Sahiner and Ibrahim 2019).

Definition 2.2. Suppose that $x \in \mathbb{D} \subset \mathbb{R}^n$ and $\varepsilon > 0$. Then, the set

$$N_\varepsilon(x) = \{y : \|y - x\| < \varepsilon\}$$

is said to be ε-neighborhood of x.

Definition 2.3. Assume $\mathbb{D} \subset \mathbb{R}^n$ and f is an objective function. If $f(x^*) \leq f(x)$ for all $x \in \mathbb{D}$, then the point $x^* \in \mathbb{D}$ is said to be a *global minimizer* of f. On the other hand, if there exists a neighborhood $N_\varepsilon(x_k^*)$ of $x_k^* \in \mathbb{D}$ such that $f(x_k^*) \leq f(x)$ for all $x \in N_\varepsilon(x_k^*)$, then the point x_k^* is called as a *local minimizer*.

Similarly, x_k^* is called as a *strict local minimizer*, if $f(x_k^*) < f(x)$ for all $x \in N_\varepsilon(x_k^*) \setminus x_k^*$. (Sahiner *et al.* 2019).

Definition 2.4. Assume that x_k^* is a local minimizer of f over $\mathbb{D}$. A set $B_k^* \subset \mathbb{D}$ is called a *basin* of f at the point x_k^* if any local minimization method starting from any point in B_k^* finds local minimizer x_k^*. Any local minimizer x_{k+1}^* of f is *higher (or lower)* than x_k^* if $f(x_{k+1}^*) \geq f(x_k^*)$ $\left(\text{or} f(x_k^*) \geq f(x_{k+1}^*)\right)$, then the basin B_{k+1}^* is said to be *higher (or lower)* than B_k^* (Sahiner and Ibrahim 2019).

3. THEORETICAL PART (SMOOTHING TECHNIQUES)

In this chapter, we study on total variation function. Total variation regularization has important applications in signal processing including denoising, deblurring, and restoration. The total variation function is one of the most significant regularization function owing to its capability to recover edges and sustain the texture of the image. A significant challenge in the practical use of total variation regularization lies in non-differentiable convex optimization, which is difficult to solve especially for large-scale problems, that means total variation is not smooth.

The objective function $f : \mathbb{D} \subset \mathbb{R}^n \rightarrow \mathbb{R}$ is said to be a *smooth function* if it is continuously differentiable at all points of domain $\mathbb{D}$ and if it has continuous derivatives in $\mathbb{D}$ up to a known limit k, then we say that this function is $\mathbb{D}_k$ smooth. If $k = 2$, then the objective function is called a twice continuously differentiable, and that is what we need in optimization problems. On the other hand, if the objective function f is non-differentiable (no matter it is continuous or discontinuous), then it is called *non-smooth*. Figure 3.1 displays the relation of some smooth and non-smooth functions. In optimization problems, if objective function is smooth, it is easy to find the local and global optimal solution, but there exists a problem if the objective function is non-smooth (no matter is continuous or discontinuous).

Recently, the non-smooth functions has attracted great attention of many mathematicians or engineers. Smoothing techniques have two main classes such as

- local smoothing techniques
- global smoothing techniques.

Smoothing techniques belonging to the first class aim to smooth the objective function locally in a neighborhood of a point at which the objective function is non differentiable (Bertsekas 1975, Zang 1980, Chen and Wan 2015, Sahiner *et al.* 2018), whereas the techniques belonging to the second class use the whole domain to approximate the function f globally. However, many important studies related to global smoothing technique have been published by (Xu *et al.* 2001, Xavier 2010, Xiao and Yu 2010).

In this thesis, we introduce a new smoothing technique to make the total variation differentiable and smooth which will be regarded as the absolute value function

$$A(x) := |x|$$

throughout the study (Figure 3.2).

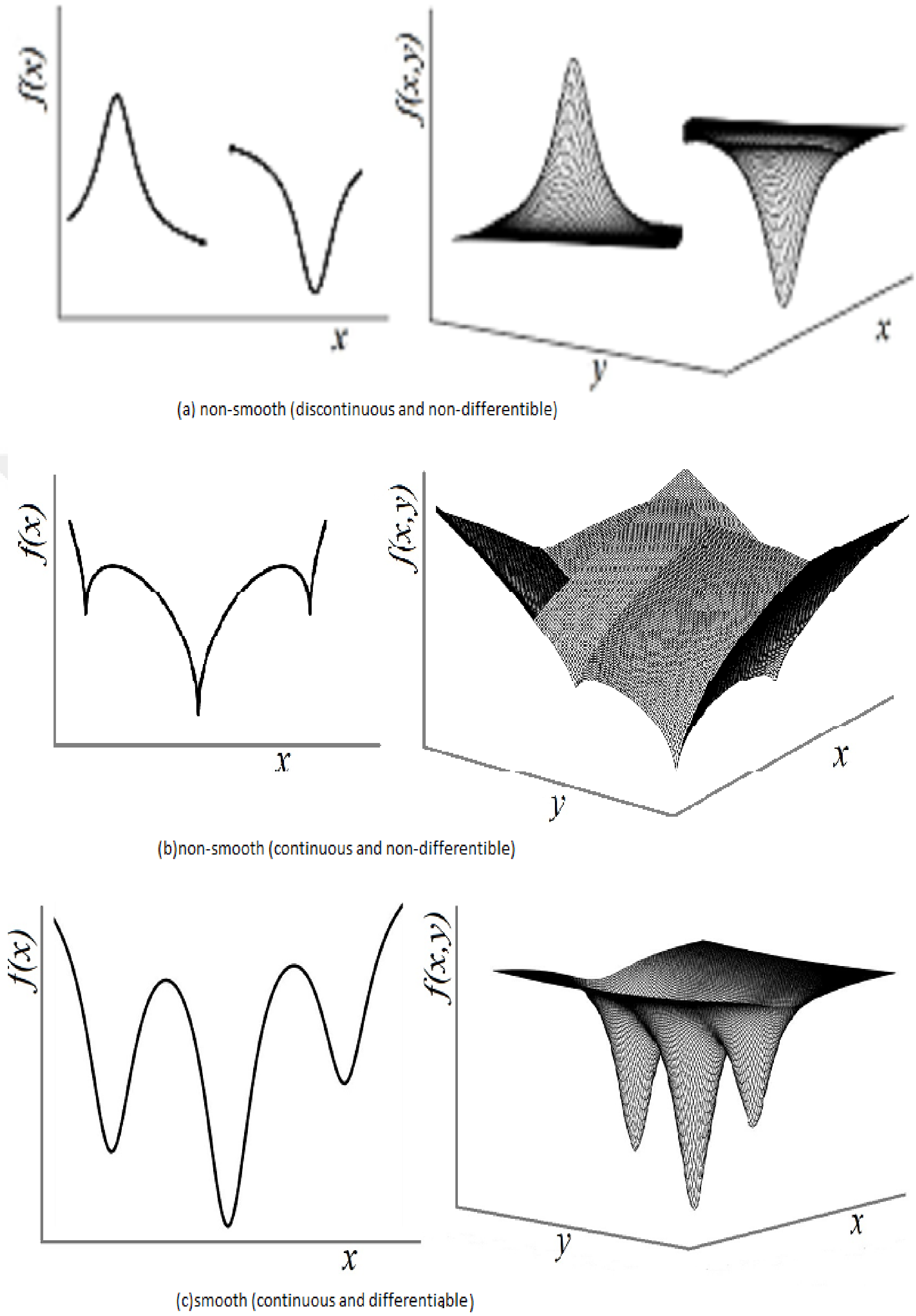

Figure 3.1 Examples of smooth and non-smooth functions

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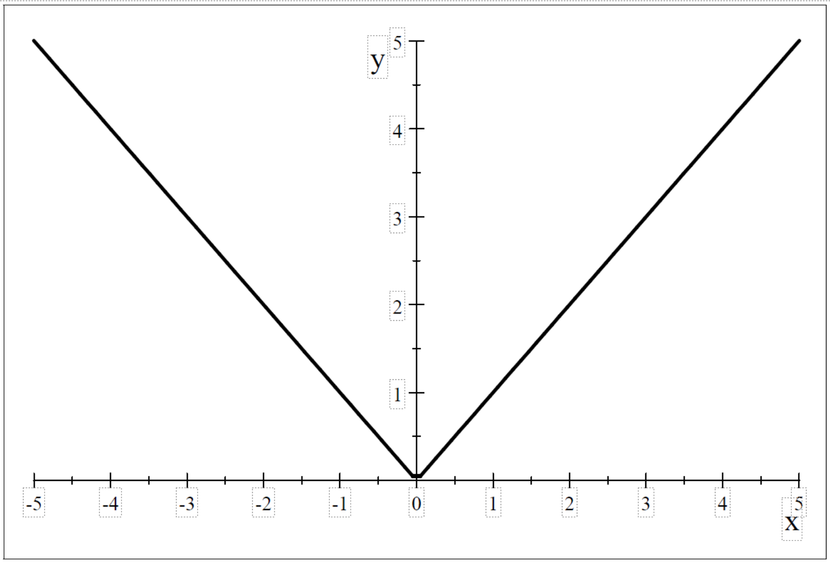

Figure 3.2 Continuous function (absolute value function)

First of all, our aim is to make the function A smooth, hence, we can rewrite the absolute value function A as follows:

$$A(x) = |x| = xS(x),$$

where

$$S(x) = \begin{cases} 1, & x \geq 0 \\ -1, & x < 0 \end{cases}.$$

Clearly, the function S is not smooth (see Figure 3.3) and neither is the function A. In order to smooth A, its enough to smooth the function S.

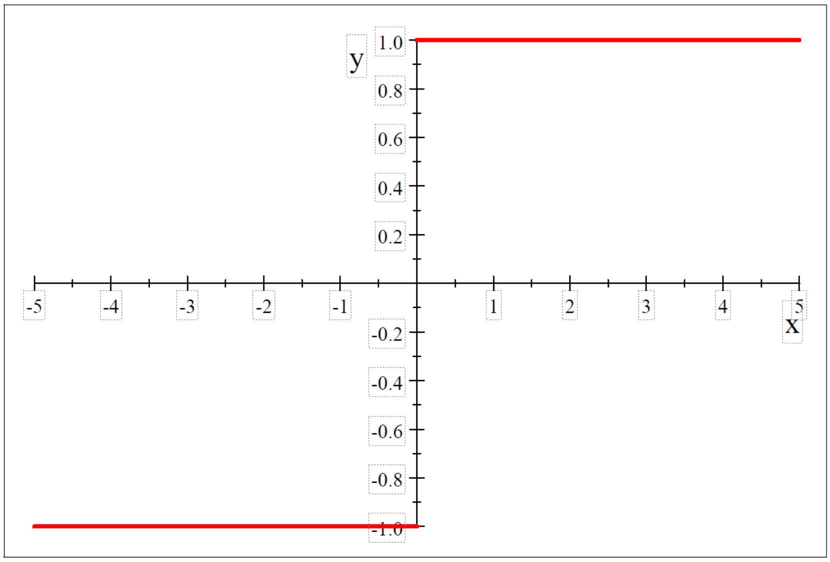

Figure 3.3 The graph of the discontinuous function S

To smooth $S(x)$, we can use another function $\breve{S}$ satisfying the following properties:

i. $\lim_{x\to\infty} \breve{S}(x,\beta) = 1$,
ii. $\lim_{x\to-\infty} \breve{S}(x,\beta) = -1$,
iii. $\forall x \in \mathbb{R}, \breve{S}'(x,\beta) > 0$,
iv. $\forall x \in (-\infty,0), \breve{S}''(x,\beta) < 0$ and $\forall x \in [0,\infty), \breve{S}''(x,\beta) \geq 0$.

Now, let us consider the function

$$\breve{S}(x,\beta) = \frac{x}{\sqrt{\beta + x^2}}$$

as the smoothing function of our methodology, where $\beta > 0$ is smoothing parameter. Clearly, the function $\breve{S}(x,\beta)$ satisfies the condition

$$\lim_{\beta\to 0} \breve{S}(x,\beta) = \begin{cases} 1 & x > 0, \\ 0 & x = 0, \\ -1 & x < 0. \end{cases}$$

Finally, we obtain the smoothing version for $S(x)$ by using the equality

$$\breve{A}(x,\beta) = x\breve{S}(x,\beta).$$

The parameter β is used to squeeze the function $\breve{S}(x,\beta)$. It means that when $\beta \rightarrow 0$, then the function $\breve{S}(x,\beta) \rightarrow S(x)$. The graphs of the functions $\breve{S}(x,\beta)$ and $S(x)$ with different values of β are displayed in Figure 3.4. Moreover, the graphs of $A(x)$ and $\breve{A}(x,\beta)$ with various β values can be seen in Figure 3.5.

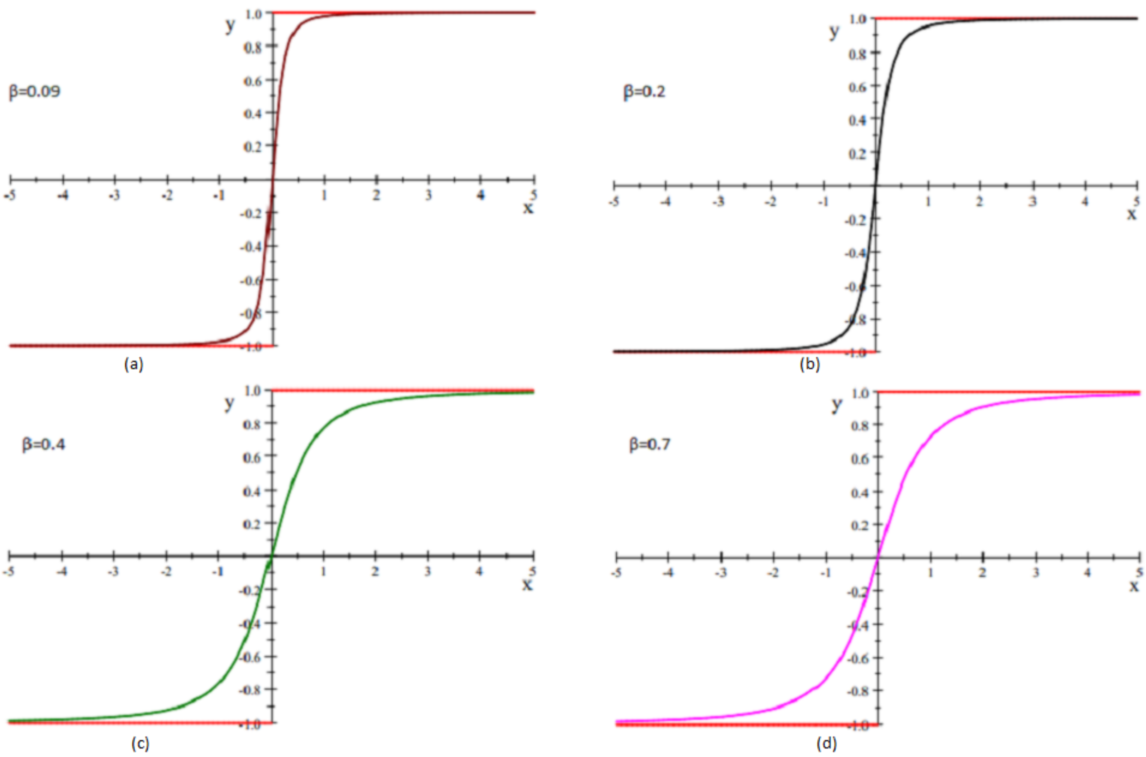

Figure 3.4 The graph of $\breve{S}(x,\beta)$ (red, black, green, pink and solid) and $S(x)$ (black and solid) a, b, c and d with $\beta = 0.09, 0.2, 0.4, 0.7$, respectively

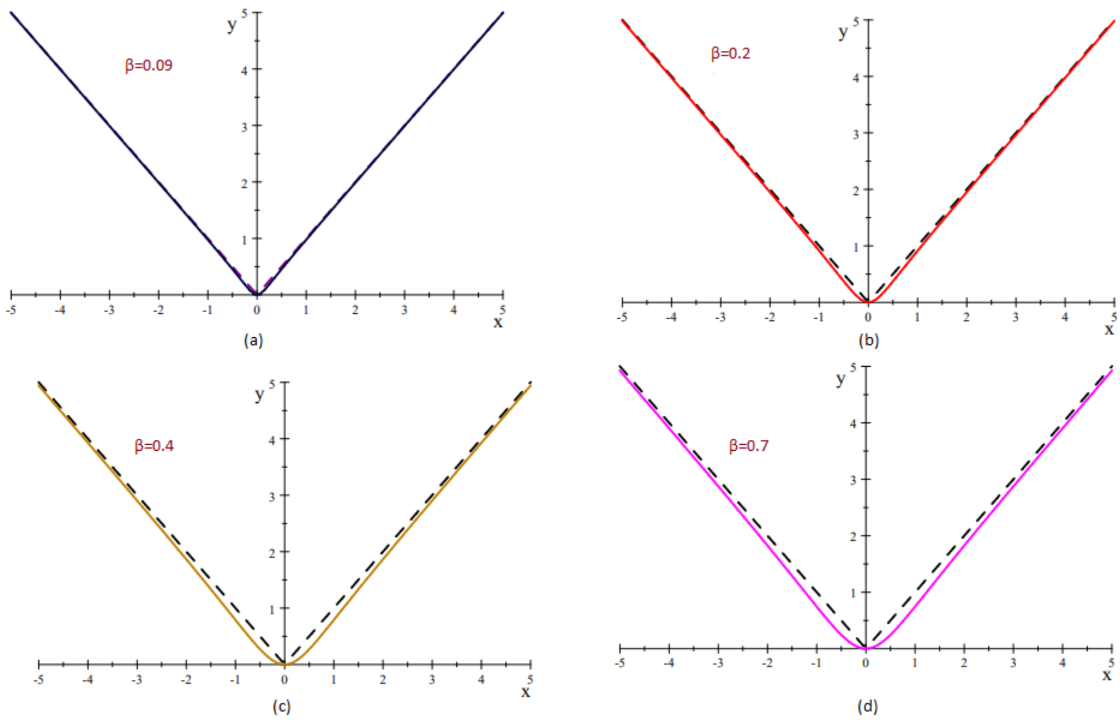

Figure 3.5 The graph of $\breve{A}(x,\beta)$ (blue, red, yellow, pink and solid) and $A(x)$ (black and dot) a,b,c and d with $\beta = 0.09, 0.2, 0.4, 0.7$, respectively

Lemma 3.5. Let $\breve{S}(x,\beta)$ is a smooth version of $S(x)$, then

$$\|\breve{S}(x,\beta) - S(x)\|_{L_1} \leq \frac{20}{3}\beta.$$

Proof. Since $x \in (-\infty, \infty)$, then we obtain the inequality as follows.

$$\begin{aligned}
\|\breve{S}(x,\beta) - S(x)\|_{L_1} &= \int_{-\infty}^{\infty} \left|\breve{S}(x,\beta) - S(x)\right| dx \\
&= \int_{-\infty}^{0} \left|\breve{S}(x,\beta) - (-1)\right| dx + \int_{0}^{\infty} \left|\breve{S}(x,\beta) - 1\right| dx \\
&= 4\beta \ln 2 + 4\beta \ln 2 \\
&\leq \frac{20}{3}\beta.
\end{aligned}$$

□

Theorem 3.6. Let the function $\breve{A}(x,\beta)$ is continuously differentiable on $\mathbb{R}^n$ and let it be a smooth version of $A(x)$, then

$$\|\breve{A}(x,\beta) - A(x)\|_{L_1} \leq \frac{23}{3}\beta^2.$$

Proof.

$$
\begin{aligned}
\|\breve{A}(x,\beta)-A(x)\|_{L_1} &= \int_{-\infty}^{\infty} |\breve{A}(x,\beta)-A(x)|dx \\
&= \int_{-\infty}^{\infty} |x\breve{S}(x,\beta)-xS(x)|dx \\
&= \int_{-\infty}^{\infty} x|\breve{S}(x,\beta)-S(x)|dx \\
&= \int_{-\infty}^{0} x|\breve{S}(x,\beta)-(-1)|dx + \int_{0}^{\infty} x|\breve{S}(x,\beta)-1|dx \\
&\leq \frac{23}{3}\beta^2.
\end{aligned}
$$

□

Theorem 3.7. Let $\breve{A}(x,\beta)$ be the smoothing function of the function $A(x)$, then the folllowing condition is satisfied:

$$\lim_{\beta\to 0} \breve{A}(x,\beta) = A(x).$$

Proof. Since the function $\breve{S}(x,\beta)$ is smooth, then the function $\breve{A}(x,\beta)$ is also smooth for any $\beta > 0$. It can be clearly obtained by Theorem 3.6 that $\breve{A}(x,\beta)$ approaches to $A(x)$ as β approaches to zero. □

4. TOTAL VARIATION APPLICATION IN IMAGE DENOISING

A denoising image is a crucial part of computer vision and image processing. The majority of technologies are used to preserve, recover, or transfer images so that image noise may be reduced and the original image can be recaptured. In this process, considered images are all images in size $n \times n$. Now, assume that the original image is $x' \in R^{n^2}$, and let $\delta \in R^{n^2}$ denote the Gaussian noise. Any image $y \in R^{n^2}$ degraded can be written as follows:

$$y = x' + \delta. \tag{4.1}$$

The inverse of the model in Equation (4.1) can be used to remove noise from the image, with regularization techniques attempting to integrate both the terms data-fidelity and the model in Equation (4.1) into a single objective function, and the appropriate solutions for this function can be thought of as a minimization problem as follows:

$$min_x = \{\|y - x\| + \lambda J(x)\}, \tag{4.2}$$

where $J(x)$ is a regularization term and λ is a positive scalar which plays a role as an equalizer in Equation (4.2). When it comes to regularization approaches, the most important thing to keep in mind is making the right decision for the given picture model and having enough mathematical definition. Several various techniques for defining the recommended image have been proposed in the literature (Bruckstein *et al.* 2009, Ao *et al.* 2013). Due to its capacity to restore edges and maintain picture, the total variation function is one of the most essential regularization function. Rudin and Osher are the first using the total variation function as an image denoising regularizer (Rudin and Osher 1994).

The method was then extended and applied to deblurring and segmentation (He *et al.* 2012, Chan *et al.* 2006). Depending on the property of the total variation function, Equation (4.2) can be written as follows

$$min_x = \{\|y - x\| + TVJ(x)\} \tag{4.3}$$

and

$$TV(x) = \int \| D(x) \| \, dx = \sum_i \sqrt{(D_i^h x)^2 + (D_i^v x)^2},$$

where $D_i^h;D_i^v$ denote first-order horizontal and vertical finite-difference of x at (i-th) pixels. The minimizer is unique because the total variation function defined in Equation (4.3) is convex. However, because of the structure of the norm-term, which is hard to minimize and whose gradient flow is not well-defined, this function is non-differentiable. Researchers were obliged to investigate alternatives as a result of this challenge (Chambolle 2005). There are several ways for solving the total variation function, for additional information, the references Liu 2001, Marquina and Osher 2000, Sheffield 1985 can be investigated.

In various works (Chambolle 2004, Guo *et al.* 2009, Yang *et al.* 2009), the gradient descent approach was recommended as one of the significant ways for solving optimization with a twofold formulation for denoising total variation function. We employ the method established in previous chapter to create total variation function differentiable in this chapter, and then we apply it to picture denoising. To make total variation function differentiable, let $\|x\| := \|D(x)\|$. It is needed to utilize the outcomes and features of Chapter 3 to make the total variation function smooth which is defined in Chapter 1 as

$$TV(x) = \int \|D(x)\| dx.$$

Thus, the gradient of the above function can be calculated as

$$\nabla TV(x) = \; div\left(\frac{D(x)}{\|D(x)\|}\right).$$

It can be easily noticed that, at a pixel x, if $D(x)$ is equal to zero, then the gradient of total variation function becomes undefined which cause big difficulties to minimize the total variation function. Since gradient flow is not well defined, some methods are applied to the total variation function (Chan *et al.* 1999). In this case, the total variation function $TV(x)$ can be replaced by

$$TV(x,\beta) = \int D(x)\breve{S}(x,\beta) dx,$$

where

$$\breve{S}(x,\beta) = \frac{D(x)}{\sqrt{\beta+(D(x))^2}}$$

where $\beta > 0$. The gradient of the smoothed total variation function is

$$\nabla TV(x,\beta) = div\left(\frac{\beta}{((D(x))^2+\beta)^{\frac{3}{2}}}\right).$$

Now, the problem set out in Equation (4.3) can be reformulated using smooth total variation function as

$$min_x = \{\|y-x\| + TV(x,\beta)\} \tag{4.4}$$

By using Equation (4.4) with the help of gradient descent minimization method as in the next algorithm, we can obtain a good image denoising solution.

4.1 Algorithm

Step 1- Set $k=0$, choose $\beta > 0$, $\lambda > 0$, $\varepsilon > 0$ as a stopping condition, and $\tau > 0$ as a step.

Step 2- Let $x_1 = y$, y is a noisy image.

Step 3- Calculate a better solution $x^{(r)+1}$ by
$x^{(r)+1} = x_1 - \tau(y - x_1 + \lambda\nabla TV_\beta(x_1))$.

Step 4- If $\|x^{(r)+1} - x_1\| > \varepsilon$, then take $x_1 = x^{(r)+1}$, set $r = r+1$ and go to Step 3, else stop the algorithm and go to Step 5.

Step 5- Take $x^{(r)+1}$ as the best solution of image denoising operations.

In this chapter, we use some test images size of 256x256 as shown in Figure 4.1 to show the advantages of the presented method MSTV. Figure 4.2 - Figure 4.9 represent the gradient of the smooth the total variation function on test images with different β values.

The denoising results are illustrated in Figure 4.10 - Figure 4.17. The efficiency of the result is counted in Table 4.1 by peak signal-noise ratio PSNR. The higher PSNR values gives the higher quality of the restoration. We compared our method MSTV proposed in this thesis with some current image restore algorithms, such as FASTA denoising total variation (Beck and Teboulle 2009), the SA-DCT method in (Foi *et al.* 2007), the non-sampled contourlet denoising method in (Da Cunha *et al.* 2006) (DNS-CT), and generalized TV regularization (GTV) (Wu *et al.* 2017).

Figure 4.1 Some common test images (a) cameraman (b) church (c) satellite (d) Lenna (e) Barbara (f) pepper (g) cell and (h) Ibrahim

.

Figure 4.2 The gradient by MSTV on cameraman image

Figure 4.3 The gradient by MSTV on church image

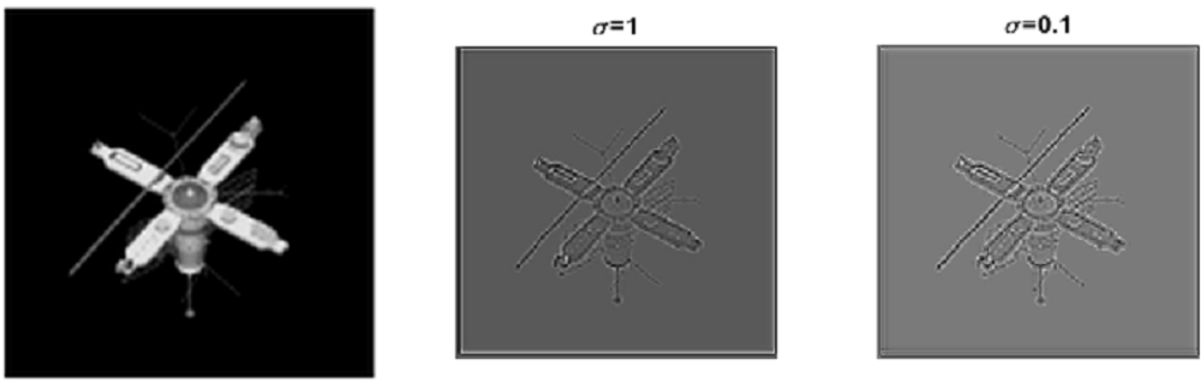

Figure 4.4 The gradient by MSTV on satellite image

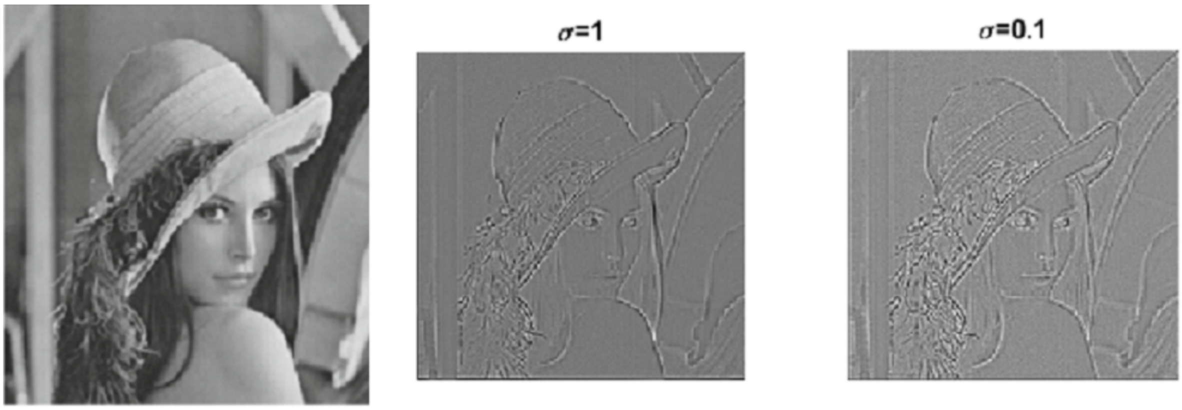

Figure 4.5 The gradient by MSTV on Lenna image

Figure 4.6 The gradient by MSTV on Barbara image

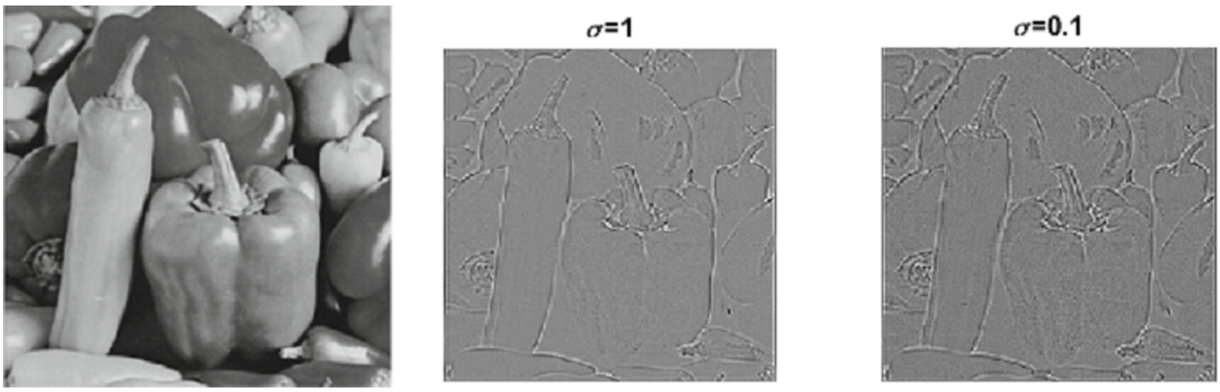

Figure 4.7 The gradient by MSTV on pepper image

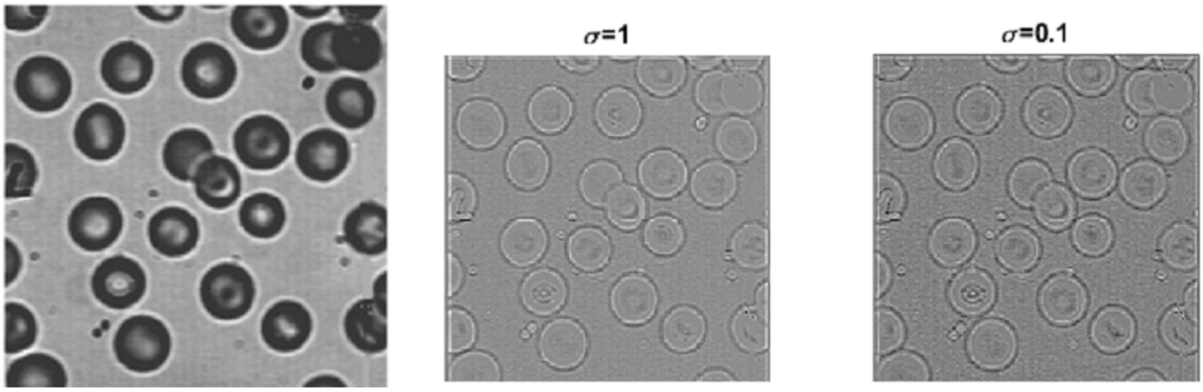

Figure 4.8 The gradient by MSTV on cell image

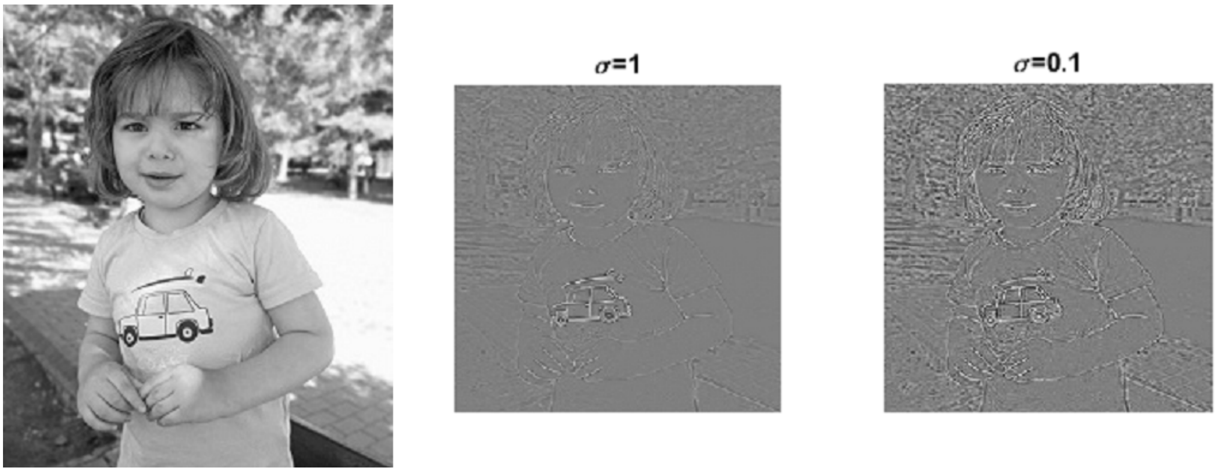

Figure 4.9 The gradient by MSTV on Ibrahim image

Figure 4.10 The denoising test with cameraman image using MSTV method

Figure 4.11 The denoising test with church image using MSTV method

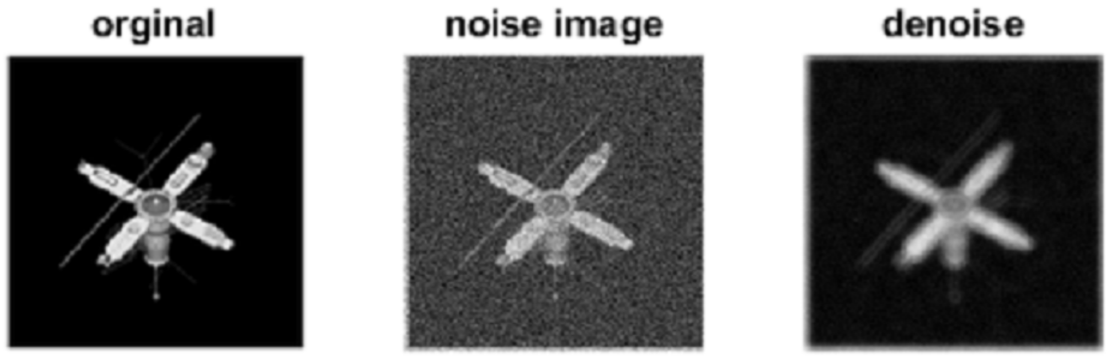

Figure 4.12 The denoising test with satellite image using MSTV method

Figure 4.13 The denoising test with Lenna image using MSTV method

Figure 4.14 The denoising test with Barbara image using MSTV method

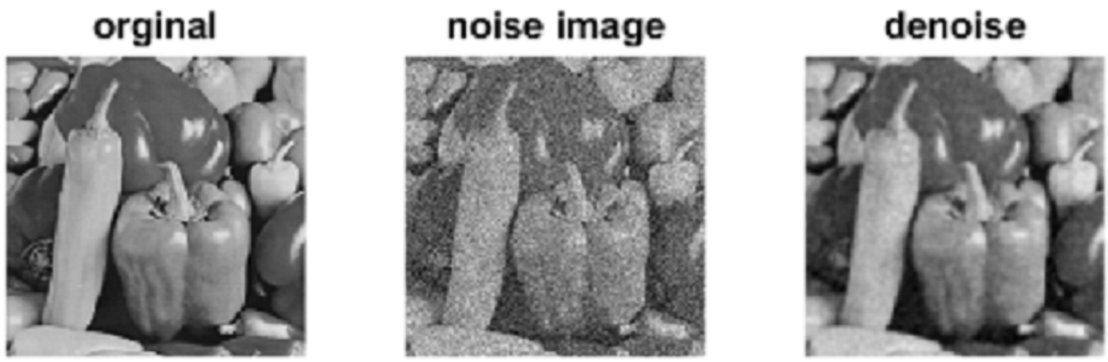

Figure 4.15 The denoising test with pepper image using MSTV method

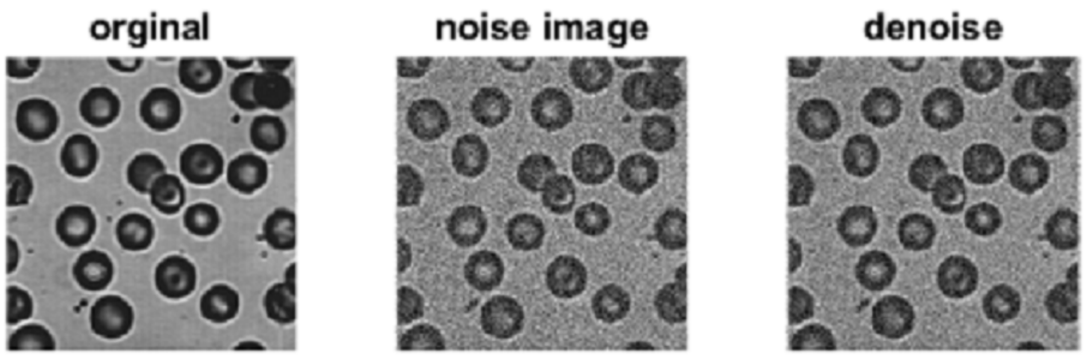

Figure 4.16 The denoising test with cell image using MSTV method

Table 4.1 The PSNR value by various methods (unit:db).

	MSTV	**TVF**	**SA_DCT**	**DNSCT**	**GTV**
Cameraman	28.4246	28.6591	30.0145	28.4210	30.3569
Barbara	28.0462	29.3121	30.8906	29.1285	31.9690
Peppers	28.6213	30.2731	31.8967	29.9971	31.7693
Lena	29.3214	29.4092	30.9197	29.1131	31.6027

Figure 4.17 The denoising test with Ibrahim image using MSTV method

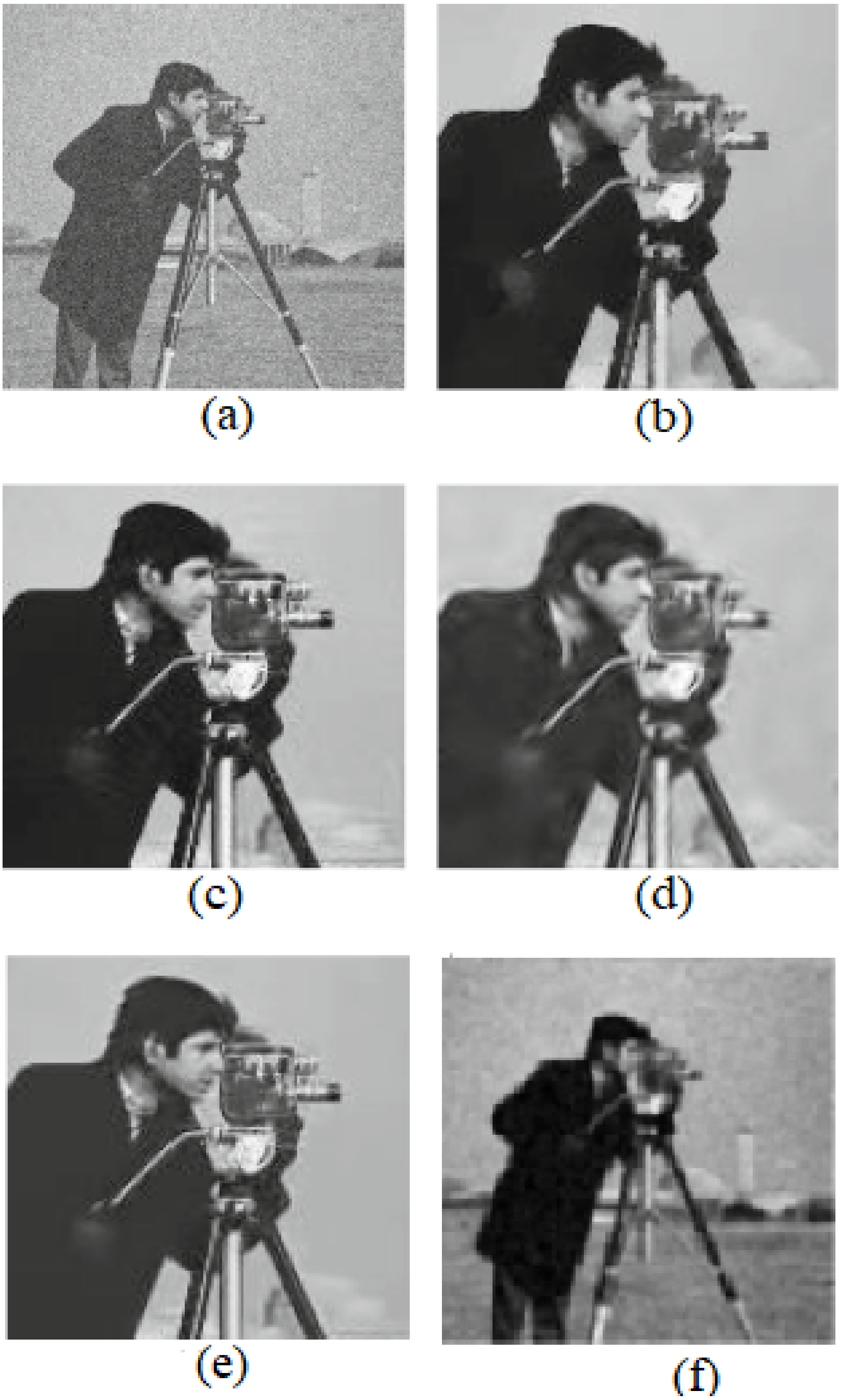

Figure 4.18 The denoising experiment with cameraman image (a) Noisy image (b) TVF (c) SA-DCT (d) DNSCT (e) GTV (f) MSTV

Figure 4.19 The denoising experiment with Barbara image (a) Noisy image (b) TVF (c) SA-DCT (d) DNSCT (e) GTV (f) MSTV

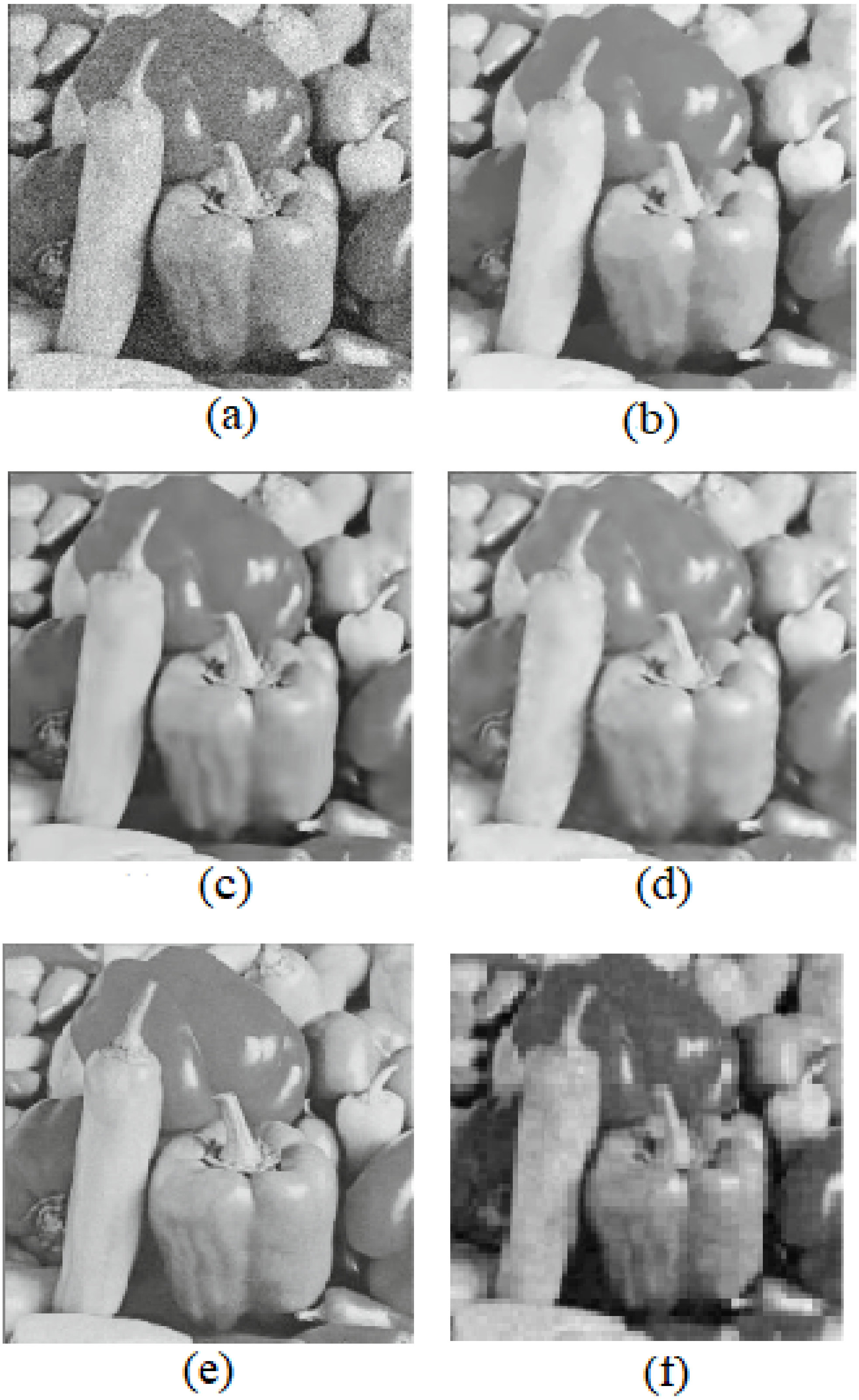

Figure 4.20 The denoising experiment with pepper image (a) Noisy image (b) TVF (c) SA-DCT (d) DNSCT (e) GTV (f) MSTV

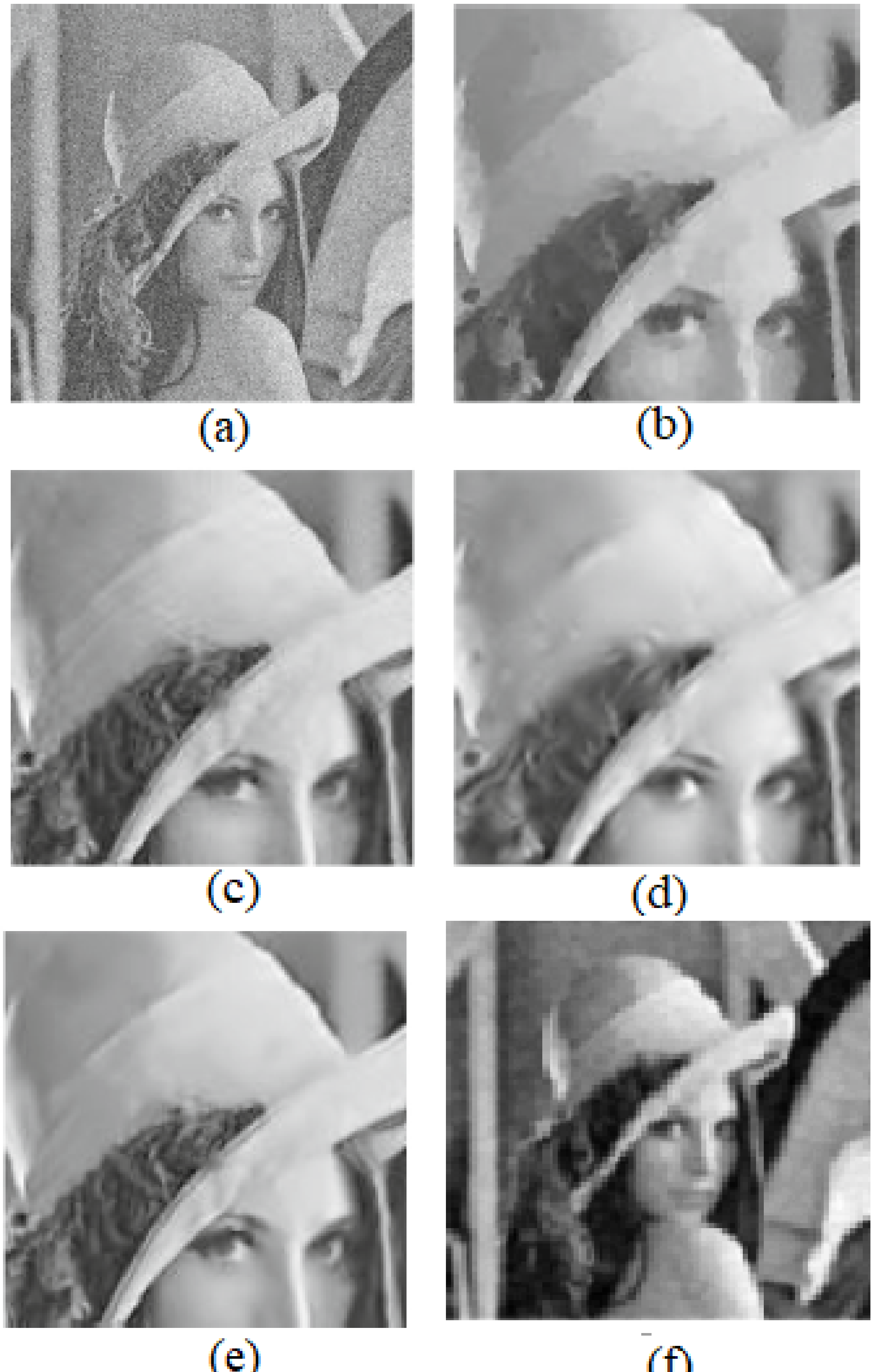

Figure 4.21 The denoising experiment with Lenna image (a) Noisy image (b) TVF (c) SA-DCT (d) DNSCT (e) GTV (f) MSTV

5. CONCLUSIONS AND RECOMMENDATION

In the new decade, the image processing attract the attention of scientists due to the expanding application area in various disciplines and necessity in modeling real processes, this leads to a requirement of developing theory. As in other fields of applied mathematics, denoising and image recovery is one of the most important techniques in moving image systems and play an essential role in many computer vision applications.

In this thesis, a new smoothing technique is used to make the total variation function smooth (continuous and differentiable) and then applied to reduce noise in the image. Experimental results proved that the proposed algorithm is effective in denoising, as shown in the fourth chapter of this thesis. As for future work, the proposed algorithm can be applied as an extended study of many real-life problems such as data mining, chemical processes, aerospace industries and image processing such as deblurring and segmentation.

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