ISTANBUL TECHNICAL UNIVERSITY ★ GRADUATE SCHOOL

SOME RESULTS ON THE SUMS OF UNIT FRACTIONS

Ph.D. THESIS

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Department of Mathematics Engineering

Mathematics Engineering Programme

MARCH 2025

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BİRİM KESİR TOPLAMLARI ÜZERİNE BAZI SONUÇLAR

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Çağatay ALTUNTAŞ, a Ph.D. student of ITU Graduate School student ID 509182201, successfully defended the dissertation entitled "SOME RESULTS ON THE SUMS OF UNIT FRACTIONS", which he prepared after fulfilling the requirements specified in the associated legislations, before the jury whose signatures are below.

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To my family,

FOREWORD

First, I would like to express my profound gratitude to my advisors, Assoc. Prof. Dr. Ergün YARANERİ and Assoc. Prof. Dr. Haydar GÖRAL. I am particularly grateful to Assoc. Prof. Dr. Haydar GÖRAL for his mentorship and collaboration, which significantly influenced the direction and outcomes of this thesis.

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March 2025

Çağatay ALTUNTAŞ
(Research Assistant)

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SYMBOLS

$\mathbb{N}$	: The set of natural numbers
$\mathbb{Z}$	: The set of integers
$\mathbb{Q}$	: The set of rational numbers
$\mathbb{P}$	: The set of all prime numbers
$p^k \parallel n$	: The integer p^k divides n but p^{k+1} does not
$\nu_p(a)$	: The p-adic valuation of an integer a
$gcd(m,n)$	: The greatest common divisor of m and n; (m,n) may also be used
$\mathscr{O}_K$	: The ring of integers of the number field K
$N(I)$	: The norm of the ideal $I \subset \mathscr{O}_K$
$\mathbb{F}_p$	: The finite field with p elements
$\Re(s)$	: The real part of $s \in \mathbb{C}$
$\zeta(s)$	: The Riemann zeta function
$\zeta_K(s)$	: The Dedekind zeta function of the number field K
$h_K(n)$	: The n^{th} Dedekind Harmonic Number
$e_{\mathfrak{p}_i}$	: The ramification index of the prime ideal $\mathfrak{p}_i$
$f_{\mathfrak{p}}$	: The inertial degree of the prime ideal $\mathfrak{p}$
$\pi(x)$	: The prime counting function, the number of primes less than or equal to x
$\pi_K(x)$	: The counting function for prime ideals with norms bounded by x in K
$O(\cdot)$	: The big-O notation
$o(\cdot)$	: The little-o notation
$\deg(f)$	: The degree of a polynomial f
$\mathbb{Q}(\sqrt{d})$	: Quadratic number field generated by the square root of d
Δ_K	: Discriminant of the number field K
$\chi_{\Delta_K}(\cdot)$	: Dirichlet character with modulus $\lvert\Delta_K\rvert$
$L(s,\chi_{\Delta_K})$	: Dirichlet L-function associated with the character χ_{Δ_K}
$a * b$	: Dirichlet convolution of the arithmetical functions a and b
$\left(\frac{\cdot}{p}\right)$	: Kronecker symbol mod p

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SOME RESULTS ON THE SUMS OF UNIT FRACTIONS

SUMMARY

A unit fraction is a rational number having 1 in its numerator and any positive integer in its denominator. Our purpose in this thesis is that exploring various sums of unit fractions in different aspects.

The sum of unit fractions is a broad subject which comes with great variety of problems and allows one to apply many techniques. An elementary example of such sums could be the harmonic numbers. Given a positive integer n, the n^{th} harmonic number is defined as

$$H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}.$$

As $H_n \sim \log n$, one may ask if there is a positive integer n so that H_n is an integer. The answer for this question is no unless n is 1 and one can prove that by considering the 2-adic valuation of H_n. For a given prime p, evaluating the p-adic order of a sum of unit fractions can be used to determine whether it may be an integer or not. Even if the answer is known, one can continue to investigate the sums' p-adic properties which lead us some other directions.

We begin by presenting a generalization of the harmonic numbers called the Dedekind harmonic numbers. In order to define them, we take a number field K and then consider the sum of reciprocals of norms of ideals of $\mathcal{O}_K$, the ring of integers of this number field K, in which their norms are bounded by a given positive integer n. They are indeed a generalization of the harmonic numbers which can be seen by setting $K = \mathbb{Q}$. We first show that these numbers are not integers after a while. Then, we provide this specific upper bound for some quadratic number fields to guarantee that they are non-integer. Furthermore, under the Riemann hypothesis, we obtain the non-integerness of differences of these numbers together with uniform bounds for quadratic number fields and derive an asymptotic result. These results are presented in Chapter 3 and they are based on our work [1].

We then continue with another example of the sums of unit fractions called the hyperharmonic numbers. The hyperharmonic numbers were long known to be non-integer and conjectured in [2] that such an example does not exist. Various results [2–6] pointed out that it may not be possible to find an hyperharmonic integer, yet, it was shown in [7] that they in fact exists. In the paper [2] that proposed the conjecture, there was a question: Can two hyperharmonic numbers of different indices and different orders be equal?

A partial answer to a more generalized version of this question will be given in Chapter 4 via a geometric approach with the help of related problems in arithmetic geometry. Afterwards, an analytic approach will be followed and we deduce that the differences of distinct hyperharmonic numbers are almost never an integer. The results provided in this chapter are derived from our work [8].

For any given prime number p, the set denoted by $J(p)$ was introduced in [9]. This set consists of the indices of the harmonic numbers whose numerators are divisible by this prime p in their lowest terms. The authors showed that this set always contains the integers $p-1, p(p-1), p^2-1$ whenever $p>2$. Moreover, the authors conjectured that the set is finite for each prime number. The size of this set for several prime numbers was calculated [10] and even upper bounds for a counting function for this set was given [11, 12]. We, in Chapter 5, generalize the set $J(p)$ to the generalized harmonic numbers. The generalized harmonic numbers are a sum of unit fractions where they have some positive integer powers s of the positive integers in their denominators. We define the generalizations $J(p,s)$ and $J(p^s,s)$ of $J(p)$, deduce some finiteness results, provide congruence relations and eventually obtain an upper bound for the counting function for $J(p,s)$. Moreover, we provide an explicit criterion that implies the finiteness of our set, together with computational results, and then point out the subjects that may reveal more about the finiteness of $J(p,s)$, by introducing Bernoulli and Euler numbers together with the irregular primes. All of these results are drawn from our work [13].

BİRİM KESİR TOPLAMLARI ÜZERİNE BAZI SONUÇLAR

ÖZET

Birim kesir toplamları, geniş bir problem yelpazesine sahip olan ve birçok tekniğin uygulanmasına olanak tanıyan kapsamlı bir konudur. Bu tür toplamların temel bir örneği harmonik sayılar olabilir. Pozitif bir n tam sayısı için, n. harmonik sayı

$$H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$$

olarak tanımlanır.

Harmonik sayıların büyüme hızı $H_n \sim \log n$ olduğundan, H_n'yi tam sayı yapacak bir pozitif n tam sayısı var mı diye sorulabilir. Bu sorunun yanıtı, $n = 1$ durumu hariç olumsuzdur ve bunu göstermek için H_n'nin 2-sel değerlemesine bakmak yeterlidir. Genel olarak, birim kesir toplamlarının tamsayı olup olmadığını kontrol etmek için, p bir asal sayı olmak üzere, p-sel değerleme kullanılabilir. Tamsayı olmalarından bağımsız olarak, bu toplamların p-sel özellikleri de araştırmaya değerdir ve bizi farklı yönlere götürür.

Bu tezde ilk olarak, birim kesir toplamlarının bir örneği olan harmonik sayıların bir genellemesini, Dedekind harmonik sayılarını tanıtacağız. Herhangi bir K sayı cismi ile onun tamsayılar halkası $\mathscr{O}_K$ ve bir n pozitif tamsayısı için, n. Dedekind harmonik sayısı

$$h_K(n) = \sum_{\substack{0 \neq I \subseteq \mathscr{O}_K \\ N(I) \leq n}} \frac{1}{N(I)}$$

olarak tanımlanır. Buradaki birim kesirlerin paydası, tamsayılar halkası $\mathscr{O}_K$'nın 0'dan farklı ideallerinden normu en fazla n olanlarından oluşmaktadır. Ek olarak, eğer sayı cismimizi rasyonel sayılar alırsak, n. Dedekind harmonik sayımız $h_{\mathbb{Q}}(n)$'nin aslında H_n olduğunu görebiliriz, yani bu sayılar gerçekten de harmonik sayıların bir genellemesidir.

Dedekind harmonik sayılarını tanıttıktan sonra, K bir sayı cismi olmak üzere, $h_K(n)$'nin bir n_K tamsayısından sonra tamsayı olamayacağını yani yalnızca sonlu n tamsayısı için tamsayı olabileceğini kanıtlayacağız. Devamında ise kuadratik sayı cisimlerine odaklanacağız ve n_K sınırının tam olarak kaç olduğunu bulmaya çalışacağız. Dahası, Dedekind zeta fonksiyonları için Riemann hipotezi altında göstereceğiz ki, belirli bir yerden sonra, iki farklı Dedekind harmonik sayısının farkı tamsayı olamaz. Son olarak, aynı hipotez altında kuadratik sayı cisimlerinde

tamsayı olmama durumunu inceleyeceğiz. Bu bölümdeki sonuçlar, [1] çalışmamızdan oluşturulmuştur.

Harmonik sayıların bir diğer genellemesi hiperharmonik sayılardır. Bir $r \geq 2$ tamsayısı için, $h_n^{(1)} = H_n$ olmak üzere, r. dereceden n. hiperharmonik sayı, yinelemeli olarak

$$h_n^{(r)} = \sum_{i=1}^{n} h_i^{(r-1)},$$

şeklinde tanımlanır.

Bu sayılar, birçok özellik ve problem ile gelir. Bu sayıların tamsayı olup olmaması bu problemlerden biridir. Örneğin [2] çalışmasında, hiçbir (n,r) tamsayı çifti için bu sayıların tamsayı olamayacağı iddia edilip her $n > 1$ ve $r = 2,3$ için bu iddia kanıtlanmıştır. Sonrasında, her $n > 1$ tamsayısı ve $r \leq 25$ tamsayısı dahil olmak üzere birçok n ve r ikilisi için de iddianın doğru olduğu gösterilmiştir [3,4]. Problem üzerindeki çalışmalar devam etmiş, r sayısı için üst sınır $25'$ten 20001'e çıkarılmıştır [5]. Aynı çalışmada,

$$S(x) = |\{(n,r) \in [1,x] \times [1,x] \colon h_n^{(r)} \notin \mathbb{Z}\}|$$

fonksiyonu için

$$S(x) = x^2 + O\left(x^{\frac{2.475}{1.475}}\right)$$

sağlandığı yani tamsayı olmayan hiperharmonik sayıların birince dörtlüde tam asimptotik yoğunluğa sahip olduğu gösterilmiştir. Buradaki hata terimi [6] çalışmasında iyileştirilmiştir. Mezö'nün iddiasını kuvvetlendiren bu çeşitli sonuçlara rağmen, hiperharmonik tamsayıların olduğu, hatta sonsuz tane oldukları [7] çalışmasında gösterilmiştir.

Tamsayılık sorusu dışında, Mezö'nün aynı [2] yayında ele aldığı şu problem de incelenebilir:

Problem. Hangi $n \neq m$ ve $r \neq s$ tamsayıları için

$$h_n^{(r)} = h_m^{(s)}$$

sağlanır?

Bu soru Bölüm 4'de incelenip kısmi bir cevap elde edilecek. Bölümün ilk kısmi ise aşağıdaki teoreme adanmış olacak.

Teorem. Herhangi iki $n > m \geq 4$ tamsayısı için $(n-1, m-1) = 1$ sağlansın. Herhangi bir γ rasyonel sayısı için,

$$h_n^{(r)} - h_m^{(s)} = \gamma \tag{1}$$

eşitliğini sağlayan yalnızca sonlu sayıda r,s pozitif tamsayıları bulunur. Ek olarak, $(n,m) \in \{(3,2),(4,2),(4,3)\}$ ve $\gamma \in \mathbb{Z}$ için (1) denkleminin çözümü yoktur.

Teoremimizi kanıtlamak için, geometrik bir yöntem kullanarak bu sonluluk problemimizi aritmetik geometrideki ilgili bir soruya bağlayacağız. Teoremdeki (n,m) ikilileri, $[1,x] \times [1,x]$ karesinde önemli bir kısmı kapsamaktadır. Daha açık bir ifade ile, bu karede, arasında asal olan (n,m) ikililerini sayan

$$C(x) = |\{(n,m) \in [1,x] \times [1,x] \colon n,m \in \mathbb{Z}^{>0},\ \ (n,m) = 1\}|$$

fonksiyonu için

$$\lim_{x\to\infty} \frac{C(x)}{x^2} = \frac{6}{\pi^2}$$

sağlandığı bilinmektedir [14].

Aynı bölümde, analitik bir yöntem kullanarak hiperharmonik farkların neredeyse hiçbir zaman tamsayı olamayacağını da göstereceğiz. Yani, x pozitif bir reel sayı olmak üzere, $[1,x]^4$ dört boyutlu küpündeki (n,m,u,v) dörtlülerini dikkatlice sayarak bunlara karşılık gelen

$$h_n^{(u)} - h_m^{(v)}$$

farklarının tamsayı olamayacağını göstereceğiz. Sonuç olarak, aşağıdaki teoremi elde edeceğiz:

Teorem. $Q(x)$ fonksiyonu,

$$|\{(n,m,u,v) \in [1,x]^4 \colon n,m,u,v \in \mathbb{Z}^{>0},\ \ h_n^{(u)} - h_m^{(v)} \notin \mathbb{Z}\}|$$

olarak tanımlansın. O zaman, herhangi bir $\varepsilon > 0$ gerçel sayısı için

$$Q(x) = x^4 + O_\varepsilon\left(x^{\frac{59}{18}+\varepsilon}\right)$$

sağlanır. Dahası, Riemann hipotezi altında

$$Q(x) = x^4 + O\left(x^3 \log^3 x\right)$$

sağlanır.

Bu bölümde sunulan sonuçlar, [8] çalışmamızdan oluşturulmuştur.

Bir p asal sayısı için, $J(p)$ kümesi

$$J(p) = \{n \in \mathbb{N} \colon \nu_p(H_n) \geq 1\}$$

olarak verilmiştir [9]. Çalışmada, her p asal sayısı için bu kümenin sonlu olduğu sanısı ortaya atılıp $p \leq 7$ asal sayıları için karşılık gelen $J(p)$ kümeleri elde edilmiştir. Her $p > 2$ asal sayısı için $p-1, p(p-1), p^2-1$ elemanlarının her zaman bu kümede olduğu da gösterilmiş ve eğer $J(p)$ yalnızca bu elemanlardan oluşuyorsa bu p asal sayılarına harmonik asallar denmiştir. Sonrasında, p asal sayısı $83, 127, 397$ sayılarından farklı ve 550'den küçük iken $J(p)$ kümesinin eleman sayısı hesaplanmıştır [10]. Bu küme için

$$J_p(x) = |J(p) \cap [1,x]|, \ \ x \in \mathbb{R}^{\geq 1}$$

şeklinde bir sayaç fonksiyonu tanımlanıp bir üst sınır verilmiş [11], sonrasında bu üst sınır geliştirilmiştir [12].

Herhangi n ve s pozitif tamsayıları için s. dereceden n. genelleştirilmiş harmonik sayı

$$H_n^{(s)} = \sum_{k=1}^{n} \frac{1}{k^s}$$

olarak tanımlanır. Bu sayılar, bariz $n = s = 1$ durumu hariç tamsayı değildir. Yine de, genelleştirilmiş harmonik sayılar, harmonik sayıların sağladıklarına benzer denklikleri sağlar. Örnek olarak,

$$H_{p-1} \equiv 0 \pmod{p}$$

denkliğinin her $p \geq 3$ asalı için sağlandığı bilinmektedir [15]. Eğer $p > 5$ ise bu denklik mod p^2'de sağlanır [16]. Benzer bir sonuç, Bölüm 5'de ele alınacaktır. Bu denklik hakkında bir derleme için [17] çalışması incelenebilir.

Biz de, genelleştirilmiş harmonik sayılarda, $s \geq 1$ iken $J(p)$ kümesinin

$$J(p,s) := \{n \in \mathbb{Z}^{>0} \colon H_n^{(s)} \equiv 0 \pmod{p}\}$$

ve

$$J(p^s,s) := \{n \in \mathbb{Z}^{>0} \colon H_n^{(s)} \equiv 0 \pmod{p^s}\}$$

genellemelerini inceleyeceğiz. İlk olarak, $p-1 \nmid s$ kısıtı altında, bu kümelerden birinin sonluluğunun diğerinin sonluluğunu verdiğini göstereceğiz. $J(p,s)$ kümesinin yapısını anladıktan sonra, her $x \geq 1$ gerçel sayısı için

$$J_{p,s}(x) = |J(p,s) \cap [1,x]|$$

sayaç fonksiyonunu tanımlayacağız. $J(p,s)$ kümesinin bir parçalanışını elde edip, parçalanıştaki bölümler için dikkatli bir şekilde üst sınırlar bulacağız. Böylece, $J(p,s)$ kümesinin sayaç fonksiyonu için

$$J_{p,s}(x) \leq 3x^{\frac{2}{3}+\frac{1}{25\log p}+\frac{\log s}{3\log p}+\frac{\log s}{3\log x}}$$

üst sınırını bulacağız. Bu üst sınırın bir sonucu olarak, $p > se^{\frac{3}{25}}$ iken

$$J_{p,s}(x) = o(x)$$

eşitliğinin sağlandığını da göstereceğiz.

Aynı bölümde hesaplamalı sonuçlara da değineceğiz. Daha açık olmak gerekir ise, makul bir koşul altında $J(p,s)$ kümesinin sonlu olduğunu kanıtlayıp, bu kümenin sadece p'den küçük pozitif tamsayılardan oluşabileceğini göstereceğiz. Sonrasında, bu koşulu sağlamayan p asal sayıları ve s tamsayılarını arayacağız. Birkaç örnek bulduktan sonra, $J(p,s)$ kümesinin sonluluğu hakkında kayda değer sonuçlar verebilecek Bernoulli ve Euler sayıları ile düzensiz asallar hakkında bir analiz yapacağız. Bu bölümdeki sonuçlar, [13] çalışmamıza dayanmaktadır.

1. INTRODUCTION

The sums of unit fractions is a broad subject that comes with a great variety of problems and allows one to apply many techniques. An elementary example of such sums could be the harmonic numbers. Given a positive integer n, the n^{th} harmonic number is defined as

$$H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}.$$

As $H_n \sim \log n$, one may ask if there is a positive integer n so that H_n is an integer. The answer to this question is no unless n is 1, and one can prove that by considering the 2-adic valuation of H_n. For a given prime p, evaluating the p-adic order of a sum of unit fractions can be used to determine whether it may be an integer or not. Even if the answer is known, one can continue to investigate the sums' p-adic properties, which lead us to some other directions.

In this thesis, we first present a generalization of the harmonic numbers, which we call the Dedekind harmonic numbers. Given any number field K with the ring of integers $\mathscr{O}_K$ and any positive integer n, we define the n^{th} Dedekind harmonic number by

$$h_K(n) = \sum_{\substack{0 \neq I \subseteq \mathscr{O}_K \\ N(I) \leq n}} \frac{1}{N(I)},$$

so that the sum is taken over the non-zero ideals of $\mathscr{O}_K$ with norms up to n. We first demonstrate that only finitely many of these are integers:

Theorem A. For any number field K, there exists $n_K \in \mathbb{Z}^{>0}$ such that $h_K(n) \notin \mathbb{Z}$ for any $n \geq n_K$. In particular, the positive integer n_K depends only on K.

Subsequently, we establish an explicit form of this result for quadratic fields.

Theorem B. (i) Let d be an integer and $K = \mathbb{Q}(\sqrt{d})$ be a quadratic number field where

$$d \not\equiv 1, 17 \pmod{24}$$

is square-free. Then for any $n \geq 4$, the corresponding Dedekind harmonic number is non-integer.

(ii) Let d be an integer and $K = \mathbb{Q}(\sqrt{d})$ be a quadratic number field where

$$d \equiv 1 \pmod{24}$$

is square-free. Then, $h_K(n)$ is non-integer for any $n \geq 4$ if

- $n \in [2^m, 2^{m+1})$ for some even integer $m \geq 2$, or,
- $n \in [2^m, 2^{m+1})$ for some positive integer $m \geq 3$ with $m \equiv 3 \pmod 4$, or
- $n \in [3^m, 3^{m+1})$ for some positive integer $m \not\equiv 2 \pmod 3$.

(iii) Let d be an integer and $K = \mathbb{Q}(\sqrt{d})$ be a quadratic number field where

$$d \equiv 17 \pmod{24}$$

is square-free. Then, $h_K(n)$ is non-integer for any $n \geq 9$ if

- $n \in [2^m, 2^{m+1})$ for some even integer $m \geq 2$, or,
- $n \in [2^m, 2^{m+1})$ for some positive integer $m \geq 3$ with $m \equiv 3 \pmod 4$, or
- $n \in [3^m, 3^{m+1})$ for some even integer $m \geq 2$.

Moreover, if we assume the Riemann hypothesis for Dedekind zeta functions, we show that, given enough terms, the difference between two Dedekind harmonic numbers will eventually be non-integral, and we prove non-integrality for quadratic number fields in another uniform manner along with an asymptotic result. These results are based on our work [1].

Another example of the sums of unit fractions is the hyperharmonic numbers. They are defined recursively as

$$h_n^{(r)} = \sum_{i=1}^{n} h_i^{(r-1)},$$

where $r \geq 2$ and $h_n^{(1)} = H_n$, the n^{th} harmonic number. We call $h_n^{(r)}$ the n^{th} hyperharmonic number of order r.

These numbers arise with plenty of fruitful properties. In particular, one can study the integerness features of them. It was claimed in [2] that there is not any (n, r) couple such that $h_n^{(r)}$ is an integer and it was shown that the claim is true for $r = 2, 3$ and for any $n > 1$. In [3,4], this result was strengthened, showing that $h_n^{(r)}$ is not an integer for any $n > 1$ and $r \leq 25$, along with several (n,r) pairs so that the corresponding $h_n^{(r)}$ is non-integer. Studies continued on this problem, where, for example, the upper bound for r is improved from 25 to 20001 in [5]. In the same paper, it was shown that

$$S(x) = |\{(n,r) \in [1,x] \times [1,x] \colon h_n^{(r)} \notin \mathbb{Z}\}|$$

satisfies

$$S(x) = x^2 + O\left(x^{\frac{2.475}{1.475}}\right),$$

which means the hyperharmonic numbers that are non-integers have full asymptotic density (in the first quadrant). Subsequently, the error term was enhanced in [6]. Although numerous results were supporting Mezö's conjecture, it was shown in [7] that there are infinitely many hyperharmonic numbers which are integers. Alternatively, one may examine the problem below, which was also presented by Mezö [2].

Problem. What are the integers $n \neq m$ and $r \neq s$ such that the equation

$$h_n^{(r)} = h_m^{(s)}$$

is satisfied?

In Chapter 4, we obtain a partial answer to a generalized version of this question and the first part of the chapter will be devoted to the following theorem.

Theorem D. Assume that $n > m \geq 4$ be any integers with $\gcd(n-1, m-1) = 1$. Then, for any $\gamma \in \mathbb{Q}$, the equation

$$h_n^{(r)} - h_m^{(s)} = \gamma \tag{1.1}$$

is satisfied only for finitely many $r, s \in \mathbb{Z}^{>0}$. Furthermore, there is no solution to (1.1) for $(n,m) \in \{(3,2),(4,2),(4,3)\}$ and any integer γ.

In order to prove our theorem, we employ a geometric approach that links our finiteness problem to a related question in arithmetic geometry. We note that our theorem covers a significant proportion of (n,m) tuples in $[1,x] \times [1,x]$. That is, as shown in [14], the function

$$C(x) = |\{(n,m) \in [1,x] \times [1,x] \colon n,m \in \mathbb{Z}^{>0}, \text{ with } \gcd(n,m) = 1\}|$$

satisfies

$$\lim_{x\to\infty}\frac{C(x)}{x^2}=\frac{6}{\pi^2}.$$

Moreover, in the same chapter, we then follow an analytic approach and show that the hyperharmonic differences are almost never an integer. In particular, we carefully count the integer tuples (n,m,u,v) within $[1,x]^4$ such that the hyperharmonic difference

$$h_n^{(u)}-h_m^{(v)}$$

does not yield an integer. Consequently, we establish the following theorem.

Theorem E. Let $Q(x)=|\{(n,m,u,v)\in[1,x]^4\colon\ n,m,u,v\in\mathbb{Z}^{>0},\ h_n^{(u)}-h_m^{(v)}\notin\mathbb{Z}\}|$. Then, for any positive real number $\varepsilon>0$ the equality

$$Q(x)=x^4+O_\varepsilon\left(x^{\frac{59}{18}+\varepsilon}\right)$$

holds. Furthermore, under the Riemann hypothesis, we have

$$Q(x)=x^4+O\left(x^3\log^3 x\right).$$

The results that we present in this chapter are based on our work [8].

Given a prime number p, the set $J(p)$ was defined in [9] as

$$J(p)=\{n\in\mathbb{N}\colon\ \nu_p(H_n)\geq 1\}.$$

It was conjectured by the authors of the paper that $J(p)$ is finite for any prime number p and obtained the sets $J(p)$ for primes $p\leq 7$. They also showed that the integers

$$p-1,p(p-1),p^2-1$$

are always in the set whenever $p>2$ and called the primes p harmonic if $J(p)$ consists only of those elements. In [10], the size of $J(p)$ was calculated for primes $p<550$, with the exclusion of $p\in\{83,127,397\}$. Later, an upper bound for the counting function of $J(p)$, namely,

$$J_p(x)=|J(p)\cap[1,x]|,\ \ x\in\mathbb{R}^{\geq 1}$$

was obtained in [11] and then improved in [12]. Now, for any positive integers n and s, the n^{th} generalized harmonic number of order s is defined as

$$H_n^{(s)}=\sum_{k=1}^{n}\frac{1}{k^s}.$$

They are not integers except for the trivial case, 1. Nevertheless, the generalized harmonic numbers satisfy equivalences similar to those found in harmonic numbers. As shown in [15], we know that

$$H_{p-1} \equiv 0 \pmod{p}$$

holds for $p \geq 3$. Moreover, the congruence is satisfied modulo p^2 where primes $p \geq 5$ by [16]. An analogous result for $H_{p-1}^{(s)}$ will be provided in Chapter 5 and we refer to [17] for a various generalizations of this congruence.

We consider the generalizations of $J(p)$ for the generalized harmonic numbers as

$$J(p,s) := \{n \in \mathbb{Z}^{>0} \colon H_n^{(s)} \equiv 0 \pmod{p}\}$$

together with

$$J(p^s,s) := \{n \in \mathbb{Z}^{>0} \colon H_n^{(s)} \equiv 0 \pmod{p^s}\}$$

for any positive integer s. For instance, we will see that the finiteness of one will imply the finiteness of the other, assuming $p-1 \nmid s$. After we understand the structure of $J(p,s)$, we will introduce the corresponding counting function

$$J_{p,s}(x) = |J(p,s) \cap [1,x]|$$

for any $x \in \mathbb{R}^{\geq 1}$. We obtain a partition of the set and we carefully give upper bounds for the blocks in the partition. Eventually, we get the following result.

Theorem F. For any prime number p, positive integer s and real number $x \geq 1$ we have

$$J_{p,s}(x) \leq 3x^{\frac{2}{3}+\frac{1}{25\log p}+\frac{\log s}{3\log p}+\frac{\log s}{3\log x}}.$$

Moreover, if $p > se^{\frac{3}{25}}$, then

$$J_{p,s}(x) = o(x).$$

In the same chapter, we will continue with computational results. In particular, we prove under a plausible condition that $J(p,s)$ is finite. In fact, it only consists of the elements that are strictly smaller than the prime number p:

Theorem G. Suppose that p is a prime and $s \geq 2$ is an integer satisfying $p-1 \nmid s$. If

$$\nu_p\left(H_r^{(s)}\right) \leq s-1$$

is satisfied for any $r \in \{1,2,\ldots,p-1\}$, then $J(p,s) \subset \{1,2,\ldots,p-1\}$. In fact, $J(p,s)$ is finite.

Then, we look for the primes p and the integers s in which our condition fails. After establishing several such values, we start a discussion involving Bernoulli and Euler numbers together with the irregular primes that may reveal crucial results on the finiteness of $J(p,s)$. The results established in this chapter are based on our work [13].

2. PRELIMINARIES

Let p be a prime. The p-adic valuation of an integer n is given as

$$\nu_p(n) = \begin{cases} k & \text{if } p^k \parallel n \\ \infty & \text{if } n = 0. \end{cases}.$$

The infinity above means that the number 0 has greater p-adic valuation than any other integer. For any rational number $\frac{u}{v}$, we can extend the definition as

$$\nu_p\left(\frac{u}{v}\right) = \nu_p(u) - \nu_p(v).$$

Then, $\nu_p(ab) = \nu_p(a) + \nu_p(b)$ and $\nu_p(a+b) \geq \min\{\nu_p(a), \nu_p(b)\}$ hold for any rationals a and b. Moreover, we have

$$\nu_p(a+b) = \min\{\nu_p(a), \nu_p(b)\}$$

whenever $\nu_p(a) \neq \nu_p(b)$ which we use frequently throughout the thesis.

An arithmetical function is a function $a(n) : \mathbb{Z}^{>0} \to \mathbb{C}$, for instance, the Euler's totient function

$$\varphi(n) = |\{1 \leq m \leq n \colon (m,n) = 1\}|.$$

We call an arithmetical function f multiplicative if $f(mn) = f(m)f(n)$ for any $(m,n) = 1$. If they are not coprime but still satisfy the equality we call f completely multiplicative.

The Dirichlet convolution of the arithmetical functions $a(n)$ and $b(n)$ is defined as

$$a(n) * b(n) = (a*b)(n) = \sum_{d|n} a(d) b\left(\frac{n}{d}\right)$$

and if a, b are multiplicative, then their Dirichlet convolution is also multiplicative.

A series of the form

$$\sum_{n=1}^{\infty} \frac{a(n)}{n^s}$$

is called a Dirichlet series, where $a(n)$ is an arithmetical function and s is a complex number. The Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

is a Dirichlet series and converges absolutely for all $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$.

Let $f(x)$ be any function and $g(x)$ be a function where $g(x) > 0$ for all $x \geq x_0$ for some $x_0 \in \mathbb{R}$. We write $f(x) = O\big(g(x)\big)$ or $f(x) \ll g(x)$ if there exists $B \in \mathbb{R}^{>0}$ such that for any $x \geq x_0$,

$$|f(x)| \leq Bg(x)$$

is satisfied. If the constant B depends on some parameter ε, we write

$$f(x) = O_\varepsilon\big(g(x)\big) \quad \text{or} \quad f(x) \ll_\varepsilon g(x).$$

If $f_1(x), f_2(x)$ and $g(x)$ are functions with $g(x) > 0$ for any $x \geq x_0$ for some $x_0 \in \mathbb{R}$, we mean by $f_1(x) = f_2(x) + O\big(g(x)\big)$ that $f_1(x) - f_2(x) = O\big(g(x)\big)$. Moreover, we write $f(x) = o\big(g(x)\big)$ if $\frac{f(x)}{g(x)}$ approaches 0 as x tends to infinity. In other words, if for every positive real number c there exists $x_0 \in \mathbb{R}$ such that for all $x \geq x_0$ we have

$$|f(x)| \leq cg(x).$$

If we write $f_1(x) = f_2(x) + o\big(g(x)\big)$ then we mean that

$$f_1(x) - f_2(x) = o\big(g(x)\big).$$

We write $f(x) \sim g(x)$ whenever

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1,$$

and say that $f(x)$ and $g(x)$ are asymptotically equal.

3. DEDEKIND HARMONIC NUMBERS

In this section, we introduce a sum of unit fractions, which is a generalization of the harmonic numbers, called the Dedekind harmonic numbers. The unit fractions in those numbers will have norms of some ideals inside a ring in their denominators.

3.1 Background

A finite field extension K of the rational numbers $\mathbb{Q}$ is called a number field. We call an element of K as an algebraic integer if it is a root of a non-zero polynomial over the integers. These elements give us a ring, which we denote by $\mathscr{O}_K$. This ring is a Dedekind domain, namely, it is Noetherian, integrally closed and each prime ideal is maximal. Hence, each proper ideal that is non-zero can be factorized as a product of prime ideals in a unique way.

For each non-zero ideal I of $\mathscr{O}_K$, the quotient $\mathscr{O}_K/I$ is finite and we define the norm of I as $N(I) = |\mathscr{O}_K/I|$. We note that the norm is multiplicative.

Let $\mathfrak{p} \neq 0$ be a prime ideal of $\mathscr{O}_K$. Then $\mathfrak{p} \cap \mathbb{Z}$ is a prime ideal $p\mathbb{Z}$ of $\mathbb{Z}$ for some prime number p. We say that $\mathfrak{p}$ lies above the prime number p. In addition, $\mathscr{O}_K/\mathfrak{p}$ is a finite field extension of $\mathbb{F}_p$ and hence,

$$N(\mathfrak{p}) = |\mathscr{O}_K/\mathfrak{p}| = p^{f_\mathfrak{p}}$$

for some integer $f_\mathfrak{p}$ called the inertial degree of $f_\mathfrak{p}$.

On the other hand, for any prime number p in $\mathbb{Z}$, we consider the ideal generated by p in $\mathscr{O}_K$. We write

$$p\mathscr{O}_K = \mathfrak{p}_1^{e_{\mathfrak{p}_1}} \dots \mathfrak{p}_\ell^{e_{\mathfrak{p}_\ell}}$$

for some prime ideals $\mathfrak{p}_i$ and positive integers $e_{\mathfrak{p}_i}$. The exponent $e_{\mathfrak{p}_i}$ is called the ramification index of $\mathfrak{p}_i$ and if $e_{\mathfrak{p}_i} > 1$ for some i, we say that the prime number p ramifies.

If K is a number field of degree n, then there are n distinct embeddings $\sigma_1, \ldots, \sigma_n$ of K into $\mathbb{C}$. The norm of an element α is defined as

$$N(\alpha) = \prod_{i=1}^{n} \sigma_i(\alpha).$$

The relation between the norm of an ideal generated by α and the norm of α is that $N(\alpha\mathcal{O}_K) = |N(\alpha)|$. Hence, we can write

$$\begin{aligned} N(p\mathcal{O}_K) &= N(\mathfrak{p}_1^{e_{\mathfrak{p}_1}} \ldots \mathfrak{p}_\ell^{e_{\mathfrak{p}_\ell}}) = N(\mathfrak{p}_1^{e_{\mathfrak{p}_1}}) \ldots N(\mathfrak{p}_\ell^{e_{\mathfrak{p}_\ell}}) = N(\mathfrak{p}_1)^{e_{\mathfrak{p}_1}} \ldots N(\mathfrak{p}_\ell)^{e_{\mathfrak{p}_\ell}} \\ &= p^{f_{\mathfrak{p}_1}{}^{e_1}} \ldots p^{f_{\mathfrak{p}_\ell}{}^{e_\ell}} = |N(p)| = p^n. \end{aligned}$$

That is, we have the identity

$$n = \sum_{i=1}^{\ell} e_{\mathfrak{p}_\mathrm{i}} f_{\mathfrak{p}_\mathrm{i}}. \tag{3.1}$$

If all $e_{\mathfrak{p}_\mathrm{i}} = f_{\mathfrak{p}_\mathrm{i}} = 1$ then we say p splits completely and if $\ell = 1$ and $e_1 = 1$ we say p is inert.

3.2 Dedekind Harmonic Numbers

Throughout this part, unless indicated otherwise, K will stand for a number field, p a prime number, and $\mathbb{P}$ is the set of prime numbers. Now, we will introduce our generalization of the harmonic numbers.

Definition 1. We define the n^{th} Dedekind Harmonic Number $h_K(n)$ as

$$\sum_{\substack{0 \neq I \subseteq \mathcal{O}_K \\ N(I) \leq n}} \frac{1}{N(I)}, \tag{3.2}$$

so that the sum is taken over all the ideals of $\mathcal{O}_K$ that are non-zero with norms bounded by n.

Observe that the sum (3.2) is finite since for any $n \geq 1$, the set

$$\{I \subseteq \mathcal{O}_K, I \neq 0 : N(I) \leq n\}$$

is finite via (3.1).

The construct of this analogue of harmonic numbers originates from the Dedekind zeta function of a number field. For any number field K, the Dedekind zeta function of K is defined as

$$\zeta_K(s) = \sum_{\substack{0 \neq I \\ I \subseteq \mathscr{O}_K}} \frac{1}{N(I)^s}$$

for any $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$. Note that by setting $K = \mathbb{Q}$, we obtain $\zeta_{\mathbb{Q}}(s) = \zeta(s)$ and $h_{\mathbb{Q}}(n) = h_n$.

Moreover, as $s \to 1^+$, the sum diverges so that it is meaningful to consider whether $h_K(n)$ is integral or not. Also, since there may be some distinct ideals I, J of $\mathscr{O}_K$ with the same norm, we can rewrite $h_K(n)$ as

$$h_K(n) = \sum_{j=1}^{n} \frac{a_j}{j}$$

where

$$a_j = |\{0 \neq I \subseteq \mathscr{O}_K \colon N(I) = j\}|.$$

Note that for some number field K, we may have $a_2 = 0$ meaning that there is not any ideal with norm 2. Similarly, if there is not any ideals with norm 3, we get $a_3 = 0$. Hence we have

$$h_K(1) = h_K(2) = h_K(3) = 1.$$

However, let us show that the Dedekind harmonic number $h_K(n)$ cannot be an integer, if n is large enough.

Theorem A. [1, Theorem A] For any number field K, there exists $n_K \in \mathbb{Z}^{>0}$ such that $h_K(n) \notin \mathbb{Z}$ for any $n \geq n_K$. In particular, the positive integer n_K depends only on K.

Proof. Let K be a number field of degree d and let us set $\pi_A = \{p \in \mathbb{P} \colon a_p \neq 0\}$ and $\pi_B = \{p \in \mathbb{P} \colon a_p \neq 0\}$. Notice that $\mathbb{P} = \pi_A \cup \pi_B$. For any prime number p, we know by (3.1) that $a_p \leq d$. So, our aim is to establish that there exists a prime number p with $a_p \neq 0$ that is large enough. Consequently, we need a generalization of the prime number theorem [14]:

$$\pi(x) = |\{p \in \mathbb{P} \colon p \leq x\}| \sim \frac{x}{\log x}.$$

The prime ideal theorem [18] states that

$$\pi_K(x) = |\{\mathfrak{p} \subset \mathscr{O}_K, \text{prime}\colon N(\mathfrak{p}) \leq x\}| \sim \frac{x}{\log x}.$$

Now, we can write

$$\pi_K(x) = \sum_{\substack{\mathfrak{p} \\ N(\mathfrak{p})=p\leq x}} 1 + \sum_{\substack{\mathfrak{p} \\ N(\mathfrak{p})=p^2\leq x}} 1 + \cdots + \sum_{\substack{\mathfrak{p} \\ N(\mathfrak{p})=p^d\leq x}} 1.$$

For any $1 \leq i \leq d$, the sum

$$\sum_{\substack{\mathfrak{p} \\ N(\mathfrak{p})=p^i\leq x}} 1$$

counts the number of prime ideals $\mathfrak{p}$ with norm $N(\mathfrak{p}) = p^i \leq x$, which is by definition, a_{p^i}. Hence for each i, we use the sum

$$\sum_{\substack{p\in\mathbb{P} \\ p\leq\sqrt[i]{x}}} a_{p^i}.$$

and write

$$\pi_K(x) = \sum_{\substack{p\in\mathbb{P} \\ p\leq x}} a_p + \sum_{\substack{p\in\mathbb{P} \\ p\leq\sqrt{x}}} a_{p^2} + \cdots + \sum_{\substack{p\in\mathbb{P} \\ p\leq\sqrt[d]{x}}} a_{p^d}.$$

Now, notice that (3.1) yields $a_{p^i} \leq \frac{d}{i}$ so that

$$\sum_{\substack{p\in\mathbb{P} \\ p\leq\sqrt[i]{x}}} a_{p^i} \leq \pi(\sqrt[i]{x})\frac{d}{i} \tag{3.3}$$

and we have

$$\pi_K(x) = \sum_{\substack{p\in\mathbb{P} \\ p\leq x}} a_p + O_d(\sqrt{x}).$$

Therefore, as $\pi_K(x) \sim \frac{x}{\log x}$, we have

$$s(x) = \sum_{\substack{p\in\mathbb{P} \\ p\leq x}} a_p \sim \frac{x}{\log x}.$$

Next, we check the limit $\frac{s(2x)}{s(x)}$ as $x \to \infty$. We have

$$\lim_{x\to\infty} \frac{s(2x)}{s(x)} = \lim_{x\to\infty} \frac{\frac{2x}{\log 2x}}{\frac{x}{\log x}} = 2 \neq 0.$$

Thus, $s(2x) - s(x) \to \infty$ as $x \to \infty$. That is,

$$\sum_{p \in (x,2x] \cap \mathbb{P}} a_p = \sum_{\substack{x<p\leq 2x \\ p\in\pi_A}} a_p$$

diverges as $x \to \infty$. Furthermore,

$$\pi_A(2x) - \pi_A(x) = \sum_{\substack{x<p\leq 2x \\ p\in\pi_A}} 1 = \sum_{\substack{x<p\leq 2x \\ p\in\pi_A}} \frac{a_p}{a_p} \geq \sum_{\substack{x<p\leq 2x \\ p\in\pi_A}} \frac{a_p}{d} = \frac{1}{d} \sum_{\substack{x<p\leq 2x \\ p\in\pi_A}} a_p = \frac{1}{d}\big(s(2x) - s(x)\big)$$

which diverges as $s(2x) - s(x)$ diverges when $x \to \infty$. Hence, there exist $n_K \in \mathbb{N}$ such that for any $n \geq n_K$, there is a prime $p \in (\frac{n}{2}, n]$ with $a_p \neq 0$. We can choose n_K large enough so that $n_K > 2d$.

Now, let $n \geq n_K$ be an integer and let $p \in \left(\frac{n}{2}, n\right]$ be a prime with $a_p \neq 0$. Then, as $2p > n$, we have

$$h_K(n) = 1 + \cdots + \frac{a_p}{p} + \cdots + \frac{a_n}{n}$$

as the only multiple of p up to n is p itself with $a_p \neq 0$. Finally, since $n \geq n_K > 2d$, we have

$$1 \leq a_p \leq d < \frac{n}{2} < p$$

so that the p-adic valuation of $h_K(n)$ is exactly -1 which yields $h_K(n) \notin \mathbb{Z}$. □

3.3 Explicit Computations

In the previous part, we showed that for any number field K, there exists a positive integer n_K such that for any $n \geq n_K$, the corresponding Dedekind harmonic number $h_K(n)$ is not an integer. That is, $h_K(n)$ is non-integer if n is large enough. Now, we will investigate some number fields K and try to obtain n_K's explicitly.

Let d be a square-free integer and let $K(\sqrt{d})$ be a quadratic number field. It is known that the discriminant Δ_K of K is given as

$$\Delta_K = \begin{cases} 4d & \text{if } d \equiv 2,3 \pmod 4 \\ d & \text{if } d \equiv 1 \pmod 4. \end{cases}$$

The discriminant Δ_K is an invariant of the number field K that encodes information, such as determining whether a prime p is ramified. Let p be a prime number. Define

$$\chi_{\Delta_K}(p) = \begin{cases} -1 & \text{if } p \text{ is inert} \\ 0 & \text{if } p \text{ ramifies} \\ 1 & \text{if } p \text{ splits.} \end{cases}$$

The function χ_{Δ_K} is a Dirichlet character with modulus $|\Delta_K|$. Extending it to the integers, we have the Dirichlet L-function

$$L(s,\chi_{\Delta_K}) = \sum_{n=1}^{\infty} \frac{\chi_{\Delta_K}(n)}{n^s}$$

with $s \in \mathbb{C}$ and converges absolutely for $\mathrm{Re}(s) > 1$. The Euler product $\prod_{\mathfrak{p}\,\text{prime}} \left(1 - \frac{1}{N(\mathfrak{p})^s}\right)^{-1}$ of $\zeta_K(s)$ and the definition of χ_{Δ_K} yields the identity

$$\zeta_K(s) = \zeta(s) L(s,\chi_{\Delta_K}).$$

Next, let us write

$$\zeta_K(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s} = \zeta(s) L(s,\chi_{\Delta_K}) = L(s,\chi_1) L(s,\chi_{\Delta_K})$$

where χ_1 is the trivial character, namely, $\chi_1(n) = 1$ for any $n \in \mathbb{Z}^{>0}$. As $\zeta_K(s)$ is a product of two Dirichlet series we can write

$$a_n = (1 * \chi_{\Delta_K})(n) \tag{3.4}$$

where 1 is the unit function, $*$ is the Dirichlet convolution of the arithmetical functions 1 and χ_{Δ_K}. Hence, we have that

$$a_n = \sum_{b|n} \chi_{\Delta_K}(b).$$

The character $\chi_{\Delta_K}(n)$ is, in fact, the Kronecker symbol $\left(\frac{\Delta_K}{n}\right)_K = \left(\frac{\Delta_K}{n}\right)$ which has the following properties:

(i) $\left(\frac{\Delta_K}{p}\right) = 0$ if $p \mid d$,

(ii) $\left(\frac{\Delta_K}{2}\right) = \begin{cases} -1 & \text{if } \Delta_K \equiv 5 \pmod 8, \\ 1 & \text{if } \Delta_K \equiv 1 \pmod 8 \end{cases}$

(iii) $\left(\frac{\Delta_K}{p}\right)$ is the Legendre symbol modulo p for any $p > 2$,

(iv) $\left(\frac{\Delta_K}{-1}\right) = \begin{cases} -1 & \text{if } \Delta_K < 0, \\ 1 & \text{if } \Delta_K > 0 \end{cases}$

(v) $\left(\frac{\Delta_K}{n}\right)$ is completely multiplicative.

In addition, we note that a_n is multiplicative as an arithmetical function, namely, for any positive integers u and v we have $a_{uv} = a_u a_v$ whenever $(u,v) = 1$. That is because $a_n = (1 * \chi_{\Delta_K})(n)$ and we know that the Dirichlet convolution of two arithmetical functions is also arithmetical.

Now, we are ready to compute n_K's for several fields. Let us use χ for χ_{Δ_K} in short. For any positive integer n, our aim is to find a prime p and so that $h_K(n)$ has a negative p-adic valuation for any $n \geq n_K$ making it non-integer. So, we will deduce Theorem B.

Theorem B. (i) Let d be an integer and $K = \mathbb{Q}(\sqrt{d})$ be a quadratic number field where

$$d \not\equiv 1, 17 \pmod{24}$$

is square-free. Then for any $n \geq 4$, the corresponding Dedekind harmonic number is non-integer.

(ii) Let d be an integer and $K = \mathbb{Q}(\sqrt{d})$ be a quadratic number field where

$$d \equiv 1 \pmod{24}$$

is square-free. Then, $h_K(n)$ is non-integer for any $n \geq 4$ if

- $n \in [2^m, 2^{m+1})$ for some even integer $m \geq 2$, or,
- $n \in [2^m, 2^{m+1})$ for some positive integer $m \geq 3$ with $m \equiv 3 \pmod{4}$, or
- $n \in [3^m, 3^{m+1})$ for some positive integer $m \not\equiv 2 \pmod{3}$.

(iii) Let d be an integer and $K = \mathbb{Q}(\sqrt{d})$ be a quadratic number field where

$$d \equiv 17 \pmod{24}$$

is square-free. Then, $h_K(n)$ is non-integer for any $n \geq 9$ if

- $n \in [2^m, 2^{m+1})$ for some even integer $m \geq 2$, or,
- $n \in [2^m, 2^{m+1})$ for some positive integer $m \geq 3$ with $m \equiv 3 \pmod{4}$, or
- $n \in [3^m, 3^{m+1})$ for some even integer $m \geq 2$.

Proof. Case 1. $K = \mathbb{Q}(\sqrt{d})$ for some $d \equiv 2, 3 \pmod{4}$. We have $\Delta_K = 4d$ so, $2 | \Delta_K$. Therefore,

$$\chi(2) = \left(\frac{\Delta_K}{2}\right) = 0$$

and in fact, $\chi(2^i) = 0$ for any $i \geq 1$ as χ is completely multiplicative. As a consequence,

$$a_{2^m} = \sum_{i|2^m} \chi(i) = \chi(1) + \chi(2) + \cdots + \chi(2^m) = 1 + 0 + \cdots + 0 = 1.$$

Now, let us take any positive integer $n \geq 2$ and let us write

$$2^m \leq n < 2^{m+1},$$

for some integer $m \geq 1$. Then we can write the Dedekind harmonic number $h_K(n)$ as

$$\begin{aligned} h_K(n) &= 1 + \frac{a_2}{2} + \frac{a_3}{3} + \cdots + \frac{a_{2^m}}{2^m} + \cdots + \frac{a_n}{n} \\ &= 1 + \frac{1}{2} + \frac{a_3}{3} + \cdots + \frac{1}{2^m} + \cdots + \frac{a_n}{n} \end{aligned}$$

so that the 2-adic valuation of $h_K(n)$ is $-m \leq -1$. Thus, n_K can be chosen as 2 in this case. In other words, the Dedekind harmonic number $h_K(n)$ is not an integer for any $n \geq 2$ for any $K = \mathbb{Q}(\sqrt{d})$ with $d \equiv 2, 3 \pmod 4$.

Case 2. $K = \mathbb{Q}(\sqrt{d})$ for some $d \equiv 1 \pmod 4$.

In this case, $\chi(2) = \pm 1$ as

$$\chi(2) = \left(\frac{\Delta_K}{2}\right) = \begin{cases} -1 & \text{if } \Delta_K \equiv 5 \pmod 8, \\ 1 & \text{if } \Delta_K \equiv 1 \pmod 8) \end{cases}$$

Therefore, for any positive integer $m \geq 0$, we have the following cases.

(i) If $\chi(2) = -1$, then,

$$a_{2^m} = \sum_{i|2^m} \chi(i) = \chi(1) + \chi(2) + \chi(4) + \cdots + \chi(2^m) = 1 - 1 + 1 - \cdots + (-1)^m \tag{3.5}$$

so that $a_{2^m} = 0$ if m is odd and $a_{2^m} = 1$ if m is even.

(ii) If $\chi(2) = 1$ then,

$$a_{2^m} = \sum_{i|2^m} \chi(i) = \chi(1) + \chi(2) + \chi(4) + \cdots + \chi(2^m) = 1 + 1 + \cdots + 1 = m + 1.$$

Hence, our process that we follow in Case 1 is not enough in Case 2. In fact, $\chi(2) = -1$ yields $a_2 = \chi(1) + \chi(2) = 1 - 1 = 0$ so that

$$\begin{aligned} h_K(1) &= \frac{a_1}{1} = 1 \\ h_K(2) &= 1 + \frac{a_2}{2} = 1 \end{aligned}$$

so that n_K must be at least 3. As a result, we need to know the value of $a_3 = 1 + \chi(3)$. Similarly, if $\chi(2) = 1$ then $a_2 = \chi(1) + \chi(2) = 1 + 1 = 2$ such that $h_K(1) = 1$ and $h_K(2) = 1 + \frac{2}{2} = 2$. Hence, either case, we need to know the value of $\chi(3)$. That is, if $K = \mathbb{Q}(\sqrt{d})$ for some square-free integer $d \equiv 1 \pmod 4$ then to be able to obtain some information about n_K, we need the value of

$$\chi(3) = \left(\frac{\Delta_K}{3}\right),$$

which leads a condition on Δ_K.

In short, one need to analyze 6 cases, namely, when $\chi(2) \in \{-1, 1\}$ and when $\chi(3) \in \{-1, 0, 1\}$. We start with the case $\chi(2) = -1$, or $d \equiv 5 \pmod 8$.

Case 2.1 $\chi(2) = -1$ and $\chi(3) = -1$.
For any integer $\ell \geq 0$, we have

$$a^{2\ell} = 1, \;\; a^{2\ell+1} = 0, \;\; a_3 = 0, \;\; a_{3.2^\ell} = 0$$

by (3.5) and the multiplicity of the arithmetical function a_n. Now, take any integer $n \geq 4$ and write

$$2^{2m} \leq n < 2^{2m+2}$$

for some $m \geq 1$. Then,

$$\begin{aligned} h_K(n) &= \left(1 + \frac{a_2}{2} + \frac{a_3}{3}\right) + \left(\frac{a_4}{4} + \cdots + \frac{a_8}{8} + \cdots + \frac{a_{12}}{12} + \cdots + \frac{a_{15}}{15}\right) + \cdots + \frac{a_n}{n} \\ &= (1 + 0 + 0) + \left(\frac{1}{4} + \cdots + \frac{0}{8} + \cdots + \frac{0}{12} + \cdots + \frac{a_{15}}{15}\right) + \cdots + \frac{a_n}{n} \end{aligned}$$

so that we can write $h_K(n)$ as

$$(1 + 0 + 0)$$

plus blocks of the form

$$\left(\frac{a_{2\ell}}{2^{2\ell}} + \frac{a_{2^{2\ell}+1}}{2^{2\ell}+1} + \cdots + \frac{a_{2^{2\ell+1}}}{2^{2\ell+1}} + \cdots + \frac{a_{3\cdot 2^{2\ell}}}{3 \cdot 2^{2\ell}} + \cdots + \frac{a_{2^{2\ell+2}-1}}{2^{2\ell+2}-1}\right) \tag{3.6}$$

plus the last block

$$\left(\frac{1}{2^{2m}} + \cdots + \frac{a_n}{n}\right).$$

However, each block (3.6) has 2-adic valuation $-2\ell < 0$ and the last block has $-2m < -2\ell$ for any ℓ which yields that $h_K(n) \notin \mathbb{Z}$ for any $n \geq 4$ in this case. In other words, n_K can be chosen 4.

Case 2.2 $\chi(2) = -1$ and $\chi(3) = 0$.

In this case, we have

$$a_{2^{2\ell}} = 1, \;\; a_{2^{2\ell+1}} = 0, a_{3^\ell} = 1 \text{ and } a_{2\cdot 3^\ell} = 0.$$

Let $n \geq 3$ be an integer with $3^m \leq n < 3^{m+1}$ for some integer $m \geq 1$. We have

$$\begin{aligned} h_K(n) &= \left(1 + \frac{a_2}{2}\right) + \left(\frac{a_3}{3} + \cdots + \frac{a_6}{6} + \cdots + \frac{a_8}{8}\right) + \cdots + \left(\frac{1}{3^m} + \cdots + \frac{a_n}{n}\right) \\ &= \left(1 + 0\right) + \left(\frac{1}{3} + \cdots + \frac{0}{6} + \cdots + \frac{0}{8}\right) + \cdots + \left(\frac{1}{3^m} + \cdots + \frac{a_n}{n}\right) \end{aligned}$$

so that $h_K(n)$ consists of blocks of the form

$$\left(\frac{a_{3^\ell}}{3^\ell} + \cdots + \frac{a_{2\cdot 3^\ell}}{2\cdot 3^\ell} + \cdots + \frac{a_{3^{\ell+1}-1}}{3^{\ell+1}-1}\right) = \left(\frac{a_{3^\ell}}{3^\ell} + \cdots + \frac{0}{2\cdot 3^\ell} + \cdots + \frac{1}{3^{\ell+1}-1}\right)$$

where each block has 3-adic valuation $-\ell < 0$. As a result,

$$v_3\big(h_K(n)\big) = v_3\left(\frac{1}{3^m} + \cdots + \frac{a_n}{n}\right) = -m < 0$$

so that $h_K(n)$ can be taken 3 in this case.

Case 2.3 $\chi(2) = -1$ and $\chi(3) = 1$.

We have

$$a_{2^{2\ell}} = 1, \;\; a_{2^{2\ell+1}} = 0, a_3 = 2, \;\; a_{3.2^{2\ell}} = 1.$$

Again let us take any $n \geq 4$ and write $2^m \leq n < 2^{2m} \leq n < 2^{2m+2}$ so that we have

$$h_K(n) = \left(1 + 0 + \frac{2}{3}\right)$$

plus the blocks

$$\begin{aligned} &\left(\frac{a_{2^{2\ell}}}{2^{2\ell}} + \cdots + \frac{a_{2^{2\ell+1}}}{2^{2\ell+1}} + \cdots + \frac{a_{3\cdot 2^{2\ell}}}{3\cdot 2^{2\ell}} + \cdots + \frac{a_{2^{2\ell+2}-1}}{2^{2\ell+2}-1}\right) \\ &= \left(\frac{1}{2^{2\ell}} + \cdots + \frac{0}{2^{2\ell+1}} + \cdots + \frac{2}{3\cdot 2^{2\ell}} + \cdots + \frac{a_{2^{2\ell+2}-1}}{2^{2\ell+2}-1}\right) \end{aligned}$$

for some $\ell \geq 1$. Each block has 2-adic valuation $-2\ell < 0$ and the last block has $-2m < 0$. Hence, for any $n \geq 4$ we get $h_K(n) \notin \mathbb{Z}$ and in fact, n_K can be taken 3 as $h_K(3) = \frac{5}{3}$.

Case 2.4 $\chi(2) = 1$ and $\chi(3) = 0$.

In this case, $d \not\equiv 1, 17 \pmod{24}$ and we have $a_{3^\ell} = 1$ for any $\ell \in \mathbb{Z}^{\geq 0}$. Hence, the same argument in Case 2.2 works so that n_K can be chosen 3. Now, let us summarize our work up to this point.

Let $K = \mathbb{Q}(\sqrt{d})$ be a quadratic number field where $d \not\equiv 1, 17 \pmod{24}$. Then for any $n \geq 4$, the corresponding Dedekind harmonic number is non-integer so we deduce the first part of Theorem B.

Case 2.5 $\chi(2) = 1$ and $\chi(3) = 1$.

Suppose that $d \equiv 1 \pmod{24}$. Then we have $\chi(2) = \chi(3) = 1$ and

$$a_{2^\ell} = a_{3^\ell} = \ell + 1, \;\; a_{2^{\ell_1}3^{\ell_2}} = (\ell_1 + 1)(\ell_2 + 1)$$

for any $\ell \geq 0$. Given a positive integer n, if $2^m \leq n < 2^{m+1}$ for some even $m \geq 1$ or $3^m \leq n < 3^{m+1}$ for some positive integer m with $m \neq 2 \pmod{3}$ holds, then we conclude that $h_K(n) \notin \mathbb{Z}$.

That is because in the first scenario, we can group the fractions with blocks

$$\left(\frac{a_{2^\ell}}{2^\ell} + \cdots + \frac{a_{2^{\ell+1}}}{2^{\ell+1}}\right).$$

For the last block, we have

$$\left(\frac{a_{2^m}}{2^m} + \cdots + \frac{a_n}{n}\right) = \left(\frac{m+1}{2^m} + \cdots + \frac{a_n}{n}\right)$$

and as m is even, the 2-adic valuation of the last block is $-m < 0$ which has the smallest 2-adic valuation among all the fractions implying $\nu_2\big(h_K(n)\big) = -m < 0$. Similarly, by grouping the terms with the powers of 3, the last block will be

$$\left(\frac{a_{3^m}}{3^m} + \cdots + \frac{a_n}{n}\right) = \left(\frac{m+1}{3^m} + \cdots + \frac{a_n}{n}\right)$$

and if $m \not\equiv 2 \pmod{3}$, we get $\nu_3\big(h_K(n)\big) = -m < 0$.

Now, let $n \geq 4$ be an integer. We showed that if $2^{2m} \leq n < 2^{2m+1}$ for some $m \geq 1$ then $h_K(n)$ is not an integer. However, in general, we have

$$n \in [2^{2m}, 2^{2m+1}) \text{ or } n \in [2^{2m+1}, 3 \cdot 2^{2m}), \text{ or } n \in [3 \cdot 2^{2m}, 2^{2m+2}).$$

If $n \in [2^{2m+1}, 3 \cdot 2^{2m})$, namely,

$$2^{2m+1} \leq n < 3 \cdot 2^{2m}$$

for some $m \geq 1$, then we have

$$\begin{aligned} h_K(n) &= 1 + \cdots + \frac{a_{2^{2m}}}{2^{2m}} + \cdots + \frac{a_{2^{2m+1}}}{2^{2m+1}} + \cdots + \frac{a_n}{n} \\ &= 1 + \cdots + \frac{2m+1}{2^{2m}} + \cdots + \frac{2m+2}{2^{2m+1}} + \cdots + \frac{a_n}{n} \\ &= \frac{3m+2}{2^{2m}} + \big(\text{the terms with 2-adic valuation} > -2m\big). \end{aligned}$$

So if m is odd, we obtain that $\nu_2\big(h_K(n)\big) = -2m < 0$ and $h_K(n) \notin \mathbb{Z}$.

If $n \in [3 \cdot 2^{2m}, 2^{2m+2})$ for some $m \geq 1$, then we have

$$\begin{aligned} h_K(n) &= 1 + \cdots + \frac{a_{2^{2m}}}{2^{2m}} + \cdots + \frac{a_{2^{2m+1}}}{2^{2m+1}} + \cdots + \left(\frac{a_{3 \cdot 2^{2m}}}{3 \cdot 2^{2m}}\right) + \cdots \frac{a_n}{n} \\ &= 1 + \cdots + \frac{2m+1}{2^{2m}} + \cdots + \frac{2m+2}{2^{2m+1}} + \cdots + \left(\frac{2 \cdot (2m+1)}{3 \cdot 2^{2m}}\right) + \cdots + \frac{a_n}{n} \\ &= \frac{13m+8}{3 \cdot 2^{2m}} + \big(\text{the terms with 2-adic valuation} > -2m\big). \end{aligned}$$

Hence, if m is odd then $\nu_2\big(h_K(n)\big) = -2m < 0$ and we get $h_K(n) \notin \mathbb{Z}$.

So if $n \in [2^{2m+1}, 3 \cdot 2^{2m})$ or $n \in [3 \cdot 2^{2m}, 2^{2m+2})$ for some odd integer $m \geq 1$ then $h_K(n)$ is non-integer. In short, as m is odd we can write

$$n \in [2^{2m+1}, 2^{2m+2}) = [2^{4m'+3}, 2^{4m'+4}) \implies h_K(n) \notin \mathbb{Z}.$$

To sum up, we showed that

(i) if $n \in [2^m, 2^{m+1})$ for some even integer $m \geq 2$, or,

(ii) if $n \in [2^m, 2^{m+1})$ for some positive integer $m \geq 3$ with $m \equiv 3 \pmod 4$, or

(iii) if $n \in [3^m, 3^{m+1})$ for some positive integer $m \not\equiv 2 \pmod 3$

then $h_K(n) \notin \mathbb{Z}$. In particular, $n_K = 4$ for any number field $K = \mathbb{Q}(\sqrt{d})$ with $d \equiv 1 \pmod{24}$. This proves the second part of Theorem B.

Case 2.6 $\chi(2) = 1$ and $\chi(3) = -1$.
Finally, we have the case $d \equiv 17 \pmod{24}$ so that $\chi(2) = 1$ and $\chi(3) = -1$ which yields

$$a_{2^\ell} = \ell + 1, \ \ a_{3^{2\ell}} = 0, \ \ a_{3^{2\ell+1}} = 1, \ \ a_{2 \cdot 3^\ell} = 2a_{3^\ell}.$$

A similar approach is applicable in this situation, as demonstrated in our paper [1, Theorem B], and thus we conclude that if $K = \mathbb{Q}(\sqrt{d})$ for some square-free $d \equiv 17 \pmod{24}$ then $h_K(n)$ is non-integer for any $n \geq 9$ whenever

(i) $n \in [2^m, 2^{m+1})$ for some even integer $m \geq 2$, or,

(ii) $n \in [2^m, 2^{m+1})$ for some positive integer $m \geq 3$ with $m \equiv 3 \pmod 4$, or

(iii) $n \in [3^m, 3^{m+1})$ for some even integer $m \geq 2$

and the proof is complete. □

3.4 Analytic Results on the Dedekind Harmonic Numbers

The Dedekind zeta function $\zeta_K(s)$ may be extended to the whole complex plane [19, Chapter 8]. Then, we can obtain an extension of the Riemann hypothesis for the Dedekind zeta function, which says that if $\zeta_K(s) = 0$ and $0 < \Re(s) < 1$ holds, then we have $\Re(s) = \frac{1}{2}$. This hypothesis is called the Extended Riemann Hypothesis which we use ERH for short. Now, let us continue with a fact on the prime ideals on short intervals.

Fact 2. [20, Theorem 2] Assuming ERH holds for K, there exist absolute constants $x_0, b_1, b_2 > 0$ so that for $x \geq x_0$ and

$$b_1(d_K \log x + \log|\Delta_K|)\sqrt{x} \leq y \leq x,$$

one has

$$\pi_K(x+y) - \pi_K(x) \geq b_2 \frac{y}{\log x},$$

where d_K denotes the degree of K.

Now, with the help of this fact, we are able to prove our last main result.

Theorem C. [1, Theorem C] Let K be a number field with degree d_K.

(i) Suppose ERH holds for K. Then, there exists $A, B > 0$ so that

$$h_K(n) - h_K(m) \notin \mathbb{Z}$$

for any positive integers $n > m \geq B$ whenever

$$n - m \geq A(d_K \log m + \log|\Delta_K|)\sqrt{m}$$

holds.

(ii) Suppose ERH holds for all quadratic number fields $\mathbb{Q}(\sqrt{d})$, with d being a square-free integer. Then for any $b \in (0,1)$ there exists a constant N_b so that for any $n \geq N_b$ and $|d| \leq e^{b\sqrt{\frac{n}{2}}}$, the corresponding $h_{K_d}(n) \notin \mathbb{Z}$.

(iii) Suppose ERH holds for all quadratic number fields $\mathbb{Q}(\sqrt{d})$, with d being a square-free integer and let F denotes the set of all square-free integers. Let

$$D(x) = |\{(d,m) \in [-x,x] \times [1,x] \colon d \text{ is square-free}, h_{K_d}(m) \notin \mathbb{Z}\}|$$

so that it counts the number of tuples $(d,m) \in F \times \mathbb{Z}^{>0}$ inside $[-x,x] \times [1,x]$ such that $h_{K_d}(m)$ is non-integer. Then we have

$$D(x) = 2xF(x) + O(x \log^2 x)$$

with $F(x) = |F \cap [1,x]|$. Equivalently, for almost every such pair (d,m), the corresponding Dedekind harmonic number $h_{K_d}(m)$ is non-integer since $F(x) \sim \frac{6}{\pi^2}$ and

$$D(x) \sim 2xF(x).$$

Proof. Let K be a number field, and assume that ERH holds for K. Let x_0, b_1, b_2 denote the constants from Fact 2. Now, suppose that g is a function satisfying

$$Cb_1(d_K \log x + \log|\Delta_K|)\sqrt{x} \leq g \leq x$$

where $C = \frac{2\sqrt{2}}{b_1 b_2} + 1$. Hence, there exists $x_c \geq \max\{x_0, d_K\}$ so that for all $x \geq x_c$ we have $Cb_1(d_K \log x + \log \Delta_K)\sqrt{x} \leq x$. Next, recall that we have

$$\pi_K(x) = \sum_{\substack{p \in \mathbb{P} \\ p \leq x}} a_p + \sum_{\substack{p \in \mathbb{P} \\ p \leq \sqrt{x}}} a_{p^2} + \cdots + \sum_{\substack{p \in \mathbb{P} \\ p \leq \sqrt[d]{x}}} a_{p^d}$$

and using (3.3), let us write

$$\pi_K(x) = \sum_{p \leq x} a_p + E(x),$$

where

$$|E(x)| \leq \frac{d_K}{2}\sqrt{x} + \frac{d_K}{3}\sqrt[3]{x} + \cdots + \sqrt[d_K]{x}.$$

□

Now, we can choose the above x_c big enough so that for every $x \geq x_c$, we have

$$|E(x)| \leq d_K \sqrt{x}.$$

Hence, using Fact 2,

$$\pi_K(x+g)-\pi_K(x)=\sum_{x<p\le x+g} a_p \ge \frac{b_2}{\log x}g-2d_K\sqrt{x+g}$$

and as $Cb_1(d_K\log x+\log|\Delta_K|)\sqrt{x}\le g$, we have

$$\begin{aligned}\frac{b_2}{\log x}g-2d_K\sqrt{x+g} &\ge \frac{b_2}{\log x}Cb_1(d_K\log x+\log|\Delta_K|)\sqrt{x}-2d_K\sqrt{x+g}\\ &> 2\sqrt{2}d_K\sqrt{x}-2d_K\sqrt{2x}+\frac{Cb_1b_2\log|\Delta_K|\sqrt{x}}{\log x}>0.\end{aligned}$$

Consequently, for any integer $m\ge x_c$, there exists a prime number p with $a_p\ne 0$ between m and $m+g=M$, provided that $m\ge g\ge A(d_K\log m+\log|\Delta_K|)\sqrt{m}$ for some absolute constant A. That yields

$$\nu_p\big(h_K(M)-h_K(m)\big)<0$$

so that the difference cannot be an integer.

Note that if $g>m$, we can proceed in the same manner as in the proof of Theorem A and the proof of the first part is done.

Now, suppose ERH holds for all $\mathbb{Q}(\sqrt{d})$ with d being a square-free integer and let $c\in(0,1)$. We know by Fact 2 that there exists constants $x_0,b_1,b_2>0$ so that for any $x\ge x_0$ and $b_1(2\log x+\log|\Delta_K|)\sqrt{x}\le g\le x$ we have

$$\pi_K(x+g)-\pi_K(x)\ge b_2\frac{g}{\log x}.$$

We know by Section 3.3 or [1, Theorem B] that if $d\not\equiv 1,17 \pmod{24}$ then for any $n\ge n_K=4$, then $h_K(n)\notin\mathbb{Z}$. Thus, to obtain a uniform bound for all quadratic fields, we may assume that $d\equiv 1 \pmod{24}$ or $d\equiv 17 \pmod{24}$. In either case, we have $|\Delta_K|=|d|$. Notice that if $|d|\le e^{b\sqrt{x}}$, then $\log|d|\le b\sqrt{x}$. Moreover, one can find $x_b\ge x_0$ such that for any $x\ge x_b$, we have $b_1(2\log x+b\sqrt{x})\sqrt{x}\le x$ with $\frac{b_2x}{\log x}-4\sqrt{2x}$. Now, if we set $g=x$, then we get

$$\sum_{x<p\le 2x} a_p>0 \tag{3.7}$$

similar to the first part of the proof, for any $\mathbb{Q}(\sqrt{d})$ satisfying $|d|\le e^{b\sqrt{x}}$.

Next, let $n_b>\max\{x_b,4\}$ be a positive integer. Let $n\ge n_b$ and $|d|\le e^{b\sqrt{\frac{n}{2}}}$. The inequality (3.7) implies that there is a prime $\frac{n}{2}<p_d\le n$ with $a_p\ne 0$. Then, by writing

$$h_{\mathbb{Q}(\sqrt{d})}(n)=1+\frac{a_2}{2}+\cdots+\frac{a_p}{p_d}+\cdots+\frac{a_n}{n}$$

we see that the only multiple of p_d is itself. Moreover, as $p_d > \frac{n}{2} > 2$ and $a_{p_d} \leq 2$ we conclude that

$$\nu_p\big(h_{Q(\sqrt{d})}(n)\big) = -1.$$

The proof of the second part is now done.

Lastly, let us set $b = \frac{1}{2}$ in the previous part of the theorem. Then, there exists $n_0 > 0$ so that for all $n \geq n_0$ with $|d| \leq e^{\frac{1}{2}\sqrt{\frac{n}{2}}}$ we get $h_{\mathbb{Q}(\sqrt{d})}(n) \notin \mathbb{Z}$. Let $x > n_0$ be large enough so that $d = e^{\frac{1}{2}\sqrt{\frac{n}{2}}}$ and $d = x$ intersect. In particular, they intersect when $n = 8\log^2 x$, as in Figure 3.1.

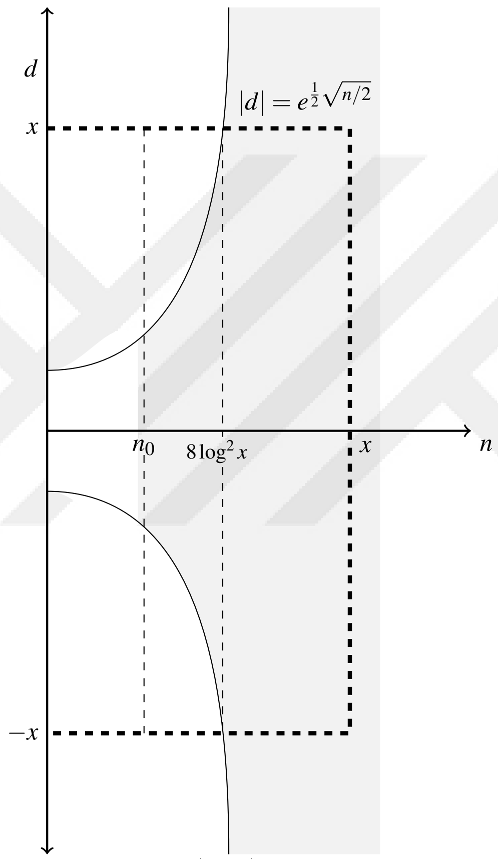

Figure 3.1 : The area highlighting (d,n) tuples where $h_{\mathbb{Q}(\sqrt{d})}(n)$ is non-integer.

Next, let us set $F(x) = |F \cap [1,x]|$ where F is the set of all square-free integers. We can write

$$D(x) - 2F(x)x \ll 8(\log^2 x)(2F(x)).$$

Hence,

$$D(x) = 2xF(x) + O(x\log^2 x)$$

and $D(x) \sim 2xF(x)$. Finally, as $F(x) \sim \frac{6}{\pi^2}$ we have $D(x) \sim \frac{12}{\pi^2}x^2$ which means almost all such tuples (d,n) yield non-integer Dedekind harmonic numbers.

4. HYPERHARMONIC NUMBERS AND THEIR DIFFERENCES

An example of the sums of unit fractions is the hyperharmonic numbers, which is a generalization of the harmonic numbers. They are defined recursively as follows:

$$h_n^{(r)} = \sum_{i=1}^{n} h_i^{(r-1)},$$

where $r \geq 2$ and $h_n^{(1)} = H_n$, the n^{th} harmonic number. We call $h_n^{(r)}$ as the n^{th} hyperharmonic number of order r. Our aim in this chapter will be to work on the differences of hyperharmonic numbers and obtain various results via geometric or analytic methods.

4.1 Geometric Results

In this section, we consider whether the difference of two different hyperharmonic numbers can be 0 or not. In other words, we look for the non-trivial solutions of

$$h_n^{(r)} = h_m^{(s)}. \tag{4.1}$$

Now, let us state our first lemma, which we use frequently throughout the thesis.

Lemma 3. Let n be any positive integer. We let

$$f_n(x) = \prod_{i=0}^{n-1} (x+i).$$

For any positive integer r, one has

$$h_n^{(r)} = \frac{f_n'(r)}{n!}.$$

Proof. Observe that

$$\log f_n(r) = \log\left(\prod_{i=0}^{n-1}(r+i)\right)$$

and taking derivative yields

$$\frac{f_n'(r)}{f_n(r)} = \sum_{i=0}^{n-1} \frac{1}{r+i}.$$

Conway and Guy showed in [21] that the hyperharmonic numbers satisfy

$$h_n^{(r)} = \binom{n+r-1}{r-1}(h_{n+r-1} - h_{r-1}).$$

Therefore,

$$h_n^{(r)} = \frac{(r-1)!\big(r\dots(n+r-1)\big)}{(r-1)!n!}\left(\frac{1}{r}+\cdots+\frac{1}{n+r-1}\right)$$

so that we have the product of

$$\frac{f_n(r)}{n!} \text{ and } \frac{f_n'(r)}{f_n(r)}.$$

Hence, the proof is done. □

So, instead of (4.1), we will work with

$$\frac{f_n'(r)}{n!} = \frac{f_m'(s)}{m!} \tag{4.2}$$

which is an equation of polynomials with rational coefficients.

To address the question, we apply [22, Theorem 1.1]. Now, (4.2) can be expressed as

$$p(x) = q(y) \tag{4.3}$$

for some $p(x), q(x) \in \mathbb{Q}[x]$ with $\deg p(x) = n-1$ and $\deg q(y) = m-1$ as f_n has degree n. Then, we can set

$$F(x,y) := p(x) - q(y) = 0 \tag{4.4}$$

and work with F. The equation $F(x,y) = 0$ is said to have infinitely many rational solutions with a bounded denominator if there exists a positive integer D so that (4.4) has infinitely many solutions $(x,y) \in \mathbb{Q}\times\mathbb{Q}$ where Dx, Dy belongs to $\mathbb{Z}$.

Below, we list five standard pairs of polynomials $\big(p(x), q(x)\big)$ over $\mathbb{Q}$, following [22]. Suppose a, A, u, v are non-zero rational numbers, m, s, t are positive integers, and $g(x)$ is a non-zero polynomial.

(i) The 1^{st} type. A tuple

$$(x^m, ax^\ell g(x)^m)$$

or reversed, $(ax^\ell g(x)^m, x^m)$ is a standard pair of the 1^{st} type, with $0 \le \ell < m,\ \gcd(\ell, m) = 1$ and $\ell + \deg g(x) > 0$.

(ii) The 2^{nd} *type.* A tuple

$$(x^2, (ux^2+v)g(x)^2)$$

or reversed, is a standard pair of the 2^{nd} type.

The polynomial

$$D_m(x,\beta) = \sum_{i=0}^{\lfloor m/2 \rfloor} \frac{m}{m-i}\binom{m-i}{i}(-\beta)^i x^{m-2i}$$

is called the m^{th} Dickson polynomial (of first type) where the parameter β is a rational number, as given in [23].

(iii) The 3^{rd} *type.* A tuple

$$\left(D_s\left(x,a^t\right), D_t\left(x,a^s\right)\right)$$

where $\gcd(s,t) = 1$ is a standard pair of the 3^{rd} type.

(iv) The 4^{th} *type.* A tuple

$$(u^{-s/2}D_s(x,u), -v^{-t/2}D_t(x,v))$$

where $\gcd(s,t) = 2$ is a standard pair of the 4^{th} type.

(v) The 5^{th} *type.* A tuple

$$((Ax^2-1)^3, 3x^4-4x^3)$$

or reversed, is a standard pair of the 5^{th} type.

Next, it is known that if the polynomials in (4.3) are one of the standard pairs over $\mathbb{Q}$ as above, then (4.3) has infinitely many solutions in $\mathbb{Q}$ with a bounded denominator via [22, p. 2].

Theorem 4 ([22, Theorem 1.1]). Suppose that $p(x)$ and $q(x)$ are two non-constant polynomials with rational coefficients. Then, the followings are equivalent.

(i) The equation (4.3) has infinitely many solutions in $\mathbb{Q}$ with a bounded denominator.

(ii) The polynomials $p(x)$ and $q(x)$ can be decomposed as $p = \varphi \circ p_1 \circ \lambda$ and $q = \varphi \circ q_1 \circ \mu$ where $\lambda(x), \mu(x) \in \mathbb{Q}[x]$ which are linear, $\varphi(x) \in \mathbb{Q}[x]$ and (p_1, q_1) is a standard pair over $\mathbb{Q}$ so that the $p_1(x) = q_1(y)$ has infinitely many solutions that are rational which have bounded denominators.

In particular, we have the upcoming fact from [22].

Fact 5 ([22, Remark 1.2.ii]). If the polynomials p and q have coprime degrees, then φ is linear and (p_1, q_1) is a standard pair of the 1^{st} or 3^{rd} type over the rationals.

Proposition 6. Let n be a positive integer and set

$$f_n(x) := \prod_{i=0}^{n-1}(x+i).$$

Assume that $n > 3$. Then, for any $a, b, c, d, \gamma \in \mathbb{Q}$, one cannot express $\dfrac{f_n'(x)}{n!} + \gamma$ as

$$a(cx+d)^{n-1} + b,$$

provided that $a, c \neq 0$.

Proof. Suppose that $n > 3$ is a positive integer. Observe that

$$\begin{aligned} f_n(x) &= x(x+1)\dots(x+n-1) \\ &= x^n + \big(1 + \dots + (n-1)\big)x^{n-1} \\ &\quad + \left(\sum_{1 \le i_1 < i_2 \le n-1} i_1 i_2\right) x^{n-2} + \dots \\ &\quad + \left(\sum_{i=1}^{n-1} (n-1)!/i\right) x^2 + (n-1)!x. \end{aligned}$$

One can explicitly obtain the coefficient of x^{n-2} as

$$\sum_{1 \le i_1 < i_2 \le z} i_1 i_2 = \frac{(z-1)z(z+1)(3z+2)}{24},$$

so, we obtain

$$\begin{aligned} f_n(x) = x^n + \frac{(n-1)n}{2}x^{n-1} + \frac{(n-2)(n-1)n(3n-1)}{24}x^{n-2} + \dots \\ + (n-1)!h_{n-1}x^2 + (n-1)!x. \end{aligned}$$

Next, by taking derivatives of both sides we get

$$f_n'(x) = nx^{n-1} + \frac{(n-1)^2 n}{2}x^{n-2} + \frac{(n-2)^2(n-1)n(3n-1)}{24}x^{n-3} + \dots$$
$$+ 2(n-1)!h_{n-1}x + (n-1)!.$$

Next, assume that

$$\frac{f_n'(x)}{n!} + \gamma = a(cx+d)^{n-1} + b$$

is satisfied, where a,b,c,d and γ are some rational numbers with $a,c \neq 0$. Let us write

$$\frac{f_n'(x)}{n!} = a(cx+d)^{n-1} + b - \gamma. \tag{4.5}$$

and compare the coefficients on both sides. As $n > 3$, we are able to examine the first three monomials.

For x^{n-1}. We have

$$\frac{n}{n!} = ac^{n-1}$$

so,

$$ac^{n-1} = \frac{1}{(n-1)!}. \tag{4.6}$$

For x^{n-2}. The equation

$$\frac{(n-1)^2 n}{2n!} = (n-1)ac^{n-2}d$$

yields

$$ac^{n-2}d = \frac{1}{2(n-2)!}. \tag{4.7}$$

For x^{n-3}. The equality

$$\frac{(n-2)^2(n-1)n(3n-1)}{24n!} = a\binom{n-1}{2}c^{n-3}d^2$$

implies that

$$ac^{n-3}d^2 = \frac{(n-2)^2(3n-1)}{12n!}. \tag{4.8}$$

Then, if we multiply (4.7) with c, we get

$$ac^{n-1}d = \frac{c}{2(n-2)!}. \tag{4.9}$$

So, by plugging (4.6) into (4.9), we have

$$\frac{d}{(n-1)!} = \frac{c}{2(n-2)!}$$

such that we get

$$\frac{c}{d} = \frac{2}{n-1}. \tag{4.10}$$

Furthermore, via (4.7) and (4.8), one has

$$\frac{ac^{n-2}d}{ac^{n-3}d^2} = \frac{1}{2(n-2)!}\frac{12n!}{(n-2)^2(3n-1)}.$$

As a result, we may write

$$\frac{c}{d} = \frac{6(n-1)n}{(n-2)^2(3n-1)}. \tag{4.11}$$

Next, if we combine (4.10) and (4.11), we obtain

$$\frac{2}{n-1} = \frac{6(n-1)n}{(n-2)^2(3n-1)}.$$

Consequently,

$$7n^2 - 13n + 4 = 0$$

must be satisfied. However, it is not possible as $n > 3$ and the proof is complete. □

Proposition 7. For any positive integer $n > 5$, let

$$f_n(x) := \prod_{i=0}^{n-1}(x+i)$$

and let D_m denotes the m^{th} Dickson polynomial (of the first kind). Then, $\frac{f_n'(x)}{n!} + \gamma$ cannot be expressed as

$$aD_{n-1}(cx+d,\alpha) + b$$

for any $a,b,c,d,\alpha,\gamma \in \mathbb{Q}$ with $a,c \neq 0$.

Proof. Let $n > 5$ be a positive integer. Let us write $f_n(x)$ as

$$\begin{aligned} f_n(x) &= x \dots (x+n-1) \\ &= x^n + \left(\sum_{j=1}^{n-1} j\right) x^{n-1} + \left(\sum_{1\le i_1<i_2\le n-1} i_1 i_2\right) x^{n-2} \\ &\quad + \left(\sum_{1\le i_1<i_2<i_3\le n-1} i_1 i_2 i_3\right) x^{n-3} \\ &\quad + \left(\sum_{1\le i_1<i_2<i_3<i_4\le n-1} i_1 i_2 i_3 i_4\right) x^{n-4} + \dots + (n-1)!x. \end{aligned}$$

We know from the proof of previous proposition that

$$\sum_{1\leq i_1<i_2\leq t} i_1 i_2 = \frac{(t-1)t(t+1)(3t+2)}{24}$$

holds. It can be verified for any $t \geq 3$ that

$$\sum_{1\leq i_1<i_2<i_3\leq t} i_1 i_2 i_3 = \frac{(t-2)(t-1)t^2(t+1)^2}{48}$$

and also

$$\sum_{1\leq i<j<k<\ell\leq t} ijk\ell = \frac{(t-3)(t-2)(t-1)t(t+1)(15t^3+15t^2-10t-8)}{5760}$$

holds.

Hence, we can write $f_n(x)$ as

$$\begin{aligned} f_n(x) = x^n &+ \frac{(n-1)n}{2}x^{n-1} + \frac{(n-2)(n-1)n(3n-1)}{24}x^{n-2} \\ &+ \frac{(n-3)(n-2)(n-1)^2n^2}{48}x^{n-3} \\ &+ \frac{(n-4)(n-3)(n-2)(n-1)n(15n^3-30n^2+5n+2)}{5760}x^{n-4} + \dots \\ &+ (n-1)!x. \end{aligned}$$

By differentiating, we obtain

$$\begin{aligned} f_n'(x) = nx^{n-1} &+ \frac{(n-1)^2n}{2}x^{n-2} + \frac{(n-2)^2(n-1)n(3n-1)}{24}x^{n-3} \\ &+ \frac{(n-3)^2(n-2)(n-1)^2n^2}{48}x^{n-4} \\ &+ \frac{(n-4)^2(n-3)(n-2)(n-1)n(15n^3-30n^2+5n+2)}{5760}x^{n-5} + \dots \\ &+ (n-1)!. \end{aligned}$$

Then, assume that

$$\frac{f_n'(x)}{n!} + \gamma = aD_{n-1}(cx+d,\alpha) + b$$

is satisfied for some $a,b,c,d,\alpha,\gamma \in \mathbb{Q}$ with $a,c,\alpha \neq 0$. So, we have

$$\frac{f_n'(x)}{n!} = aD_{n-1}(cx+d,\alpha) + b - \gamma. \tag{4.12}$$

Now, we will compare the coefficients of $x^{n-1},\dots,x^{n-5}$ on each side of (4.12). We have by definition that

$$D_{n-1}(cx+d,\alpha) = \sum_{i=0}^{\lfloor\frac{n-1}{2}\rfloor} \frac{n-1}{n-1-i}\binom{n-1-i}{i}(-\alpha)^i(cx+d)^{n-1-2i},$$

so we expand the sum for only $i \in \{0,1,2\}$.

We have

$$D_{n-1}(cx+d,\alpha) = \frac{n-1}{n-1}\binom{n-1}{0}(-\alpha)^0(cx+d)^{n-1} + \frac{n-1}{n-2}\binom{n-2}{1}(-\alpha)^1(cx+d)^{n-3} + \frac{n-1}{n-3}\binom{n-3}{2}(-\alpha)^2(cx+d)^{n-5} + \dots + \frac{n-1}{n-1-\lfloor\frac{n-1}{2}\rfloor}\binom{n-1-\lfloor\frac{n-1}{2}\rfloor}{\lfloor\frac{n-1}{2}\rfloor}(-\alpha)^{\lfloor\frac{n-1}{2}\rfloor}(cx+d)^{n-1-2\lfloor\frac{n-1}{2}\rfloor},$$

and hence, we have

$$D_{n-1}(cx+d,\alpha) = (cx+d)^{n-1} - (n-1)\alpha(cx+d)^{n-3} + \frac{(n-1)(n-4)}{2}\alpha^2(cx+d)^{n-5} + E_1(x).$$

Then, by expanding the terms, we can write $aD_{n-1}(cx+d,\alpha)+b-\gamma$ in (4.12) as

$$aD_{n-1}(cx+d,\alpha)+b-\gamma = ac^{n-1}x^{n-1} + (n-1)ac^{n-2}dx^{n-2} + (n-1)ac^{n-3}\left(\frac{n-2}{2}d^2-\alpha\right)x^{n-3} + (n-1)(n-3)ac^{n-4}d\left(\frac{n-2}{6}d^2-\alpha\right)x^{n-4} + \frac{(n-4)(n-1)}{2}ac^{n-5}\left[\frac{(n-3)(n-2)}{12}d^4-(n-3)d^2\alpha+\alpha^2\right]x^{n-5} + E_2(x).$$

So, we are ready to examine the coefficients in (4.12).

For x^{n-1}. We derive

$$ac^{n-1} = \frac{1}{(n-1)!}. \tag{4.13}$$

For x^{n-2}. We have

$$\frac{(n-1)^2 n}{2n!} = (n-1)ac^{n-2}d$$

such that

$$ac^{n-2}d = \frac{1}{2(n-2)!} \tag{4.14}$$

holds.

For x^{n-3}. The equality

$$\frac{(n-2)^2(n-1)n(3n-1)}{24n!} = (n-1)ac^{n-3}\left(\frac{n-2}{2}d^2-\alpha\right)$$

yields that

$$\frac{(n-2)(3n-1)}{24(n-1)(n-3)!} = ac^{n-3}\left(\frac{n-2}{2}d^2-\alpha\right). \tag{4.15}$$

For x^{n-4}. We have

$$\frac{(n-3)^2(n-2)(n-1)^2n^2}{48n!} = (n-1)(n-3)ac^{n-4}d\left(\frac{n-2}{6}d^2-\alpha\right),$$

which yields that

$$\frac{n}{48(n-4)!} = ac^{n-4}d\left(\frac{n-2}{6}d^2-\alpha\right). \tag{4.16}$$

For x^{n-5}. In (4.12), the corresponding coefficient on the left is

$$k_1 = \frac{(n-4)^2(n-3)(n-2)(n-1)n(15n^3-30n^2+5n+2)}{5760n!}$$

while we have

$$k_2 = \frac{(n-4)(n-1)}{2}ac^{n-5}\left[\frac{(n-3)(n-2)}{12}d^4-(n-3)d^2\alpha+\alpha^2\right].$$

on the right.

In this case, we disregard any cancellations and represent it briefly as

$$k_1 = k_2. \tag{4.17}$$

Now, we may express the variables in terms of c via the equations above. If we multiply (4.14) with c, we obtain

$$ac^{n-1}d = \frac{c}{2(n-2)!}.$$

Then, via (4.13) we have

$$d = \frac{n-1}{2}c. \tag{4.18}$$

Next, we can write (4.15) as

$$\frac{(n-2)(3n-1)}{24(n-1)(n-3)!} = ac^{n-3}\left(\frac{n-2}{2}\left(\frac{n-1}{2}c\right)^2-\alpha\right).$$

As a consequence, we obtain

$$\begin{aligned}
ac^{n-3}\alpha &= \frac{(n-2)(n-1)^2}{8}ac^{n-1} - \frac{(n-2)(3n-1)}{24(n-1)(n-3)!} \\
&\overset{(4.13)}{=} \frac{(n-2)(n-1)^2}{8}\frac{1}{(n-1)!} - \frac{(n-2)(3n-1)}{24(n-1)(n-3)!} \\
&= \frac{3(n-2)(n-1)^2 - (n-2)^2(3n-1)}{24(n-1)!} \\
&= \frac{(n-2)(n+1)}{24(n-1)!}.
\end{aligned}$$

Now, if we multiply both sides of the equation with c^2, we get

$$ac^{n-1}\alpha \overset{(4.13)}{=} \frac{\alpha}{(n-1)!} = \frac{c^2(n-2)(n+1)}{24(n-1)!}.$$

Hence, we obtain

$$\alpha = \frac{c^2(n-2)(n+1)}{24}. \tag{4.19}$$

We note here that (4.16) also implies (4.19). Now, we can move to the final step of the proof. Let us plug (4.18) and (4.19) into (4.17). We have

$$\begin{aligned}
k_2 &= \frac{(n-4)(n-1)}{2}ac^{n-5}\left[\frac{(n-3)(n-2)}{12}d^4 - (n-3)d^2\alpha + \alpha^2\right] \\
&= \frac{(n-4)(n-1)}{2}ac^{n-5}\left[\begin{array}{l} \frac{(n-3)(n-2)}{12}\left(\frac{n-1}{2}c\right)^4 \\ -(n-3)\left(\frac{n-1}{2}c\right)^2\left(\frac{c^2(n-2)(n+1)}{24}\right) \\ +\left(\frac{c^2(n-2)(n+1)}{24}\right)^2 \end{array}\right] \\
&= \frac{(n-4)(n-2)(n-1)}{2}ac^{n-1}\left[\begin{array}{l} \frac{(n-3)(n-1)^4}{192} - \frac{(n-3)(n-1)^2(n+1)}{96} \\ +\frac{(n-2)(n+1)^2}{576} \end{array}\right] \\
&\overset{(4.13)}{=} \frac{(n-4)(n-2)(n-1)}{2}\frac{1}{(n-1)!}\left[\begin{array}{l} \frac{(n-3)(n-1)^4}{192} - \frac{(n-3)(n-1)^2(n+1)}{96} \\ +\frac{(n-2)(n+1)^2}{576} \end{array}\right] \\
&= \frac{n-4}{2(n-3)!}\left(\frac{3n^5-27n^4+79n^3-78n^2+12n+7}{576}\right) \\
&= k_1 = \frac{(n-4)^2(n-3)(n-2)(n-1)n(15n^3-30n^2+5n+2)}{5760n!}.
\end{aligned}$$

Then, solving the equation yields

$$3n^2+14n+11=0.$$

Nevertheless, for any integer $n > 5$, we have

$$3n^2 + 14n + 11 > 0,$$

and the proof is done. $\square$

Now, in order to express our results using a geometric perspective, we will recall some essential definitions from arithmetic geometry. For further details, we refer to [24,25].

Suppose k is a field. Let $\mathbb{A}^2(k)$ denotes the affine plane, the set of tuples (x,y) for $x,y \in k$ and let $\mathbb{A}^n(k)$ denotes the affine space.

We define the usual *affine plane* as

$$\mathbb{A}^2(k) = \{(x,y) : x,y \in k\}.$$

For any positive integer n, the affine space $\mathbb{A}^n(k)$ is defined similarly.

Suppose that $x,y,z,u,v,w \in k$ such that the vectors (x,y,z) and (u,v,w) are not $(0,0,0)$. Then, we introduce a relation $\sim$ defined as

$$(x,y,z) \sim (u,v,w) \iff \lambda \in k^* \text{ for which } x = \lambda u, y = \lambda v, z = \lambda w.$$

This relation gives an equivalence relation with the equivalence classes

$$[x,y,z] = \{(u,v,w) : u,v,w \in k\colon\ (u,v,w) \neq (0,0,0) \text{ and } (x,y,z) \sim (u,v,w)\}.$$

Thus, the projective plane $\mathbb{P}^2(k)$ over k is defined as

$$\mathbb{P}^2(k) = \{[x,y,z] : x,y,z \in k \text{ and } (x,y,z) \neq (0,0,0)\}.$$

Notice that if $z \neq 0$, then $(x,y,z) \sim \left(\frac{x}{z}, \frac{y}{z}, 1\right)$.

Hence, we have

$$\mathbb{P}^2(k) = \{[x,y,1] : x,y \in k\} \cup \{[u,v,0] : u,v \in k\}.$$

The elements of the set $\{[u,v,0] : u,v \in k\}$ given above are referred to as the points at infinity.

Moreover, a curve in $\mathbb{A}^2(k)$ is the set of $\bar{k}$-solutions of some polynomial $f(x,y) \in k[x,y]$.

In order to define a curve in $\mathbb{P}^2(k)$, it is necessary to have a homogeneous polynomial. A polynomial $F(x,y,z)$ is called homogeneous (of degree d) if for every monomial $x^{e_1}y^{e_2}z^{e_3}$ in F, the equality $e_1+e_2+e_3 = d$ is satisfied.

A projective curve in $\mathbb{P}^2(k)$ is the set of all $\bar{k}$-solutions of a homogeneous $F(x,y,z) \in k[x,y,z]$ that is non-constant.

We will use $\mathbb{A}^2$ and $\mathbb{P}^2$ to denote the affine and projective planes when the field k is clear from the context. Now, suppose that we have a curve $C : f(x,y) = 0$ in $\mathbb{A}^2$.

The curve C is extended to a curve $\widehat{C}$ in the projective space as follows. Suppose d is the total degree of f, namely, the maximum of the degrees of the monomials. Now, we introduce

$$\widehat{C} : F(x,y,z) = z^d f\left(\frac{x}{z}, \frac{y}{z}\right) = 0.$$

We call the curve $\widehat{C}$ the projectivization of C. Notice that if we have a point (x,y) on C, then the point $[x,y,1]$ lies on the curve $\widehat{C}$.

An affine curve

$$C : f(x,y) = 0$$

is called singular at the point P in C if the partial derivatives $\frac{\partial f}{\partial x}(P) = f_x(P)$ and $\frac{\partial f}{\partial y}(P) = f_y(P)$ vanish.

Moreover, a projective curve $C' : F(x,y,z) = 0$ is singular at $Q \in C'$ if all the partial derivatives F_x, F_y, F_z vanish at the point Q.

On the other hand, we call C' non-singular (or smooth) at Q. If the curve is non-singular everywhere, then we call C' a smooth curve. Finally, we remark that similar definitions hold for affine curves.

From this point forward, let us assume that $k = \mathbb{C}$. Assume that C represents an affine curve and P denotes a point on this curve.

If P has integer coordinates, we refer to it as an integral point on C and if the coordinates belong to the rationals, then we call P a rational point on C. The set of integral and rational points on C will be denoted by $C(\mathbb{Z})$ and $C(\mathbb{Q})$, respectively.

Furthermore, let C be a projective curve defined by the equation $F(x,y,z) = 0$. Then, if F has rational coefficients, we say that C is a rational curve.

Moreover, for any curve C, there is an invariant $g \in \mathbb{Z}^{\geq 0}$ known as genus, based on the number of its singularities (see [26, Chapter 8]). If we have a non-singular projective curve C over the rationals of degree d, then the genus-degree formula

$$g = \frac{(d-1)(d-2)}{2}. \tag{4.20}$$

is satisfied. For a non-singular rational curve C with $g > 0$, it was shown by Siegel in 1929 that $C(\mathbb{Z})$ is finite (see [27]). Then, in 1983, Faltings improved the result. It was proven in [28] that given a smooth curve C that is rational with genus $g > 1$, the set $C(\mathbb{Q})$ is finite. This result is also referred to as the Mordell Conjecture.

Now, let $n,m,r,s \in \mathbb{Z}^{>0}$ and $a \in \mathbb{Q}$ satisfying

$$h_n^{(r)} - h_m^{(s)} = a.$$

Then via Lemma 3, we have

$$h_n^{(r)} - h_m^{(s)} = \frac{f_n'(r)}{n!} - \frac{f_m'(s)}{m!} = a. \tag{4.21}$$

Next, suppose without loss of generality that $n \geq m \geq 2$ and let us rewrite (4.21) as

$$f_n'(r) - d \cdot f_m'(s) = n!a \tag{4.22}$$

where

$$d = n(n-1)\dots(m+1). \tag{4.23}$$

So, we obtain a curve in $\mathbb{A}^2$ as

$$C_{n,m,a} : f(r,s) = f_n'(r) - d \cdot f_m'(s) - n!\, a = 0. \tag{4.24}$$

Let us remind that the polynomial $f_n'(r)$ has degree $n-1$ and the polynomial $f_m'(s)$ has degree $m-1$. Hence, the projectivization $\widehat{C}_{n,m,a}$ of $C_{n,m,a}$ in $\mathbb{P}^2$ can be defined as

$$\widehat{C}_{n,m,a} : F(r,s,t) = t^{n-1} f\left(\frac{r}{t}, \frac{s}{t}\right) = 0. \tag{4.25}$$

As a result, we have

$$
\begin{aligned}
F(r,s,t) = &\left(a_{n-1}r^{n-1} + a_{n-2}r^{n-2}t + \cdots + a_1 rt^{n-2} + a_0 t^{n-1}\right)\\
&- \left(b_{n-2}s^{n-2}t + b_{n-3}s^{n-3}t^2 + \cdots + b_1 st^{n-2} + b_0 t^{n-1}\right)\\
&- (n!a)t^{n-1} = 0
\end{aligned}
$$

where $a_i, b_j \in \mathbb{Z}^{>0}$ for $i \in \{0,\dots,n-1\}$, $j \in \{0,\dots,n-2\}$ (see Propositions 6, 7).

We can now link our work of hyper-harmonic differences to the field of arithmetic geometry. By Siegel's Theorem [27], if for some $a \in \mathbb{Q}$ and $(m,n) \in \mathbb{Z}^{>0} \times \mathbb{Z}^{>0}$ the curve $C_{n,m,a}$ is smooth with genus $g > 0$, then $C(\mathbb{Z})$ is finite. That is,

$$h_n^{(r)} = h_m^{(s)}$$

is satisfied for only finitely many tuples (r,s).

Let us now show the singularity of the projective curve $\widehat{C}_{n,m,a}$ whenever $n-m>1$.

Proposition 8. Suppose $n > m$ are positive integers and let $a \in \mathbb{Q}$. Then,

$$\widehat{C}_{n,m,a} \text{ is smooth at infinity if and only } n-m=1.$$

Proof. To begin with, let $n-m=1$. Now, let us write

$$h_n^{(r)} - h_m^{(s)} = \frac{f_n^{'}(r)}{n!} - \frac{f_m^{'}(s)}{m!} = a$$

and construct the affine curve

$$C_{n,m,a} : f(r,s) = f_n^{'}(r) - nf_{n-1}^{'}(s) - n!\,a = 0$$

as $m = n-1$. Then, we obtain the projective curve

$$\widehat{C}_{n,m,a} : F(r,s,t) = t^{n-1} f\left(\frac{r}{t}, \frac{s}{t}\right) = 0.$$

Next, we examine which points at infinity lie $\widehat{C}_{n,m,a}$. Assume $F(P) = 0$ for some $P = [r_0, s_0, 0]$. Then, we find

$$a_{n-1} r_0^{n-1} = 0,$$

but since $a_{n-1} = n \neq 0$, it follows that $r_0 = 0$. Therefore, $P = [0,1,0]$ is the only point at infinity on the curve. Furthermore, observe that we have

$$
\begin{aligned}
F_r &= (n-1)a_{n-1}r^{n-2} + (n-2)a_{n-2}r^{n-3}t + \cdots + a_1 t^{n-2},\\
F_s &= -(n-2)b_{n-2}s^{n-3}t - \cdots - b_1 t^{n-2}\ ,\\
F_t &= (n-1)(a_0 - b_0 - n!a)t^{n-2} + \cdots + (a_{n-2}r^{n-2} - b_{n-2}s^{n-2})
\end{aligned}
$$

and since $n > m > 0$ and $F_t(0,1,0) = -b_{n-2} = -n(n-1) \neq 0$, we deduce that $\widehat{C}_{n,m,a}$ is smooth at $[0,1,0]$.

Conversely, assume that $\widehat{C}_{n,m,a}$ is smooth at infinity and let $n - m = \ell > 1$. We can write

$$f_n^{'}(r) = a_{n-1}r^{n-1} + \cdots + a_1 r + a_0 \text{ and } f_m^{'}(s) = b_{m-1}s^{m-1} + \cdots + b_1 s + b_0$$

with $a_i, b_j \in \mathbb{Z}$ for $i \in \{0,1,\ldots,n-1\}$ and $j \in \{0,1,\ldots,m-1\}$.

Let us set $d_j = d \cdot b_j,\ \ j = 0,1,\ldots,m-1$ for the sake of simplicity where d is specified as in (4.23). We write then

$$\begin{aligned} F(r,s,t) = &\left(a_{n-1}r^{n-1} + a_{n-2}r^{n-2}t + \cdots + a_1 r t^{n-2} + a_0 t^{n-1}\right) \\ &- \left(d_{m-1}s^{m-1}t^{\ell} + d_{m-2}s^{m-2}t^{\ell+1} + \cdots + d_1 s t^{n-2} + d_0 t^{n-1}\right) - (n!a)t^{n-1} = 0. \end{aligned}$$

Thus, we have

$$\begin{aligned} F_r &= (n-1)a_{n-1}r^{n-2} + (n-2)a_{n-2}r^{n-3}t + \cdots + a_1 t^{n-2}, \\ F_s &= -(m-1)d_{m-1}s^{m-2}t^{\ell} - (m-2)d_{m-2}s^{m-3}t^{\ell+1} - \cdots - d_1 t^{n-2}\,, \\ F_t &= (n-1)(a_0 - d_0 - n!a)t^{n-2} + (n-2)(a_1 r - d_1 s)t^{n-3} + \cdots + a_{n-2}r^{n-2}. \end{aligned} \tag{4.26}$$

Observe that $P = [0,1,0]$ is a point on $\widehat{C}_{n,m,a}$ since

$$F(0,1,0) = 0.$$

Also, by (4.26), we get

$$F_r(P) = F_s(P) = F_t(P) = 0$$

as $\ell > 1$. Consequently, we see that our curve $\widehat{C}_{n,m,a}$ has a singularity, which is a contradiction. As a result, $n - m = 1$ must hold and the proof is complete. □

We are now ready to establish the proof of Theorem D.

Theorem D. [8, Theorem A] Assume that $n > m \geq 4$ be any integers with $\gcd(n-1, m-1) = 1$. Then, for any $\gamma \in \mathbb{Q}$, the equation

$$h_n^{(r)} - h_m^{(s)} = \gamma \tag{4.27}$$

is satisfied only for finitely many $r, s \in \mathbb{Z}^{>0}$. Furthermore, there is no solution to (4.27) for $(n,m) \in \{(3,2),(4,2),(4,3)\}$ and any integer γ.

Proof. Suppose that $n > m \geq 4$ are two integers satisfying $\gcd(n-1, m-1) = 1$. Suppose also that $\gamma \in \mathbb{Q}$ where

$$h_n^{(r)} - h_m^{(s)} = \gamma. \tag{4.28}$$

holds.

Case 1. For $n \geq 6$.

We will rely on Theorem 4 in this case. Notice that we can rewrite (4.28) as

$$\frac{f_n^{'}(r)}{n!} = \frac{f_m^{'}(s)}{m!} + \gamma \tag{4.29}$$

such that we get

$$p(r) = q(s) \tag{4.30}$$

for some $p(x), q(x)$ over $\mathbb{Q}$. Recall that the degree of the polynomial $f_n^{'}(r)$ is $n-1$, and the degree of the polynomial $f_m^{'}(s)$ is $m-1$. So, since we have $n > m \geq 4$, we know that $p(x), q(x)$ are not constant.

Now, assume that there exist infinitely many solutions $(r,s) \in \mathbb{Z}^{>0} \times \mathbb{Z}^{>0}$ for equation (4.30). Hence, there exists an infinite set of solutions that are rational with bounded denominator. Therefore, by Theorem 4, we can decompose p as $p = \varphi \circ p_1 \circ \lambda$ and q as $q = \varphi \circ q_1 \circ \mu$ such that

- λ and μ are polynomials of degree 1, having rational coefficients,
- φ is a polynomial over the rationals,
- The polynomials (p_1, q_1) are one of the standard pairs over $\mathbb{Q}$

where $p_1(x) = q_1(y)$ admits infinitely many solutions over the rationals with a bounded denominator.

Next, recall that we have

$$\gcd(n-1, m-1) = \gcd(\deg p, \deg q) = 1$$

so via Fact 5, we obtain $\deg \varphi = 1$ and the tuple (p_1, q_1) is a standard pair of the 1^{st} or 3^{rd} type over $\mathbb{Q}$. Furthermore, since the polynomials φ, λ, μ are of degree 1 and as we have $p = \varphi \circ p_1 \circ \lambda$, the degrees of p_1 and p are the same. Hence, $\deg p_1 = n-1$. Similarly, since we have $q = \varphi \circ q_1 \circ \mu$, we get $\deg q_1 = m-1$.

Now, recall that if the tuple $\big(p_1(x), q_1(x)\big)$ is 1^{st} type, then they appear in the form

$$(x^k, ax^r g(x)^k)$$

or reversed, for some $a \in \mathbb{Q} - \{0\}$, for some $0 \neq g(x) \in \mathbb{Q}[x]$ with $0 \leq r < k$, $\gcd(r,k) = 1$ and $r + \deg g(x) > 0$. Furthermore, let us define our linear polynomials φ, λ and μ. Let us set

$$\varphi(x) = ax + b, \lambda(x) = c_1 x + d_1 \text{ and } \mu(x) = c_2 x + d_2$$

where $a, b, c_1, c_2, d_1, d_2 \in \mathbb{Q}$ provided that $a, c_1, c_2 \neq 0$. Moreover, we either have $p_1(x) = x^{n-1}$ or $q_1(x) = x^{m-1}$. So, if we have $p_1(x) = x^{n-1}$, then

$$p(x) = (\varphi \circ p_1 \circ \lambda)(x) = a(c_1 x + d_1)^{n-1} + b. \tag{4.31}$$

If $q_1(x) = x^{m-1}$, we get

$$q(x) = (\varphi \circ q_1 \circ \mu)(x) = a(c_2 x + d_2)^{m-1} + b. \tag{4.32}$$

Recall that we have $n \geq 6$ and $m \geq 4$. Hence, the decompositions in (4.31) and (4.32) are not possible via Proposition 6.

Now, we investigate the case where (p_1, q_1) is of 3^{rd} type. Suppose $p_1(x) = D_{n-1}(x, \alpha)$ with α being a rational parameter that is non-zero. Indeed, α must equal a^{m-1} for some non-zero $a \in \mathbb{Q}$, though we have demonstrated the more general case. Next, suppose that $\varphi(x) = ax + b$ and $\lambda(x) = cx + d$, where $a, b, c, d \in \mathbb{Q}$ and $a, c \neq 0$. So, the polynomial $p(x)$ can be written as

$$p(x) = (\varphi \circ p_1 \circ \lambda)(x) = aD_{n-1}(cx + d, \alpha) + b. \tag{4.33}$$

However, the decomposition in (4.33) is not possible via Proposition 7. Therefore, we obtain via Theorem 4 that (4.30) is satisfied for only finitely many rational solutions having bounded denominators. In fact, (4.28) holds for only finitely many tuples $(r, s) \in \mathbb{Z}^{>0} \times \mathbb{Z}^{>0}$.

Case 2. For $n=5$ and $m=4$.

Assume that

$$h_5^{(r)}-h_4^{(s)}=\frac{f_5'(r)}{5!}-\frac{f_4'(s)}{4!}=\gamma \tag{4.34}$$

holds for some positive integers r,s and a rational number γ. Then, we obtain the affine curve

$$C_{5,4,\gamma}\ : f(r,s)=5r^4+40r^3+105r^2+100r-20s^3-90s^2-110s-6-120\gamma=0.$$

The partial derivatives of f can be obtained as $f_r=20r^3+120r^2+210r+100$ and $f_s=-60s^2-180s-110$. Then, solving $f_r=f_s=0$ yields the points

$$P_1\left(-2,\frac{\sqrt{15}-9}{6}\right),P_2\left(-2-\sqrt{3/2},\frac{\sqrt{15}-9}{6}\right),P_3\left(-2+\sqrt{3/2},\frac{\sqrt{15}-9}{6}\right),$$
$$P_4\left(-2,-\frac{\sqrt{15}+9}{6}\right),P_5\left(-2-\sqrt{3/2},-\frac{\sqrt{15}+9}{6}\right),P_6\left(-2+\sqrt{3/2},-\frac{\sqrt{15}+9}{6}\right).$$

Next, we get

$$f(P_1)=0 \text{ when } \gamma=\frac{36+25\sqrt{15}}{1080},\quad f(P_2)=0 \text{ when } \gamma=\frac{100\sqrt{15}-261}{4320}$$
$$f(P_3)=0 \text{ when } \gamma=\frac{100\sqrt{15}-261}{4320},\quad f(P_4)=0 \text{ when } \gamma=\frac{36-25\sqrt{15}}{1080}$$
$$f(P_5)=0 \text{ when } \gamma=-\frac{261+100\sqrt{15}}{4320},\quad f(P_6)=0 \text{ when } \gamma=-\frac{261+100\sqrt{15}}{4320}.$$

That is, the curve $C_{5,4,\gamma}$ is non-singular since $\gamma\in\mathbb{Q}$. The projectivization of this curve can be obtained as

$$\widehat{C}_{5,4,\gamma}: F(r,s,t)=t^4 f\left(\frac{r}{t},\frac{s}{t}\right)=0.$$

We know by Proposition 8 that this curve is non-singular at infinity and hence, we have a smooth curve. As a result, the genus degree formula (4.20) is satisfied. That is,

$$g=3>0.$$

Consequently, (4.34) has finitely many solutions $(r,s)\in\mathbb{Z}^{>0}\times\mathbb{Z}^{>0}$ via Siegel's Theorem [27] and the first part of the theorem is complete.

For the last part of the theorem, let us first suppose that $(n,m)=(3,2)$. In this case, we can write

$$h_3^{(r)}-h_2^{(s)}=\frac{f_3'(r)}{3!}-\frac{f_2'(s)}{2!}=\frac{r^2}{2}+r-s-\frac{1}{6}$$

such that the result is an integer whenever $\frac{3r^2-1}{6} \in \mathbb{Z}$. However, since $3r^2 - 1 \not\equiv 0 \pmod 3$ for any $r \in \mathbb{Z}^{>0}$, we get the result.

If $(n,m) = (4,2)$, we have

$$h_4^{(r)} - h_2^{(s)} = \frac{f_4'(r)}{4!} - \frac{f_2'(s)}{2!} = \frac{4r^3+18r^2+22r+6}{24} - \frac{2s+1}{2}.$$

Observe for any positive integer r that $\nu_p\left(h_4^{(r)}\right) = -2$ as $\nu_2\left(4r^3+18r^2+22r+6\right) = 1$. Moreover, we have $\nu_2(\frac{2s+1}{2}) = -1$ for any $s \in \mathbb{Z}^{>0}$. As a result,

$$\nu_2\left(h_4^{(r)} - h_2^{(s)}\right) = -2$$

for any r,s and we are done.

Finally, if $(n,m) = (4,3)$, we have

$$h_4^{(r)} - h_3^{(s)} = \frac{f_4'(r)}{4!} - \frac{f_3'(s)}{3!} = \frac{4r^3+18r^2+22r+6}{24} - \frac{3s^2+6s+2}{6}$$

and notice that $\nu_2(\frac{3s^2+6s+2}{6}) \geq -1$. However, as $\nu_2(\frac{4r^3+18r^2+22r+6}{24}) = -2$ holds, we deduce that

$$\nu_2\left(h_4^{(r)} - h_3^{(s)}\right) = -2$$

and thus the difference cannot be an integer.

The proof is now finished. □

Remark 9. Suppose that

$$p(x) = q(y)$$

has infinitely many solutions that are rational having bounded denominators. Then, this fact does not yield that there are infinitely many solutions that are integers. In fact, there is always a solution to

$$h_n^{(r)} = s$$

with a bounded denominator. Namely, for any $n \in \mathbb{Z}^{>0}$ and $r \in \mathbb{Z}^{>0}$, one can find some $s \in \mathbb{Q}^{>0}$, as we have

$$h_n^{(r)} = \frac{f_n'(r)}{n!} = s.$$

On the other hand, we know by [5] that $h_n^{(r)}$ is non-integer for any r and $n \in \{1,2,\ldots,32\}$. Hence, even if $h_n^{(r)} = s$ has infinitely solutions that are positive with bounded denominator for any $n \in \{1,2,\ldots,32\}$, one cannot find a positive integer solution to it.

4.2 Analytic Results

In this section, we present various integerness results on the difference of hyperharmonic numbers. We start with a simple observation.

Proposition 10. Let m be a positive integer and let $p > m$ be a prime number. Then for any positive integers r, s, the difference

$$h_p^{(r)} - h_m^{(s)}$$

is not an integer.

Proof. By Lemma 3, we know that

$$h_n^{(r)} = \frac{f_n^{'}(r)}{n!}.$$

Also, we have by definition

$$f_p(x) = x(x+1)\ldots(x+p-1) \equiv x^p - x \in \mathbb{F}_p[x].$$

This yields $f_p^{'}(x) \equiv -1$ in $\mathbb{F}_p[x]$, which means that if $f_p^{'}(r) = \sum_{k=0}^{n-1} c_k r^k$, then p divides each c_k except for the constant term, c_0. Consequently,

$$h_p^{(r)} = \frac{f_p^{'}(r)}{p!}$$

has p-adic valuation -1. As $p > m$ we have

$$\nu_p\big(h_m^{(s)}\big) = \nu_p\left(\frac{f_m^{'}(s)}{s!}\right) \geq 0$$

so that $\nu_p\left(h_p^{(r)} - h_m^{(s)}\right) = -1$ and we are done. □

Let $I_{n,r}$ denote the set of integers $\{r, \ldots, n+r-1\}$ for any $n, r \in \mathbb{Z}^{>0}$ and let $I_{n,r}(p)$ be the set of integers in $I_{n,r}$ that are divisible by p. Notice that if there is a prime $p < n$ and $I_{n,r}(p)$, then $h_n^{(r)} \notin \mathbb{Z}$ as $\nu_p\left(h_n^{(r)}\right) < 0$ which can be seen from the proof of Lemma 3.

Fact 11. Let c be a positive integer. Then, there exists $x_c \in \mathbb{R}$ which depends on c so that for any $x \geq x_c$, there is a prime number inside the interval $((1-c)x, x]$.

Proposition 12. Assume that m,u are two positive integers. Then, there is $n_c \in \mathbb{Z}^{>0}$ which depends on m,u so that for any $n \geq n_c$, we have $h_n^{(u)} - h_m^{(v)} \notin \mathbb{Z}$ for any $s \in \mathbb{Z}^{>0}$.

Proof. Suppose that $n_c \in \mathbb{Z}^{>0}$ is large enough such that $\left(\frac{2n}{3}, n\right]$ contains a prime number for all $n \geq n_c$ via Fact 11. We can also suppose that $n_c \geq \max\{\frac{3m}{2}, 3u-3\}$. Now, assume that $n \geq n_c$. As $n \geq 3u-3$, we have $\frac{2n}{3} \geq \frac{n+u-1}{2}$ and

$$\left(\frac{2n}{3}, n\right] \subseteq \left(\frac{n+u-1}{2}, n\right].$$

As a result, we can find a prime number p satisfying $\frac{n+u-1}{2} < p \leq n$ so that we have $u-1 < p$ and $n+u-1 < 2p$. That is, $I_{n,u}(p)$ is p itself and $|I_{n,u}(p)| = 1$.

Next, we can write

$$h_n^{(u)} = \binom{n+u-1}{u-1}(h_{n+u-1} - h_{u-1}) = \frac{\sum_{i=u}^{n+u-1} A_i}{n!}$$

with $A_i = \frac{\mathrm{Per}(n+u-1,u-1)}{i}$ and $i \in \{u, \ldots, n+u-1\}$. Now,

$$h_n^{(u)} - h_m^{(v)} = \frac{\sum_{i=u}^{n+u-1} A_i}{n!} - \frac{\sum_{j=v}^{m+v-1} B_j}{m!}$$

where $B_j = \frac{\mathrm{Per}(m+v-1,v-1)}{j}$ for $j \in \{v, \ldots, m+v-1\}$.

Moreover, as $|I_{n,u}(p)| = 1$, we can say that $p|A_j$ for all $j = u, \ldots, n+u-1$ except for A_p. Hence, $\nu_p\left(h_n^{(u)}\right) < 0$. Furthermore, $\frac{2n}{3} \geq m$ implies that $n \geq \frac{3m}{2}$ and $m < p \leq n$ holds. Then, $\nu_p\left(h_m^{(v)}\right) \geq 0$, so that $h_n^{(u)} - h_m^{(v)}$ has a negative valuation. The proof is now complete. □

As a corollary, we can state the upcoming remark.

Remark 13. Assume that $n,m \in \mathbb{Z}^{>0}$ and p is a prime number satisfying $m < p < n$. If for some $u \in \mathbb{Z}^{>0}$ we have $|I_{n,u}(p)| = 1$, then the difference $h_n^{(u)} - h_m^{(v)}$ is non-integer for any $v \in \mathbb{Z}^{>0}$.

Proposition 14. Suppose that n,m,u are some positive integers and $a,b \geq 1$ and $p,q \in \mathbb{P}^{>m}$ are some positive integers where one of the followings

$$(a-1)n \leq u < an, \quad \frac{n+u}{a+1} < p < n \quad \text{or} \tag{4.35}$$

$$\frac{bn}{2} < u \leq bn, \ \ \frac{n+u}{b+2} < q < \frac{u}{b} \tag{4.36}$$

hold. In either case, the difference

$$h_n^{(u)} - h_m^{(v)}$$

is not an integer for any positive integer v.

Proof. We give our proof by showing that $|I_{n,u}(p)| = |I_{n,u}(q)| = 1$ together with $p, q > m$ and get the result via Remark 13.

In the first case, $p < n$ yields $(a-1)p < (a-1)n \leq u$. The inequality $\frac{n+u}{a+1} < p$ implies $n+u < (a+1)p$ and $u < ap + p - n < ap$. Also we have $ap < n+u$ holds as otherwise we obtain $ap > p + r$ or $(a-1)p > r$, a contradiction. Hence, $(a-1)p < u < ap < n+u < (a+1)p$ holds, so $|I_{n,r}(p)| = 1$ which proves the first part.

Now, let us show that $I_{n,u}(q) = \{(b+1)q\}$. We have $q < n$ and as $bq < u$, we have $bq + q = (b+1)q < q + u < n + u$. Moreover, $n + u < (b+2)q$ yields that $(b+1)q = (b+2)q - q > n + u - q > u$. Thus, we obtain $bq < u < (b+1)q < n+u < (b+2)q$ and we get $|I_q(n,r)| = 1$. The result follows. □

Based on our observations, the following proposition emerges, providing a way to locate the intervals containing u such that $h_n^{(u)} - h_m^{(v)}$ is not an integer.

Proposition 15. Assume that $n, m \in \mathbb{Z}^{>0}$ and p is a prime number, satisfying $n > p > m$ together with $\frac{n}{2} < p$. Let u be any positive integer in $\left((t-1)p, (t+1)p - n\right]$ for some $t \in \mathbb{Z}^{>0}$. Then, $h_n^{(u)} - h_m^{(v)}$ is not an integer for any $v \in \mathbb{Z}^{>0}$.

Proof. Suppose that $t \in \mathbb{Z}^{>0}$ and $u \in \left((t-1)p, (t+1)p - n\right]$. We have $(t-1)p < u$. We also have $tp < p + u < n + u$ as $tp - p < u$. Moreover, the inequality $u \leq (t+1)p - n$ yields that $n + u - 1 < (t+1)p$ and $u < tp$. Hence, $|I_{n,u}(p)| = 1$ and since $p > m$, we obtain the result via Remark 13. □

Remark 16. Let $n \in \mathbb{Z}^{>0}$ and let p be a prime number satisfying $\frac{n}{2} < p < n$. Then, $\nu_p\left(h_n^{(u)}\right) \geq 0$ holds if and only if $u \in \left((t+1)p - n, tp\right]$ for some $t \in \mathbb{Z}^{>0}$.

Proof. Suppose n is a positive integer and p a prime satisfying $\frac{n}{2} < p < n$. Let us set

$$I_t := \big((t+1)p-n, tp\big] \text{ and } J_t := \big((t-1)p, (t+1)p-n\big]$$

for $t \in \mathbb{Z}^{>0}$. So, we have

$$I_t \cup J_t = \big((t-1)p, tp\big].$$

Let $u > 0$ be an integer. Then, $u \in I_t \cup J_t$ for some $t \in \mathbb{Z}$. Now, if $u \in J_t$, then $\nu_p\left(h_n^{(u)}\right) < 0$ as we know from the proof of Proposition 15. If $u \in I_t = \big((t+1)p-n, tp\big]$ then we get $(t+1)p-n = tp+p-n < u$ so $tp+p-1 < n+u-1$ holds.

As a result, we have

$$u \le tp \text{ and } tp+p = (t+1)p \le n+u-1.$$

Moreover, since $|I_{n,u}| = n < 2p$, we deduce that $I_{n,u}(p) = \{tp, (t+1)p\}$.

Now, we can write

$$h_n^{(u)} = \frac{\sum_{i=u}^{n+u-1} A_i}{n!}$$

where $A_i = \frac{\mathrm{Per}(n+u-1,u-1)}{i}$ for $i \in \{u,\ldots,n+u-1\}$.

Note that p divides A_i for all $i \in \{u,\ldots,n+u-1\}$ and we know that $\nu_p(n!) = 1$ by our assumption. Hence, we obtain

$$\nu_p\left(h_n^{(u)}\right) \ge 0$$

and we are done. □

Remark 17. Let n be a positive integer and let $p_{\langle n\rangle}$ be the greatest prime number which is smaller than n. Then we have

$$\nu_{p_{\langle n\rangle}}\left(h_n^{(u)}\right) \ge 0$$

if and only if $u \in \big((t+1)p_{\langle n\rangle} - n, tp_{\langle n\rangle}\big]$ for some $t \in \mathbb{Z}^{>0}$.

We are now prepared to prove Theorem E, needing only the following fact.

Fact 18. Let p_k denotes the k^{th} prime number. Then, for any real number $\varepsilon > 0$, we have

$$\sum_{p_k \le x} (p_{k+1} - p_k)^2 \ll_\varepsilon x^{\frac{23}{18}+\varepsilon}$$

by [29]. Moreover, if we assume the Riemann hypothesis, then by [30] we have

$$\sum_{p_k \leq x} (p_{k+1} - p_k)^2 \ll x \log^3 x.$$

Theorem E. [8, Theorem B] Let

$$Q(x) = |\{(n,m,u,v) \in [1,x]^4 \colon n,m,u,v \in \mathbb{Z}^{>0},\ \ h_n^{(u)} - h_m^{(v)} \notin \mathbb{Z}\}|.$$

Then, for any positive real number $\varepsilon > 0$ the equality

$$Q(x) = x^4 + O_\varepsilon\left(x^{\frac{59}{18}+\varepsilon}\right)$$

holds. Furthermore, under the Riemann hypothesis, we have

$$Q(x) = x^4 + O\left(x^3 \log^3 x\right).$$

Proof. Let n_c be a positive integer. Let us set

$$T(x) = |\{(n,m,u,v) \in [1,x]^4 \colon n_c \leq n, m \leq n, h_n^{(u)} - h_m^{(v)} \notin \mathbb{Z}\}|$$

and for any positive integer n, let us set

$$R_n(x) = |\{(m,u,v) \in [1,x]^3 \colon m \leq n, h_n^{(u)} - h_m^{(v)} \in \mathbb{Z}\}|.$$

Then, we have

$$T(x) + \sum_{n_c \leq n \leq x} R_n(x) = \frac{1}{2}x^4 + O(x^3) \tag{4.37}$$

as we only count the quadruples satisfying $m \leq n$ inside $[1,x]^4$. Moreover, since the cases $m \leq n$ and $n \leq m$ are symmetric, we can write

$$Q(x) = 2T(x) + O(x^3).$$

Observe that we have

$$R_n(x) = O\Bigg(\sum_{v\leq x}\sum_{u\leq x}\sum_{\substack{m\leq n \\ h_n^{(u)}-h_m^{(v)}\in\mathbb{Z}}} 1\Bigg) = O\Bigg(\sum_{v\leq x}\sum_{u\leq x}\sum_{\substack{m< p_{\langle n\rangle} \\ h_n^{(u)}-h_m^{(v)}\in\mathbb{Z}}} 1 + \sum_{v\leq x}\sum_{u\leq x}\sum_{\substack{p_{\langle n\rangle}\leq m\leq n \\ h_n^{(u)}-h_m^{(v)}\in\mathbb{Z}}} 1\Bigg). \tag{4.38}$$

Now, it follows from Remark 17 that

$$\nu_{p_{\langle n\rangle}}(h_n^{(u)}) \geq 0$$

if and only if $u \in ((a+1)p_{\langle n\rangle} - n, ap_{\langle n\rangle}]$ for some $a \in \mathbb{Z}^{>0}$. Notice that as $u \le x$ the number of such integers a is bounded by $\lfloor \frac{x}{p_{\langle n\rangle}} \rfloor$. In addition, for any n that is not prime, we set

$$\alpha(n) = n - p_{\langle n\rangle}$$

so that we have $\left|\{n \in \big((a+1)p_{\langle n\rangle} - n, ap_{\langle n\rangle}\big] \cap \mathbb{Z}\}\right| \le \alpha(n)$.

Let n be a positive integer and let (m,u,v) be an element in $R_n(x)$ with $m < p_{\langle n\rangle}$ so that the corresponding difference $h_n^{(u)} - h_m^{(v)}$ is an integer. Thus, we have $v_{p_{\langle n\rangle}}\left(h_n^{(u)} - h_m^{(v)}\right) \ge 0$ and since $m \le p_{\langle n\rangle}$, the hyperharmonic number $h_m^{(v)}$ has a non-negative $p_{\langle n\rangle}$-adic order. Hence, we have

$$v_{p_{\langle n\rangle}}\left(h_n^{(u)}\right) \ge 0.$$

Now, we are ready to deal with the last error term in (4.38). For the first summand, we can write

$$\begin{aligned}
\sum_{v\le x}\sum_{u\le x}\sum_{\substack{m<p_{\langle n\rangle}\\ h_n^{(u)}-h_m^{(v)}\in\mathbb{Z}}} 1 &\le \sum_{u\le x}\sum_{v\le x}\sum_{\substack{m<p_{\langle n\rangle}\\ v_{p_{\langle n\rangle}}\left(h_n^{(u)}\right)\ge 0}} 1\\
&< \sum_{u\le x}\sum_{\substack{v\le x\\ v_{p_{\langle n\rangle}}\left(h_n^{(u)}\right)\ge 0}} p_{\langle n\rangle}\\
&\le \sum_{v\le x}\frac{x}{p_{\langle n\rangle}}d(n)p_{\langle n\rangle}\\
&\le x^2\alpha(n).
\end{aligned}$$

For the second summand, we have

$$\sum_{v\le x}\sum_{u\le x}\sum_{\substack{p_{\langle n\rangle}\le m\le n\\ h_n^{(u)}-h_m^{(v)}\in\mathbb{Z}}} \le \sum_{v\le x}\sum_{u\le x}\big(\alpha(n)+1\big) \le x^2\alpha(n)+x^2.$$

Notice that $\alpha(n) = n - p_{\langle n\rangle} \ne 0$ as n is not a prime so that the both summands above yield $O\big(x^2\alpha(n)\big)$. As a result, (4.38) becomes

$$\sum_{n\le x} R_n(x) = O\left(x^2\sum_{n\le x} d(n)\right). \tag{4.39}$$

Now, let p_i denote the i^{th} prime and let $n \in (p_i, p_{i+1}]$. Then,

$$\alpha(n) \le p_{i+1} - p_i.$$

So,

$$\sum_{n\in(p_i,p_{i+1}]} \alpha(n) \leq (p_{i+1}-p_i)^2$$

and we have

$$\sum_{n\leq x} \alpha(n) \leq \sum_{p_i\leq x} \sum_{n\in(p_i,p_{i+1}]} \alpha(n) \leq \sum_{p_i\leq x} (p_{i+1}-p_,i)^2. \tag{4.40}$$

We know by Fact 18, if $\varepsilon > 0$ is any real number,

$$\sum_{p_i\leq x} (p_{i+1}-p_i)^2 \ll_{\varepsilon} x^{\frac{23}{18}+\varepsilon}$$

holds. As a result, (4.39) can be written as

$$\sum_{n\leq x} R_n(x) = O\left(x^2 \sum_{n\leq x} \alpha(n)\right) = O_{\varepsilon}\left(x^{\frac{59}{18}+\varepsilon}\right).$$

Then, we can rewrite (4.37) and have

$$T(x) = \frac{1}{2}x^4 + O_{\varepsilon}\left(x^{\frac{59}{18}+\varepsilon}\right).$$

Consequently, we are able to write

$$Q(x) = x^4 + O_{\varepsilon}\left(x^{\frac{59}{18}+\varepsilon}\right)$$

which proves the first part of the theorem.

Furthermore, by assuming the Riemann hypothesis, Fact 18 leads to

$$\sum_{p_k\leq x} (p_{k+1}-p_k)^2 \ll x\log^3 x.$$

Combining this with (4.39) and (4.40), we obtain

$$\sum_{n\leq x} E_n(x) = O\left(x^2 \sum_{n\leq x} \alpha(n)\right) = O\left(x^3\log^3 x\right).$$

Therefore, we proceed as in the previous part, we find that

$$Q(x) = x^4 + O\left(x^3\log^3 x\right).$$

This concludes the proof. □

4.3 Integer Differences and Concluding Remarks

The results in the previous sections indicate that it is not likely to have an integer hyperharmonic difference. First, we will show for any $n \in \mathbb{Z}$ that $h_n^{(r)} - h_n^{(s)} \in \mathbb{Z}$ may hold for infinitely many r and s.

Proposition 19. Let n be a positive integer. Then,

$$h_n^{(r)} - h_n^{(s)} \in \mathbb{Z}$$

whenever $r \equiv s \pmod{n!}$ for some $r, s \in \mathbb{Z}^{>0}$. Moreover, if $p \geq 5$ is a prime then

$$h_p^{(r)} - h_p^{(s)} \in \mathbb{Z}$$

whenever $r \equiv s \pmod{(p-1)!}$.

Proof. Let us write

$$h_n^{(r)} = \frac{f_n'(r)}{n!}$$

where $f_n(x) = \prod_{j=0}^{n-1}(x+j)$. We may also write

$$f_n'(x) = \sum_{j=0}^{n-1} a_j x^j$$

for some $a_0, a_1, \ldots, a_{n-1} \in \mathbb{Z}^{>0}$. In fact, we know by Section 4.1 that

$$a_0 = (n-1)!, \ \ a_{n-2} = \frac{n(n-1)^2}{2}, \ \ a_{n-1} = n.$$

So we may write

$$\begin{aligned} h_n^{(r)} - h_n^{(s)} &= \frac{1}{n!}\left(f_n'(r) - f_n'(s)\right) \\ &= \frac{1}{n!}\sum_{j=0}^{n-1} a_j (r^j - s^j) \\ &= \frac{1}{n!}\sum_{j=1}^{n-1} a_j (r-s)(r^{j-1} + \cdots + s^{j-1}) \end{aligned}$$

and hence, the difference is an integer whenever $r \equiv s \pmod{n!}$.

Moving forward, if $n \geq 5$ is a prime number, say p, we have

$$f_p'(r) = \sum_{j=0}^{p-1} a_j r^j.$$

However, we know by Proposition 10 that $p \mid a_j$ for any $j \neq 0$. So,

$$\begin{aligned}
h_p^{(r)} - h_p^{(s)} &= f_p'(r) - f_p'(s) \\
&= \sum_{j=0}^{p-1} \frac{a_j(r^j - s^j)}{p!} \\
&= \sum_{j=1}^{p-1} pb_j(r-s)(r^{j-1} + \cdots + s^{j-1}) \\
&= \frac{1}{(p-1)!} \sum_{j=1}^{p-1} b_j(r-s)(r^{j-1} + \cdots + s^{j-1})
\end{aligned}$$

for some $b_j \in \mathbb{Z}^{>0}$.

Hence, the result follows. □

Remark 20. The modulo in the equivalence $r \equiv s \pmod{(p-1)!}$ can be improved to $\frac{(p-1)!}{2}$ with a little effort. The interested reader may consult our work [8].

Remark 21. One can derive a version of Proposition 19 for the primes $p = 2, 3$. For $p = 2$, observe that $h_2^{(r)} = r + \frac{1}{2}$. Hence, $h_2^{(r)} - h_2^{(s)} \in \mathbb{Z}$ for any positive integers r, s. Moreover, we have

$$h_3^{(r)} - h_3^{(s)} = \sum_{k=1}^{2} \frac{b_k(r-s)(r^{k-1} + \cdots + s^{k-1})}{(2)!}$$

for some $b_k \in \mathbb{Z}^{>0}$ such that the difference is an integer if r and s are both odd or even.

Now, let us revisit the problem of Mezö:

Problem. What are the integers $n \neq m$ and $r \neq s$ such that the equation

$$h_n^{(r)} = h_m^{(s)}$$

is satisfied?

For $n = 2$, we have

$$h_2^{(r)} = r + \frac{1}{2}$$

with $r \in \mathbb{Z}^{>0}$. Hence, if we can find some $m, s \in \mathbb{Z}^{>0}$ so that

$$h_m^{(s)} \in \mathbb{Z} + \frac{1}{2},$$

or an half-integer, then we may find an $r \in \mathbb{Z}^{>0}$ satisfying $h_2^{(r)} = h_m^{(s)}$. We know by Theorem 4.2 that $h_2^{(r)} = h_m^{(s)}$ does not hold for $m = 3,4$. Moreover, by Proposition 10, we have $h_5^{(r)} - h_2^{(s)} \notin \mathbb{Z}$. Surprisingly, for the next candidate $m = 6$, and with the help of SageMath [31] we find values of s such that

$$h_6^{(s)} \in \mathbb{Z} + \frac{1}{2}.$$

Then, employing Proposition 19, we are able to find infinitely many positive integers s such that r where $h_6^{(s)}$ is a half-integer. Hence, we have

$$h_6^{(r)} = h_2^{(s)} \tag{4.41}$$

for infinitely many (r,s) tuples. We provide several examples as below.

- For $r = 20$ and $s = 47501$, we have $h_6^{(r)} = h_2^{(s)} = \frac{95003}{2}$.
- For $r = 55$ and $s = 5228670$, we have $h_6^{(r)} = h_2^{(s)} = \frac{10457341}{2}$.
- For $r = 75$ and $s = 23275838$, we have $h_6^{(r)} = h_2^{(s)} = \frac{46551677}{2}$.
- For $r = 100$ and $s = 94231673$, we have $h_6^{(r)} = h_2^{(s)} = \frac{188463347}{2}$.

In addition, for $n = 6$ and $m = 3$, there are infinitely many positive integers (r,s) so that $h_6^{(r)} - h_3^{(s)}$ is an integer. For instance, if $r = 15$ we get $h_6^{(15)} = 80507/6$ and if $s = 1$ we get $h_3^{(1)} = 11/6$. Hence, we have

$$h_6^{(15)} - h_3^{(1)} = 13416.$$

In particular, if $(r,s) \in \{(15, 2i+1) \colon i \in \mathbb{Z}^{\geq 0}\}$ then

$$h_6^{(r)} - h_3^{(s)}$$

is an integer by Proposition 19.

For $m = 4$, the set yields

$$(r,s) \in \{(5, 4 + k \cdot (4!)) \mid k \in \mathbb{Z}^{\geq 0}\}$$

infinitely many integer hyperharmonic differences.

Moreover, if $m = 5$, the set $\{(6, 1 + k \cdot (4!)) \mid k \in \mathbb{Z}^{\geq 0}\}$ of (r,s) implies $h_6^{(r)} - h_5^{(s)} \in \mathbb{Z}$.

5. RESULTS ON THE GENERALIZED HARMONIC NUMBERS

One of the generalization of the harmonic numbers is the generalized harmonic numbers. For any positive integers n and s, the n^{th} generalized harmonic number of order $s \in \mathbb{Z}^{>0}$ is defined as the sum

$$\sum_{k=1}^{n} \frac{1}{k^s}.$$

As the series satisfy

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6} < 2,$$

we note that they are non-integers except for the trivial case, $n = s = 1$.

Let p be a prime number and $\frac{a}{b}$ be a rational number. We write $p \mid \frac{a}{b}$ to mean that the numerator of $\frac{a}{b}$ is divisible by p in the lowest terms. We will also use $\frac{a}{b} \equiv 0 \pmod{p}$ when p divides a and not b.

In [9], the set $J(p)$ was introduced as the set of positive integers n where the numerator of H_n is divisible by p. We consider the generalizations of this set as follows. Let s be any positive integer. We will investigate the set

$$J(p,s) := \{n \in \mathbb{Z}^{>0} \colon H_n^{(s)} \equiv 0 \pmod{p}\}$$

together with

$$J(p^s,s) := \{n \in \mathbb{Z}^{>0} \colon H_n^{(s)} \equiv 0 \pmod{p^s}\}.$$

5.1 Characteristics of $J(p,s)$ and $J(p^s,s)$

It is known by [16] that $H_{p-1} \equiv 0 \pmod{p}^2$ for any prime number $p \geq 5$. Hence, $p-1$ is in the set $J(p)$. A similar result holds for generalized harmonic numbers, as shown below.

Proposition 22. [32, Theorem 1] For any positive integer s and any prime number p, we have

$$H_{p-1}^{(s)} \equiv 0 \pmod{p}$$

provided that $p-1 \nmid s$.

Proof. Let s be any positive integer and p be a prime number with $p-1 \nmid s$. Assume that $(\mathbb{Z}/p\mathbb{Z})^{\times}$ is generated by α. Observe that $p \nmid \alpha^s - 1$ as $p-1 \nmid s$. Now, let us set

$$S = \sum_{i=1}^{p-1} \frac{1}{i^s}$$

and notice that for any $i, j \in \{1, \ldots, p-1\}$ we have $\alpha i \not\equiv \alpha j$ whenever $i \neq j$. So, $\{\alpha i \colon 1 \leq i \leq p-1\}$ gives the same set of residues. Then,

$$S = \sum_{i=1}^{p-1} \frac{1}{i^s} \equiv \sum_{i=1}^{p-1} \frac{1}{(\alpha i)^s} \equiv \alpha^{-s} S \pmod{p}.$$

Thus, $\alpha^{-s}(\alpha^s - 1)S \equiv 0 \pmod{p}$. As $p \nmid \alpha^{-s}$ and $p \nmid \alpha^s - 1$, we have

$$S = \sum_{i=1}^{p-1} \frac{1}{i^s} = H_{p-1}^{(s)} \equiv 0 \pmod{p}.$$

□

We now notice that the set $J(p,s)$ holds an essential property that is also found in the set $J(p)$.

Lemma 23. Suppose that p is a prime number, s is a positive integer and n belongs to $J(p,s)$. Let $n = pm + r$ where $m \in \mathbb{Z}^{>0}$ and $0 \leq r \leq p-1$. Then, whenever $p-1 \nmid s$ we have $m \in J(p,s)$. In fact,

$$\nu_p(H_m^{(s)}) \geq s$$

is satisfied.

Proof. Let n be a positive integer in $J(p,s)$ and $p-1 \nmid s$. We have by Proposition 22 that if $p-1 \nmid s$ holds then $H_{p-1}^{(s)} \equiv 0 \pmod{p}$. This implies that for any $k \geq 1$,

$$\frac{1}{(pk-p+1)^s} + \frac{1}{(pk-p+2)^s} + \cdots + \frac{1}{(pk-1)^s} + \frac{1}{(pk)^s} \equiv \frac{1}{(pk)^s} \pmod{p}$$

holds. Hence, writing $n = pm + r$ for some $m \in \mathbb{Z}^{>0}$ and $0 \leq r \leq p-1$ yields

$$H_n^{(s)} \equiv \frac{1}{p^s} H_m^{(s)} + H_r^{(s)} \equiv 0 \pmod{p}$$

so that $\nu_p\left(\frac{1}{p^s} H_m^{(s)} + H_r^{(s)}\right) \geq 1$ holds. However, as $0 \leq r \leq p-1$, we have that $\nu_p\left(H_r^{(s)}\right) \geq 0$. As a result, $\nu_p\left(\frac{1}{p^s} H_m^{(s)}\right) \geq 0$ must hold. So, we obtain that $m \in J(p,s)$ with $\nu_p\left(H_m^{(s)}\right) \geq s$. □

Now, we see the relation between the sets $J(p,s)$ and $J(p^s,s)$.

Proposition 24. Assume that p is a prime number. Then,

$$|J(p^s,s)| < \infty \text{if and only if} |J(p,s)| < \infty,$$

whenever $p-1 \nmid s$.

Proof. Notice that if $J(p,s) = \{n \in \mathbb{Z}^{>0} \colon H_n^{(s)} \equiv 0 \pmod p\}$ is finite, then clearly $\{n \in \mathbb{Z}^{>0} \colon H_n^{(s)} \equiv 0 \pmod{p^s}\} = J(p^s,s)$ is finite. Now, let us assume that $J(p^s,s)$ is finite but $J(p,s)$ is infinite and obtain a contradiction. Let us say

$$J(p^s,s) = \{n_1 < \cdots < n_k\}$$

and let us choose an element $n \in J(p,s)$ big enough, namely, $n > p(n_k+1)$. Then, we can write $n = pm + r$ where $0 \leq r \leq p-1$. We have by Lemma 23 that

$$H_n^{(s)} \equiv \frac{1}{p^s} H_m^{(s)} + H_r^{(s)} \equiv 0 \pmod p$$

with

$$v_p\left(H_m^{(s)}\right) \geq s$$

so that $m \in J(p^s,s)$. However, we have

$$m = \frac{n-r}{p} \geq \frac{n-p+1}{p} > \frac{p(n_k+1)-p+1}{p} = \frac{pn_k+1}{p} > n_k$$

contradicting the fact that $J(p^s,s)$ is finite. □

5.2 Upper Bound for $J(p,s)$

Let $x > 0$ be a positive real number, p a prime number, and s a positive integer. We set

$$J_{p,s}(x) = |J(p,s) \cap [1,x]|$$

so that we count the number of elements in $J(p,s)$ up to x. Our aim in this section will be to obtain an upper for this function. In fact, we will prove the following theorem.

Theorem F. [13, Theorem A] For any prime number p, positive integer s and real number $x \geq 1$ we have

$$J_{p,s}(x) \leq 3x^{\frac{2}{3}+\frac{1}{25\log p}+\frac{\log s}{3\log p}+\frac{\log s}{3\log x}}.$$

Moreover, if $p > se^{\frac{3}{25}}$, then

$$J_{p,s}(x) = o(x).$$

Now, let us suppose throughout this section that $p-1 \nmid s$.

Next, we inductively define the sets

$$J_{p,s}{}^{(1)} := \{1 \le m \le p-1 \colon H_m^{(s)} \equiv 0 \pmod{p}\},$$

$$J_{p,s}{}^{(k+1)} := \{pn+r \colon n \in J_{p,s}^{(k)},\, 0 \le r \le p-1,\;\; pn+r \in J(p,s)\} \text{ for } k \in \mathbb{Z}^{>0}.$$

Lemma 25. For any $k \in \mathbb{Z}^{>0}$, we have $J_{p,s}^{(k)} = J(p,s) \cap [p^{k-1}, p^k - 1]$.

Proof. We prove by induction on k. If $k=1$ we have

$$J_{p,s}{}^{(1)} := \{1 \le m \le p-1 \colon H_m^{(s)} \equiv 0 \pmod{p}\} = J(p,s) \cap [1, p-1].$$

Assume that the claim holds for any $i \le k$. Now, we will establish that

$$J_{p,s}^{(k+1)} = J(p,s) \cap [p^k, p^{k+1} - 1].$$

Let $z \in J_{p,s}^{(k+1)}$, so $z = pn + r \in J(p,s)$ for some $n \in J_{p,s}^{(k)}$ with $0 \le r \le p-1$. Then, as $n \in J_{p,s}^{(k)}$, we have $n \in J(p,s) \cap [p^{k-1}, p^k - 1]$, so that pn lies in the interval $[p^k, p^{k+1} - p]$. So,

$$p^k \le z = pn + r \le p^{k+1} - p + p - 1 = p^{k+1} - 1$$

and $z \in [p^k, p^{k+1} - 1]$. Conversely, if $z \in J(p,s) \cap [p^k, p^{k+1} - 1]$, then $z = pn + r$ for some $n \in \mathbb{Z}^{>0}$ and $0 \le r \le p-1$. Moreover, via Lemma 23, we have $n \in J(p,s)$. Finally, as

$$p^k \le z = pn + r \le p^{k+1} - 1$$

holds, we get $n \in [p^{k-1}, p^k - 1]$. Hence, we obtain $n \in J_{p,s}^{(k)}$. The proof is now complete. □

Corollary 26. For any prime p and positive integer s, we have

$$J(p,s) = \bigcup_{k=1}^{\infty} J_{p,s}^{(k)}.$$

We see that the set $J(p,s)$ can be partitioned. At this point, we can establish an upper bound for the elements in a short interval of the set.

Lemma 27. Suppose that p is a prime number and s is a positive integer with $p-1 \nmid s$. Let $x, y \in \mathbb{R}^{>0}$ with $1 \leq y < p$ and let

$$J(p,s) \cap [x, x+y] = \{n_1 < \cdots < n_t\}.$$

Then, for any $d \geq 1$,

$$|\{j \colon n_{j+1} - n_j = d\}| \leq s(d-1)$$

holds.

Proof. For $j = 1, \ldots, t-1$, we set $d_j = n_{j+1} - n_j$. Observe that

$$\begin{aligned} H_{n_{j+1}}^{(s)} - H_{n_j}^{(s)} &= \frac{1}{(n_j+1)^s} + \cdots + \frac{1}{(n_{j+1})^s} = \frac{1}{(n_j+1)^s} + \cdots + \frac{1}{(n_j+d_j)^s} \\ &\equiv 0 \pmod{p}. \end{aligned} \tag{5.1}$$

Moreover, for any $d \geq 1$, set $f_d(x) = (x+1)(x+2)\ldots(x+d)$. By first taking the logarithm and then differentiating the equation, we get

$$\frac{f_d'(x)}{f_d(x)} = \frac{1}{x+1} + \frac{1}{x+2} + \cdots + \frac{1}{x+d}.$$

Now, let us set

$$g_d(x) = \frac{f_d'(x)}{f_d(x)}.$$

One can show that

$$g_d^{(k)}(x) = (-1)^k k! \sum_{<j=1}^{d} (x+j)^{-(k+1)}$$

for any $k \geq 0$. Then, for any integer $s \geq 1$, we can write

$$g_d^{s-1}(x) = (-1)^{s-1}(s-1)! \left(\frac{1}{(x+1)^s} + \frac{1}{(x+2)^s} + \cdots + \frac{1}{(x+d)^s} \right)$$

such that

$$\begin{aligned} g_{d_j}^{(s-1)}(n_j) &= (-1)^{s-1}(s-1)! \left(\frac{1}{(n_j+1)^s} + \cdots + \frac{1}{(n_j+d_j)^s} \right) \\ &= (-1)^{s-1}(s-1)! \big(H_{n_j+1}^{(s)} - H_{n_j}^{(s)} \big). \end{aligned}$$

We know by (5.1) that $H_{n_j+1}^{(s)} - H_{n_j}^{(s)} \equiv 0 \pmod{p}$. Thus, we have

$$g_{d_j}^{(s-1)}(n_j) \equiv 0 \pmod{p}.$$

Now, since the numerator of $g_{d_j}^{(s-1)}(x)$ is of degree $s(d_j - 1)$, it has at most $s(d_j - 1)$ many roots modulo p. Then, if we take $d = d_j$ we get the result. □

We now extend the argument presented in [12] to generalized harmonic numbers, which will be used to count the elements in short intervals within our set.

Lemma 28. Let $y \geq \frac{8}{3}$ be a real number, and s be a positive integer with $u_1, \ldots, u_t$ being integers such that $0 \leq u_i \leq s(i-1)$ for $i = 1, \ldots, t$. Additionally, assume that

$$\sum_{i=1}^{t} iu_i \leq y.$$

Then, it follows that

$$1 + \sum_{i=1}^{t} u_i \leq \left(\frac{9}{8}\right)^{\frac{1}{3}} y^{\frac{2}{3}} s^{\frac{1}{3}}.$$

Proof. First, if $\frac{y}{s} \geq \frac{8}{3}$, then setting y to be $\frac{y}{s}$ and a_k's to be $\frac{a_k}{s}$ yields the result by [12]. However, our objective will be bounding y by a prime number p to count the number of elements of $J(p,s)$ in short intervals. Therefore, we will prove the general case.

First, if $\frac{y}{s} \geq \frac{8}{3}$, then assigning y to be $\frac{y}{s}$ and u_i's to be $\frac{u_i}{s}$ gives the conclusion as shown in [12]. Nevertheless, our goal is to bound y using a prime number p to estimate $|J(p,s)|$ within short intervals. Hence, we will establish the general case.

To start, if $u_1 + \cdots + u_t \leq 1$, then we obtain

$$1 + \sum_{i=1}^{t} u_i \leq 2 \leq \left(\frac{9}{8}\right)^{\frac{1}{3}} y^{\frac{2}{3}} s^{\frac{1}{3}}$$

since $y \geq \frac{8}{3}$ and $s \geq 1$. Thus, we can assume that $u_1 + \cdots + u_t \geq 2$.
Next, we will rewrite each pair (u_i, u_{i+1}) of integers as follows. If $u_{i_0} < s(i_0 - 1)$ and $u_{i_0+1} > 0$, then substitute u_{i_0} with $u'_{i_0} = u_{i_0} + 1$ and u_{i_0+1} with $u'_{i_0+1} = u_{i_0+1} - 1$, while keeping the remaining numbers unchanged.

Consequently, for these new integers $u'_1, \ldots, u'_t$, we have

$$0 \leq u'_i \leq s(i-1)$$

with $i = 1, \ldots, t$.
Observe that for any pair (u_i, u_{i+1}), we have

$$u'_i + u'_{i+1} = u_i + 1 + u_{i+1} - 1 = u_i + u_{i+1},$$

and furthermore,

$$iu_i^{'} + (i+1)u_{i+1}^{'} = iu_i + i + (i+1)u_{i+1} - (i+1) = iu_i + (i+1)u_{i+1} - 1 < iu_i + iu_{i+1}.$$

Thus, we derive

$$\sum_{i=1}^{t} iu_i^{'} < \sum_{i=1}^{t} iu_i \leq y \tag{5.2}$$

and

$$\sum_{i=1}^{t} u_i^{'} = \sum_{i=1}^{t} u_i.$$

Continuing this procedure, we obtain $w \leq t$ integers $v_1, \ldots, v_w$ satisfying

$$v_k = s(k-1) \text{ for } 1 \leq k < w, \text{ and } 1 \leq v_w \leq s(w-1)$$

as well as

$$\sum_{i=1}^{w} v_i = \sum_{i=1}^{t} u_i.$$

Moreover, we can express using (5.2) that

$$\sum_{i=1}^{w} iv_i \leq y.$$

Observe that this process produces a monotonically increasing sequence of non-negative integers that increments by s at each step. Ultimately, we consider only the non-zero elements of the sequence and know their precise values, except for the last one, v_w.

Now, we have

$$y \geq \sum_{i=1}^{w} iv_i = wv_w + \sum_{i=1}^{w-1} iv_i = wv_w + \sum_{i=1}^{w-1} i(s(i-1)) = wv_w + s\sum_{i=1}^{w-1}(i^2 - i)$$

such that

$$y \geq s\left(\frac{w(w-1)(w-2)}{3}\right) + wv_w. \tag{5.3}$$

Here, (5.3) implies that

$$s\left(w(w-1)(w-2)\right) + 3wv_w \leq 3y,$$

and consequently,

$$(sw(w-1)(w-2) + 3wv_w)^{2/3} s^{1/3} \leq (3y)^{2/3} s^{1/3} \tag{5.4}$$

is valid. Moreover, we have

$$\sum_{i=1}^{t} u_i = \sum_{i=1}^{w} v_i = v_w + \sum_{i=1}^{w-1} v_i = v_w + \sum_{i=1}^{w-1} s(i-1)$$
$$= v_w + \left(s\sum_{i=1}^{w-1} i-1\right) = s\left(\frac{(w-1)(w-2)}{2}\right) + v_w.$$

Next, as $1 \leq v_w \leq w-1$ and $\sum_{i=1}^{t} u_i \geq 2$, we have

$$\left(s\frac{(w-1)(w-2)}{2} + v_w\right) \geq 2,$$

which implies that $w \geq 3$. Now, we will examine two cases: when $v_w = 1$ and when $v_w \geq 2$.

Case i). $v_w = 1$.

It can be noted that (5.4) may be expressed as

$$(sw(w-1)(w-2)+3w)^{2/3}s^{1/3} \leq (3y)^{2/3}s^{1/3},$$

and if

$$2\left(1+\sum_{i=1}^{t} u_i\right) = 2\left[1+\left(s\frac{(w-1)(w-2)}{2}+1\right)\right]$$
$$= s(w-1)(w-2)+4$$
$$\leq \left(sw(w-1)(w-2)+3w\right)^{2/3}s^{1/3}$$

is satisfied, we reach a conclusion. To clarify, if

$$(s(w-1)(w-2)+4)^3 \leq s\left(sw(w-1)(w-2)+3w\right)^2 \tag{5.5}$$

holds, we finalize our argument.

Now, when $s = 1$, we derive

$$((w-1)(w-2)+4)^3 \leq (w(w-1)(w-2)+3w)^2.$$

Nonetheless, the inequality holds for any $w \geq 3$, which can be confirmed through the proof of [12, Lemma 2.3]. Thus, we can assume $s \geq 2$ for the remainder of the proof. Then, let us define

$$z = (w-1)(w-2)$$

such that $z \geq 2$ since $w \geq 3$. Consequently, the inequality (5.5) can be expressed as

$$s(swz+3w)^2-(sz+4)^3 \geq 0.$$

Therefore, our objective is now to demonstrate that

$$sw^2(sz+3)^2-(sz+4)^3 \geq 0.$$

However, for the left-hand side, we have

$$\begin{aligned}
sw^2(sz+3)^2-(sz+4)^3 &= sw^2(s^2z^2+6sz+9)-(s^3z^3+12s^2z^2+48sz+64) \\
&= (s^3w^2z^2+6s^2w^2z+9sw^2)-(s^3z^3+12s^2z^2+48sz+64) \\
&= (s^3w^2z^2-s^3z^3)+(6s^2w^2z+9sw^2-12s^2z^2-48sz-64) \\
&= (3s^3z^2w-2s^3z^2)+\left(-12s^2z^2+(6s^2w^2z-48sz)+(9sw^2-64)\right) \\
&= \left(s^2z^2(3sw-2s-12)\right)+\left(6sz\underbrace{(sw^2-8)}_{\geq 0}+\underbrace{(9sw^2-64)}_{\geq 0}\right) \geq 0
\end{aligned}$$

since $s \geq 2, w \geq 3$ and $z \geq 2$.

Case ii). $v_w \geq 2$.

First, let us assume that $w \geq \left(\frac{3y}{s}\right)^{\frac{1}{3}}$ is satisfied. We know from (5.3) that

$$y \geq s\left(\frac{w(w-1)(w-2)}{3}\right)+wv_w = B.$$

Therefore,

$$\begin{aligned}
1+\sum_{i=1}^{t} u_i &= \frac{s}{2}(w-1)(w-2)+v_w+1 \\
&= \frac{3}{w}\frac{w}{3}\left(\frac{s}{2}(w-1)(w-2)\right)+v_w+1 \\
&= \frac{3}{2w}\left(\frac{sw(w-1)(w-2)}{3}+\frac{2}{3}wv_w+\frac{2}{3}w\right) \\
&= \frac{3}{2w}(B-\frac{w}{3}v_w+\frac{2}{3}w) \\
&\leq \frac{3}{2w}(B-\frac{2}{3}w+\frac{2}{3}w) \quad \text{as } v_w \geq 2 \\
&\leq \frac{3y}{2w}.
\end{aligned}$$

Consequently, we obtain

$$1+\sum_{i=1}^{t} u_i \leq \frac{3y}{2w} \leq \frac{3}{2}\frac{y}{\left(\frac{3y}{s}\right)^{\frac{1}{3}}} = \left(\frac{9}{8}\right)^{\frac{1}{3}} y^{\frac{2}{3}} s^{\frac{1}{3}}.$$

Finally, let us check the case $w < \left(\frac{3y}{s}\right)^{\frac{1}{3}}$. Recall that $w \geq 3$ and $v_w \leq s(w-1)$ is satisfied.

Thus, we conclude that

$$\begin{aligned}
1+\sum_{i=1}^{t} u_i &= \frac{s}{2}(w-1)(w-2)+v_w+1 \\
&\leq \frac{s}{2}(w-1)(w-2)+s(w-1)+1 \\
&= \frac{1}{2}(sw^2)-\frac{sw}{2}+1 \\
&\leq \frac{1}{2}sw^2 \\
&< \frac{1}{2}s\left(\frac{3y}{s}\right)^{\frac{2}{3}} \\
&= \left(\frac{9}{8}\right)^{\frac{1}{3}} y^{\frac{2}{3}} s^{\frac{1}{3}}
\end{aligned}$$

and we are done. □

Lemma 29. Let x, y be real numbers and let p be a prime number where $\frac{8}{3} \leq y < p$. Then for any integer $s > 0$,

$$|J(p,s)\cap[x,x+y]| \leq \left(\frac{9}{8}\right)^{\frac{1}{3}} y^{\frac{2}{3}} s^{\frac{1}{3}}.$$

Proof. Let us write $J(p,s)\cap[x,x+y] = \{n_1 < \cdots < n_t\}$ and set

$$u_k = |\{1 \leq j \leq t-1 \mid n_{j+1}-n_j = k\}|.$$

Then, by Lemma 27 we have $0 \leq u_k \leq s(k-1)$. Furthermore,

$$\sum_k k u_k = \sum_{j=1}^{t-1}(n_{j+1}-n_j) \leq y$$

holds so that the assumptions of Lemma 28 hold.

Then,

$$|J(p,s)\cap[x,x+y]| = t = 1+\sum_k u_k \leq y$$

and the result follows by Lemma 28. □

We are now ready to prove the main theorem of the section, Theorem F.

Proof of Theorem F. First, for $s = 1$, we find $J(p,1) = J(p)$ and thus the result is obtained from [12]. Additionally, we may exclude the cases $p = 2,3,5$ from Table 5.1. Hence, we may take $p \geq 7$ with $s > 1$. Now, let us set

$$C = \left(\frac{9}{8}\right)^{\frac{1}{3}} (p-1)^{\frac{2}{3}} s^{\frac{1}{3}}.$$

Now, by Lemma 29, we know that

$$\left|J_{p,s}^{(1)}\right| = |J(p,s) \cap [1, p-1]| \leq C.$$

Moreover, via Lemma 25, we have

$$\left|J_{p,s}^{(k+1)}\right| = \sum_{n \in J_{p,s}^{(k)}} |J(p,s) \cap [pn, pn+p-1]| \leq \left|J_{p,s}^{(k)}\right| C.$$

As a result, we obtain

$$\left|J_{p,s}^{(k)}\right| \leq C^k$$

for any $k \in \mathbb{Z}^{>0}$.

Now, we begin our analysis of $J_{p,s}(x)$. Let m be the positive integer satisfying

$$p^{m-1} \leq x < p^m.$$

Then, we can write

$$J_{p,s}(x) = J_{p,s}(p^{m-1} - 1) + |J(p,s) \cap [p^{m-1}, x]| \tag{5.6}$$

and consider the summands one at a time. For the first summand, we can write via Lemma 25 that

$$\begin{aligned} J_{p,s}(p^{m-1} - 1) &= \sum_{k=1}^{m-1} \left|J(p,s) \cap [p^{k-1}, p^k - 1]\right| \\ &= \sum_{k=1}^{m-1} \left|J_{p,s}^{(k)}\right| \\ &\leq \sum_{k=1}^{m-1} C^k \\ &< \frac{C}{C-1} C^{m-1}. \end{aligned} \tag{5.7}$$

For the second summand, we write

$$\begin{aligned}|J(p,s)\cap[p^{m-1},x]| &\leq \sum_{\substack{n\in J_{p,s}^{(m-1)}\\ pn\leq x}} |J(p,s)\cap[pn,pn+p-1]| \\ &\leq C\sum_{\substack{n\in J_{p,s}^{(m-1)}\\ pn\leq x}} 1 = C\left|J(p,s)\cap\left[p^{m-2},\frac{x}{p}\right]\right|.\end{aligned}$$

Moreover, notice that we have

$$C\left|J(p,s)\cap\left[p^{m-2},\frac{x}{p}\right]\right| \leq C^2\left|J(p,s)\cap\left[p^{m-3},\frac{x}{p^2}\right]\right|$$

and by proceeding in this way, we derive that

$$|J(p,s)\cap[p^{m-1},x]| \leq C\left|J(p,s)\cap\left[p^{m-2},\frac{x}{p}\right]\right| \leq C^{m-1}\left|J(p,s)\cap\left[1,\frac{x}{p^{m-1}}\right]\right|. \quad (5.8)$$

Now, for $\left|J(p,s)\cap\left[p^1,\frac{x}{p^{m-1}}\right]\right|$, if we have $x<3p^{m-1}$, then we obtain

$$\left|J(p,s)\cap\left[1,\frac{x}{p^{m-1}}\right]\right| \leq 1 \leq \left(\frac{9}{8}\right)^{\frac{1}{3}}\left(\frac{x}{p^{m-1}}\right)^{\frac{2}{3}} s^{\frac{1}{3}}.$$

If we have $x\geq 3p^{m-1}$, then via Lemma 29, we get

$$\left|J(p,s)\cap\left[1,\frac{x}{p^{m-1}}\right]\right| \leq \left(\frac{9}{8}\right)^{\frac{1}{3}}\left(\frac{x}{p^{m-1}}\right)^{\frac{2}{3}} s^{\frac{1}{3}}.$$

Hence, (5.8) can be rewritten as

$$|J(p,s)\cap[p^{m-1},x]| \leq C^{m-1}\left|J(p,s)\cap\left[1,\frac{x}{p^{m-1}}\right]\right| \leq C^{m-1}\left(\frac{9}{8}\right)^{\frac{1}{3}}\left(\frac{x}{p^{m-1}}\right)^{\frac{2}{3}} s^{\frac{1}{3}}. \quad (5.9)$$

Now, via (5.7) and (5.9), we can write for (5.6) that

$$\begin{aligned}J_{p,s}(x) &= J_{p,s}(p^{m-1}-1)+|J(p,s)\cap[p^{m-1},x]| \\ &\leq \frac{C}{C-1}C^{m-1}+C^{m-1}\left(\frac{9}{8}\right)^{\frac{1}{3}}\left(\frac{x}{p^{m-1}}\right)^{\frac{2}{3}} s^{\frac{1}{3}} \\ &\leq 3C^{m-1}s^{\frac{1}{3}}\left(\frac{x}{p^{m-1}}\right)^{\frac{2}{3}} \\ &= 3\left(\left(\frac{9}{8}\right)^{\frac{1}{3}}(p-1)^{\frac{2}{3}}s^{\frac{1}{3}}\right)^{m-1} s^{\frac{1}{3}}\left(\frac{x}{p^{m-1}}\right)^{\frac{2}{3}} \\ &= 3\left(\frac{9}{8}\right)^{\frac{m-1}{3}} s^{\frac{m-1}{3}}s^{\frac{1}{3}}x^{\frac{2}{3}} \\ &= 3\left(\frac{9}{8}\right)^{\frac{m-1}{3}} s^{\frac{m}{3}}x^{\frac{2}{3}} \leq 3\left(e^{\frac{1}{25}}\right)^{m-1} s^{\frac{m}{3}}x^{\frac{2}{3}}.\end{aligned}$$

Observe that the fact $p^{m-1} \leq x < p^m$ implies that

$$m - 1 \leq \frac{\log x}{\log p}. \tag{5.10}$$

Thus,

$$J_{p,s}(x) \leq 3\left(e^{\frac{1}{25}}\right)^{\frac{\log x}{\log p}} s^{\frac{m}{3}} x^{\frac{2}{3}} = 3\left(x^{\frac{1}{25\log p}}\right) s^{\frac{m}{3}} x^{\frac{2}{3}} = 3x^{\frac{2}{3}+\frac{1}{25\log p}} s^{\frac{m}{3}}. \tag{5.11}$$

Now, we examine on $s^{\frac{m}{3}}$ to finish the proof. We have

$$s^{\frac{m-1}{3}} \leq s^{\frac{\log x}{3\log p}}$$

so that

$$s^{\log x} = e^{\log\left(s^{\log x}\right)} = e^{\log x \log s} = x^{\log s}.$$

Therefore, we get

$$s^{\frac{\log x}{3\log p}} = x^{\frac{\log s}{3\log p}}$$

which implies that

$$s^{\frac{m-1}{3}} \leq x^{\frac{\log s}{3\log p}}. \tag{5.12}$$

On the contrary, for $s^{\frac{1}{3}}$ we obtain

$$s^{\frac{1}{3}} = x^{\frac{\log s}{3\log x}}. \tag{5.13}$$

Then, by combining (5.12) and (5.13), we have

$$s^{\frac{m}{3}} \leq x^{\frac{\log s}{3\log p}+\frac{\log s}{3\log x}}.$$

Now, substituting our result into (5.11), we derive the upper bound for the set as

$$J_{p,s}(x) \leq 3x^{\frac{2}{3}+\frac{1}{25\log p}} x^{\frac{\log s}{3\log p}+\frac{\log s}{3\log x}} = 3x^{\frac{2}{3}+\frac{1}{25\log p}+\frac{\log s}{3\log p}+\frac{\log s}{3\log x}}.$$

Finally, to obtain $J_{p,s}(x) = o(x)$, one must guarantee that

$$\frac{1}{25\log p} + \frac{\log s}{3\log p} + \frac{\log s}{3\log x} < \frac{1}{3}$$

holds.

Finally, for the last part of the proof, the inequality

$$\frac{1}{25\log p} + \frac{\log s}{3\log p} + \frac{\log s}{3\log x} < \frac{1}{3}$$

must hold to have $J_{p,s}(x) = o(x)$. In other words,

$$9\log x + 75\log x \log s + 25\log p \log s < 75\log x \log p$$

must hold. However, if x sufficiently large, then $25\log p\log s$ is relatively insignificant. Thus, if the inequality

$$75\log x \log p > 9\log x + 75\log x \log s$$

is satisfied, then we obtain the result. So, by considering $\log p - \log s > \frac{9}{75}$, we obtain

$$p > se^{\frac{3}{25}}$$

and the proof is now complete. □

5.3 Computational Results on $J(p,s)$

In this section, we will continue to explore $J(p,s)$ and derive finiteness results with computational techniques. We computed $|J(p,s)|$ for some values of p and s using SageMath [31] with the code provided in the appendices.

Table 5.1 : The values of $|J(p,s)|$ for various values of p and s.

p \ s	1	2	3	4	5	6	7	8	9	10
2	0	0	0	0	0	0	0	0	0	0
3	3	0	1	0	1	0	1	0	1	0
5	3	2	1	0	1	2	1	0	1	2
7	13	3	1	2	1	0	1	2	1	2
11	638	2	3	2	3	2	1	2	1	0
13	3	2	1	2	1	4	3	2	1	2
17	3	2	3	4	1	2	3	4	4	2
19	19	2	1	2	1	2	3	4	5	2

As shown in Table 5.1, for certain values of s, the set $J(p,s)$ is non-trivial. In other words, $J(p,s)$ contains additional elements beyond $p-1$, assuming $p-1 \nmid s$. For example, let us demonstrate for $s=2$ that the elements $\frac{p-1}{2}$ and $p-1$ are in the set whenever $p>3$. Assume that $p>3$ is a prime number. We know by Proposition 22 that

$$H_{p-1}^{(s)} \equiv 0 \pmod{p}.$$

Moreover, notice that

$$\frac{1}{i^2} \equiv \frac{1}{(p-i)^2} \pmod{p} \tag{5.14}$$

is satisfied for any i with $1 \le i \le \frac{p-1}{2}$. As a result,

$$H_{p-1}^{(2)} = \sum_{i=1}^{p-1} \frac{1}{i^2} \equiv 2 \sum_{i=1}^{\frac{p-1}{2}} \frac{1}{i^2} = 2\left(H_{(p-1)/2}^{(2)}\right) \equiv 0 \pmod{p}$$

so we have $H_{(p-1)/2}^{(2)} \equiv 0 \pmod{p}$.

In particular, we can extend this fact to all even s and primes $p > 3$ satisfying $p-1 \nmid s$ since (5.14) is valid for any such values of p and s. Now, let us establish our finiteness result.

Theorem G. [13, Theorem B] Suppose that p is a prime and $s \ge 2$ is an integer satisfying $p-1 \nmid s$. If

$$\nu_p\left(H_r^{(s)}\right) \le s-1$$

is satisfied for any $r \in \{1,2,\ldots,p-1\}$, then $J(p,s) \subset \{1,2,\ldots,p-1\}$. In fact, $J(p,s)$ is finite.

Proof. For the first part, let us assume that n and $s \ge 2$ be positive integers and p be a prime number satisfying $p-1 \nmid s$. Let us write

$$p^{k-1} \le n < p^k$$

for some positive integer $k \ge 2$ so that we have

$$H_n^{(s)} = 1 + \frac{1}{2^s} + \cdots + \frac{1}{(p^{k-1})^s} + \cdots + \frac{1}{(2p^{k-1})^s} + \cdots + \frac{1}{n^s}.$$

One can notice that the terms with the least p-adic valuations are those whose denominators involve multiples of p^{k-1}. So, the p-adic valuation of $H_n^{(s)}$ is determined by them. So, suppose that there are $r \le p-1$ many such terms, namely, we have

$$p^{k-1} \le rp^{k-1} \le n < p^k.$$

This yields that

$$\begin{aligned}
&\nu_p\left(\frac{1}{(p^{k-1})^s} + \frac{1}{(2p^{k-1})^s} + \cdots + \frac{1}{(rp^{k-1})^s}\right) \\
&= \nu_p\left(\left(1 + \frac{1}{2^s} + \cdots + \frac{1}{r^s}\right)\frac{1}{(p^{k-1})^s}\right) \\
&= \nu_p\left(H_r^{(s)} \frac{1}{(p^{k-1})^s}\right) = -s(k-1) + \nu_p\left(H_r^{(s)}\right).
\end{aligned}$$

Moreover, if the assumption

$$\nu_p\left(H_r^{(s)}\right) \leq s-1$$

holds for any $r \leq p-1$, then we get

$$\begin{aligned}\nu_p&\left(\frac{1}{(p^{k-1})^s}+\frac{1}{(2p^{k-1})^s}+\cdots+\frac{1}{(rp^{k-1})^s}\right)\\&=-s(k-1)+\nu_p\left(H_r^{(s)}\right) \leq -s(k-1)+s-1=-s(k-2)-1.\end{aligned}$$

Furthermore, notice that

$$H_n^{(s)}=\underbrace{\left(1+\frac{1}{2^s}+\cdots+\frac{1}{(p^{k-2})^s}+\cdots+\frac{1}{(p^{k-1}-1)^s}\right)}_{\text{of } p\text{-adic val:} \geq -s(k-2)}+\frac{1}{(p^{k-1})^s}+\cdots+\cdots+\frac{1}{n^s}.$$

Hence, if $n \geq p$ or if $k \geq 2$, then $H_n^{(s)}$ has a negative p-adic order so that $n \notin J(p,s)$. Therefore, $n<p$ must hold and we get $J(p,s) \subset \{1,2,\ldots,p-1\}$ and the set is finite.

□

Now, let us look for cases where the condition on Theorem G fails. Let $p=5$ and $s=2$. One may verify that

- For $n=1$, we have $H_n^{(2)}=1$ and $\nu_5\left(H_n^{(2)}\right)=0$.
- For $n=2$, we have $H_n^{(2)}=\frac{5}{4}$ and $\nu_5\left(H_n^{(2)}\right)=1$.
- For $n=3$, we have $H_n^{(2)}=\frac{49}{36}$ and $\nu_5\left(H_n^{(2)}\right)=0$.
- For $n=4$, we have $H_n^{(2)}=\frac{205}{144}$ and $\nu_5\left(H_n^{(2)}\right)=1$.

Hence, the condition in the theorem, namely

$$\nu_5\left(H_n^{(s)}\right) \leq s-1=1$$

is satisfied for any $n \in \{1,2,3,4\}$. As a result, we get that

$$J(5,2) \subseteq \{1,2,3,4\}.$$

In particular, one may check the SageMath [31] code we provide in the Appendix that $J(5,2)=\{2,4\}$.

Next, let us set $p=5$ and $s=2$.

- For $n=1$, we have $H_n^{(2)} = 1$ and $\nu_7\left(H_n^{(2)}\right) = 0$.
- For $n=2$, we have $H_n^{(2)} = \frac{5}{4}$ and $\nu_7\left(H_n^{(2)}\right) = 0$.
- For $n=3$, we have $H_n^{(2)} = \frac{49}{36}$ and $\nu_7\left(H_n^{(2)}\right) = \mathbf{2}$.
- For $n=4$, we have $H_n^{(2)} = \frac{205}{144}$ and $\nu_7\left(H_n^{(2)}\right) = 0$.
- For $n=5$, we have $H_n^{(2)} = \frac{5269}{3600}$ and $\nu_7\left(H_n^{(2)}\right) = 0$.
- For $n=6$, we have $H_n^{(2)} = \frac{5369}{3600}$ and $\nu_7\left(H_n^{(2)}\right) = 1$.

Here, the condition in the theorem

$$\nu_7\left(H_r^{(2)}\right) \leq 2-1 = 1$$

fails as we have $\nu_7\left(H_r^{(2)}\right) = 2$ for $r=3$. As a result, there must be an element in $J(7,2)$ that is greater than 7. In fact, we find that

$$\nu_7\left(H_{26}^{(2)}\right) = 1,$$

so, $26 \in J(7,2)$. However, again with the help of SageMath [31], we find that there is not any other elements in the set and we get

$$J(7,2) = \{3,6,26\}.$$

Now, if $p=37$ and $s=3$, we find via SageMath [31] that

$$\nu_{37}\left(H_{36}^{(3)}\right) = 3 \nleq 2 = s-1.$$

Therefore, we expect to have some element in $J(7,3)$ that is greater than 37 and it turns out that $J(37,3) = \{4,13,23,32,36,1340,1360\}$.

Recall that we showed $\frac{p-1}{2}, p-1 \in J(p,s)$ for any all even s and primes $p>3$ satisfying $p-1 \nmid s$. Moreover, we know by Proposition 22 that $p-1 \in J(p,s)$ for any p,s with $p-1 \nmid s$. Thus, it seems reasonable to consider $r = p-1$ as a potential candidate where the inequality $\nu_p\left(H_r^{(s)}\right) \leq s-1$ could fail. Consequently, for this purpose, it is essential to determine the exact p-adic order of $H_{p-1}^{(s)}$ or to look for the corresponding congruences modulo p^k for some $k \in \mathbb{Z}^{>0}$.

For example, when $p \geq s+3$, we have

$$H_{p-1}^{(s)} \equiv \begin{cases} \frac{s}{s+1} pB_{p-1-s} & \pmod{p^2} \text{ for any even } s, \\ -\frac{s(s+1)}{2(s+2)} p^2 B_{p-2-s} & \pmod{p^3} \text{ for any odd } s, \end{cases} \tag{5.15}$$

by [33, 34], where B_j represents the j^{th} Bernoulli number, defined by the relation

$$\frac{x}{e^x-1} = \sum_{j=0}^{\infty} B_j \frac{x^j}{j!} \qquad \text{for } j = 0, 1, 2, \ldots$$

We have $B_0 = 1, B_2 = \frac{1}{6}, B_4 = \frac{1}{30}, \ldots$ and $B_1 = -\frac{1}{2}, B_3 = B_5 = B_7 = \cdots = 0$ so $B_{2j+1} = 0$ for any $j \geq 1$. Now, if we revisit the case $p = 37$ and $s = 3$, we obtain via SageMath [31] that

$$\nu_{37}(B_{32}) = \nu_p(B_{p-2-s}) = 1$$

and (5.15) implies

$$H_{36}^{(3)} \equiv 0 \pmod{p^3}.$$

The prime 37 is, in fact, an irregular prime and it is the smallest. A prime number $p > 3$ is said to be irregular if at least one of $B_2, B_4, B_6, \ldots, B_{p-3}$ is divisible by p. Furthermore, it was shown [35] that for irregular primes $p > 3$, the inequality

$$\nu_p(H_{p-1}^{(s)}) > 1$$

holds for more than half of all integers $s > 0$. Thus, focusing on these primes may be a useful strategy for further understanding our set.

In addition, we provide a brief discussion on the congruence relations involving Euler numbers. The Euler numbers are defined iteratively as

$$E_0 = 1 \text{ and } \sum_{\substack{0 \leq j \leq n \\ j \text{ even}}} \binom{n}{j} E_{n-j} = 0$$

for any positive integer n. Now, let $p > 3$ be a prime number and $s = 2$. Then, we obtain via [36] that

$$H_{\lfloor \frac{p}{4} \rfloor}^{(2)} = \sum_{k=1}^{\lfloor \frac{p}{4} \rfloor} \frac{1}{k^2} \equiv (-1)^{\left(\frac{p-1}{2}\right)} 4E_{p-3} \pmod{p}. \tag{5.16}$$

That is, if the Euler number E_{p-3} is divisible by p then the right-hand side of (5.16) is 0 and we get

$$\left\lfloor \frac{p}{4} \right\rfloor \in J(p, 2).$$

The important fact to notice is that if E_{p-3} is divisible by p^2, in other words, if

$$\nu_p\left(E_{p-3}\right) \geq 2$$

holds, then the condition in Theorem 5.3 fails. In particular, we obtian that

$$p\left\lfloor \frac{p}{4} \right\rfloor + r > p$$

is an element of $J(p,s)$ for some $r \in \{0,1,\ldots,p-1\}$. Moreover, we call such a prime p as E-irregular [37]. To sum up, the irregular prime numbers may be crucial for the finiteness of $J(p,s)$.

6. CONCLUSION

In this thesis, we cover various sums of unit fractions so that we both introduce new generalizations and some problems on the classical sums of unit fractions.

A natural example for a sum of unit fractions could be the harmonic numbers and we first delve into number fields to introduce our generalization of the harmonic numbers. We call our generalization the Dedekind harmonic numbers. These numbers arise naturally as the relation between the harmonic numbers and the Riemann zeta function, and the relation between Dedekind harmonic numbers and the Dedekind zeta function is similar. The exploration of these numbers include the problems that were studied on the sums of unit fractions over the years. For instance, a complete answer is given for the integerness of these numbers. We also provide the criterias on the integerness for various number fields, examine the integerness of their differences, supplied with an asymptotic result.

We continue with another generalization of the harmonic numbers called the hyperharmonic numbers. They also come with intriguing properties in which one can find plenty of problems to work on. The integerness of them was one of the problems that have been studied. We aim to answer a question that states if there may be some distinct hyperharmonic numbers so that their difference is an integer. An answer will be given via a geometrich approach, supplied with an asymptotic result. Moreover, we support our work with the cases that has a positive answer to the integerness question.

Furthermore, we also consider another example of the sums of unit fractions, called the generalized harmonic numbers. These numbers are known to be non-integer except for the trivial case, but they recover divisibility properties that are similar to the classical harmonic numbers satisfy. Our study in the related chapter includes a generalization of a set that was defined for the harmonic numbers by their divisibility features. We establish the structure of the set and then give an upper bound for the number of elements in that set. This upper bound comes with an asymptotic result and we finalize our work with a finiteness result.

In conclusion, this thesis contributes a comprehensive study on several sums of unit fractions, their properties, providing new methods and results that lay the foundation for further exploration of these subjects in number theory. We think that the publication of our results in three peer-reviewed papers [1], [8], [13], underscores their value in encouraging and influencing future research.

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APPENDICES

APPENDIX A : SageMath [31] code to determine the elements in $J(p,s)$.

Appendix A: The elements of $J(p,s)$ were found using the SageMath [31] code provided below.

```
def generate_sequence(prime, level):
    sequence_list = []
    if level == 1:
        sequence_list = [prime^(level-1)..prime^(level)-1]
    else:
        for element in generate_sequence(prime, level-1):
            for digit in [0..prime-1]:
                sequence_list.append(prime*element + digit)
    return sequence_list

# Set prime_base and exponent
prime_base = 7
exp = 2

element_count = 1
iteration = 1
result_sequence = []

while True:
    if element_count == 0:
        print("Computation complete")
        break
    print("Current iteration level =", iteration, ":")
    valid_elements = [
        val for val in generate_sequence(prime_base, iteration)
        if valuation(harmonic_number(val, exp), prime_base) > 0
    ]
    print(valid_elements)
    result_sequence = result_sequence + valid_elements
    element_count = len(valid_elements)
    iteration = iteration + 1

print("J(prime_base, exp) =", "J(", prime_base, ",", exp, ")")
print(result_sequence)
print("Total number of elements: ", len(result_sequence))
```

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