BETTI TABLES OF MULTIPARAMETER PERSISTENCE MODULES

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ABSTRACT

BETTI TABLES OF MULTIPARAMETER PERSISTENCE MODULES

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People still try to search for new effective methods to analyze the given data. Topological data analysis (TDA) is just one way for doing this, and multiparameter persistent homology is currently one of the favorite areas of TDA. Moreover, Betti tables is an important invariant in multiparameter persistent homology. The aim of this thesis is to give a survey about Betti tables also known as multigraded Betti numbers. In this thesis, we elaborate the article "On the Support of the Betti Tables of Multiparameter Persistent Homology Modules" [13], which gives a bound to support of the Betti tables of some special multiparameter persistence modules. After giving the necessary background about the topic, we give an equivalent definition for Betti tables of a persistence module by using Koszul complex, and we construct Koszul complexes of persistence modules iteratively. Then, a proof of bounding the support of the Betti tables of the given persistence module by using the critical cells of a discrete gradient vector field is given. Lastly, for the case $n = 2$ this result is strengthened.

Keywords: persistent homology, koszul complex, discrete morse theory, multiparameter persistence, Betti tables

ÖZ

ÇOK PARAMETRELİ ISRARLI MODÜLLERİN BETTI TABLOLARI

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İnsanlar hala verilen verileri analiz etmek için yeni etkili yöntemler aramaya çalışıyor. Topolojik veri analizi (TDA), bunu yapmanın bir yolu olup, çok parametreli ısrarlı homoloji, TDA'nın şu anda en gözde alanlarından biridir. Dahası, Betti tabloları, çok parametreli ısrarlı homolojide önemli bir değişmezdir. Bu tezin amacı, çok dereceli Betti sayıları olarak da bilinen Betti tabloları hakkında bir inceleme sunmaktır. Bu tezde, bazı özel çok parametreli ısrarlı modüllerin Betti tablolarının desteğine bir sınır getiren "On the Support of the Betti Tables of Multiparameter Persistent Homology Modules" [13] makalesini detaylandırıyoruz. Konu hakkında gerekli arka planı verdikten sonra, Koszul kompleksi kullanarak ısrarlı bir modülün Betti tabloları için eşdeğer bir tanım veriyoruz ve ısrarlı modüllerin Koszul komplekslerini tekrarlamalı bir şekilde inşa ediyoruz. Daha sonra, bir ayrık gradyan vektör alanının kritik hücrelerini kullanarak verilen ısrarlı modülün Betti tablolarının desteğini sınırlandırma kanıtını sunuyoruz. Son olarak, n = 2 durumu için bu sonucu güçlendiriliyoruz.

Anahtar Kelimeler: ısrarlı homoloji, koszul kompleksi, ayrık morse teorisi, çok de-

ğişkenli ısrarlılık, Betti tabloları

To my family

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TABLE OF CONTENTS

LIST OF FIGURES

FIGURES

CHAPTER 1

INTRODUCTION

Topological Data Analysis (TDA) is a field which was emerged in the early 2000s, and TDA brought a new perspective to the field of data analysis. Besides the ordinary data analysis methods, one also has tools from algebraic topology in topological data analysis in order to analyze the shape and structure of a data set. For example, homology is a common invariant in algebraic topology and is used to identify the different dimensional holes of topological shapes. However, instead of looking at holes for a single topological space, we can track their lifespan through topological spaces depending on a variable. This method is known as persistent homology, one of the most frequently used tools of topological data analysis. By using cell complexes, filtrations, homology modules etc., persistent homology is helpful to catch the topological features of data sets. In persistent homology, filtrations and persistence modules are two of the main objects. Filtrations are actually nested sequences of topological spaces. There are many ways of obtaining filtered spaces from point cloud data such as filtrations obtained by Vietoris-Rips complexes or Čech complexes. Beyond this, if we apply the homology functor to filtrations, we obtain persistence modules which are algebraic tools that are used to analyze topological features of filtrations.

If only one parameter is used in the filtration, then we call it a 1-parameter persistence. In 1-parameter persistent homology, we have a barcode structure of persistence modules. Barcodes give the death and birth information of the homology classes, and it is a complete invariant. Unfortunately, we don't have a structure like barcode in multiparameter persistent homology, but sometimes 1-parameter persistent homology modules may not be sufficient to analyze the given data. For example, if we have a data set with some outliers, which are points that deviate from the other points, then excluding these few outliers may cause a sharp change in the outcome. To include these outliers, sometimes we have to consider more than one parameter to filter the data. Thus, we need to use multiparameter persistent homology to analyze the data. Even if there is no invariant in the multiparameter case as strong as barcodes, we still have some useful invariants in multiparameter persistent homology, such as Betti tables of multiparameter persistence modules.

The aim of this thesis is to give a survey about Betti tables also known as multigraded Betti numbers. Based on the article "On the Support of the Betti Tables of Multiparameter Persistent Homology Module" which is written by Claudia Landi and Andrea Guidolin, we give a bound to the supports of Betti tables for some special persistence modules.

The structure of this thesis is as follows: In Chapter 2, we give the necessary background to expound bounding the supports of Betti tables. We have four sections in this background chapter; Discrete Morse theory, Koszul Complex, Persistent Homology and the relation between Discrete Morse Theory and Multifiltrations. In the first section of chapter 3, we give an equivalent definition for Betti table of a persistence module by using the content given in the Koszul complex section of the background chapter. In the second section, we look from a different perspective to the Koszul complex of a persistence module, and we show that we can construct it iteratively. Chapter 4 is the main chapter of this thesis, and we give a proof of the following result in " [13]": If we have a persistence module which is obtained by a 1-critical filtration of a finite cell-complex, and a discrete gradient vector field which is consistent with the filtration, then we can bound the support of the Betti table of this persistence module by using the closure (with respect to the least upper bound of the poset $(\mathbb{N}^n, \leq)$) of the entrance grades of the critical cell of the discrete gradient vector field. In the last chapter, the main result is strengthened, which is proved in chapter 4, for the case $n = 2$. To strengthen this result, we first show that the set of q-th homological grades is a subset of the set of the entrance grades of q-th critical cells for all $q \in \mathbb{N}$, then we bound the support of the Betti table of the persistence module with the closure of the set of homological critical grades.

CHAPTER 2

BACKGROUND

2.1 Discrete Morse Theory

In this chapter, our source will be Forman's paper titled Morse Theory for Cell Complexes [11].

In the preliminaries chapter, we gave some background about discrete Morse theory. Now we know that if we take a CW-complex with a discrete Morse function, then we will obtain some critical cells. In this chapter, we will define a complex using critical cells of the CW-complex, and we will observe that the homology of this complex (which is called the Morse complex) is actually the homology of our original CW-complex. However, we have to define a new boundary map for the Morse complex because the image of a critical cell under our usual boundary map may not be critical in general, therefore we cannot take the restriction of the usual boundary operator. Hence, we will define a new boundary map for the Morse complex. In this section, let M denote a finite CW complex with its usual boundary operator ∂, and let K denote the cells of M, so K_p will denote the set of p-cells of M. Also, we will take f as a fixed discrete Morse function in the remaining part of this chapter. Let us introduce the chain complex of M before we continue with our definitions. Firstly, fix an orientation for the cells of M, and if τ is a cell of M, then define -τ as τ but with an opposite orientation. In addition to this, let $C_i(M, \mathbb{Z})$ be the free abelian group over $\mathbb{Z}$ which is spanned by the cells of M. Also, as mentioned above, let ∂ be the usual boundary map.

Definition 1. A function $f : K \to \mathbb{R}$ on M is called a discrete Morse function if for all $\sigma \in K_p$ it satisfies:

- $f(\sigma) < f(\tau)$ whenever σ is an irregular face of $\tau^{(p+1)}$. Moreover,

$$\#\{\tau^{(p+1)} > \sigma \mid f(\tau) \leq f(\sigma)\} \leq 1$$

- $f(v) < f(\sigma)$ whenever v^{p-1} is an irregular face of σ. Moreover,

$$\#\{v^{(p-1)} < \sigma \mid f(v) \geq f(\sigma)\} \leq 1.$$

This definition is too fundamental and important for this background section, so let us give an example to better understand the definition.

Example 2.1.1. Note that we will denote our functions by f in this example.

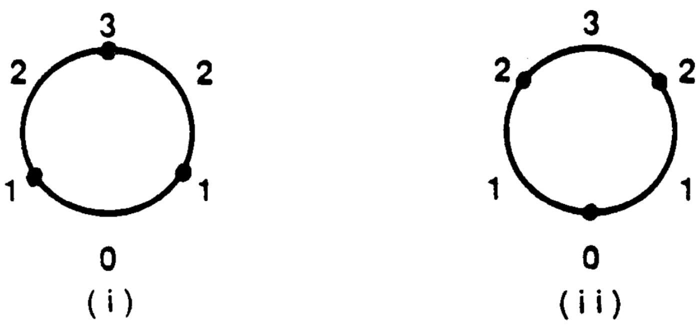

Figure 2.1: An example from [11]

If we look first at the function f on (i) in the Figure 2.1, we can see that the vertex $f^{-1}(3)$ does not satisfy the first condition of the Definition 1. Thus, the function f on (i) is not a discrete Morse function. On the other hand, if we look at the function on (ii) in the Figure 2.1, then we can see that it respects the condition of the Definition 1, so it is a discrete Morse function.

Definition 2. Let f be a discrete Morse function on M. A p-cell σ is called critical cell of index p if there is no $(p+1)$-dimensional cell τ which is a coface of σ, but $f(\tau) \leq f(\sigma)$, and if there is no $(p-1)$-dimensional cell v which is a face of σ, but $f(\sigma) \leq f(v)$.

Notation 2.1.1. Throughout the rest of this background section, let A_p denote the set of critical p-cells on M.

Now, since ∂ is the usual boundary map, we have $\partial\tau = \sum_{\sigma<\tau} \epsilon(\tau, \sigma)\sigma$ where $\epsilon(\tau, \sigma)$'s are called the incidence numbers.

On the other hand, we can define a suitable inner product $\langle \ , \ \rangle$ on C_* by considering the cells of M as an orthonormal basis. Then, we can write ∂ as follows by using the inner product

$$\partial\tau = \sum_{\sigma<\tau} \langle \partial\tau, \sigma \rangle \sigma$$

Now let us define the discrete gradient vector field V_f for the cells of M.

Definition 3. Let σ be a p-cell of M. If there is a $(p+1)$-cell τ which satisfies $\sigma < \tau$,and $f(\tau) \leq f(\sigma)$, then we define the discrete gradient vector field V_f as

$$V_f(\sigma) = -\langle \partial\tau, \sigma \rangle \tau.$$

Remark 2.1.1. In the last definition, σ must be a regular face of τ because we want to guarantee that $\langle \partial\tau, \sigma \rangle = \pm 1$

Note 1. In the remaining part of this chapter, we will use V instead of V_f.

Now, we can extend the discrete gradient vector field V linearly to a map

$V : C_p(M, \mathbb{Z}) \to C_{p+1}(M, \mathbb{Z})$ for all integers p greater than or equal to 0.

Since we gave the definition of V, now we are ready to define the discrete gradient flow Φ_f.

Definition 4. The discrete gradient flow Φ_f is defined as follows:

$$\Phi_f = 1 + \partial V + V\partial.$$

Note 2. We will also write Φ instead of Φ_f in the remaining part of this chapter.

Now we will prove the following theorems which include the main properties of V and Φ.

Theorem 2.1.2. *(i)* $V \circ V = 0$

(ii) If σ *is an oriented* p*-cell, then the number of* $(p-1)$ *cells whose image under* V *is* $\pm\sigma$ *is less than or equal to 1.*

(iii) If σ *is an oriented* p*-cell, then*

σ *is critical if and only if its image under* V *is 0, and* σ *is not in the image of* V.

Proof. (i) Let σ be any oriented p-cell. If $V(\sigma) = 0$, then composition will also be 0. If $V(\sigma) = \pm\tau$ for some $(p+1)$-cell τ, then $\sigma < \tau$ and $f(\tau) < f(\sigma)$ by the definition of V. Also, we know that f is a discrete Morse function, so there cannot be any $(p+2)$-dimensional cell θ such that $\tau < \theta$ and $f(\theta) < f(\tau)$, so $V(\tau)$ must be equal to 0. Hence, $V \circ V = 0$.

(ii) If $V(v) = \pm\sigma$ for some $(p-1)$-dimensional cell, then we know that $v < \sigma$ and $f(\sigma) < f(v)$ by the definition of V. However, we also know that such a v must be unique since f is a discrete Morse function by the definition 1 of a discrete Morse function .

(iii) Let σ be a oriented p-cell. Firstly, assume that σ is a critical cell, then there is no $(p-1)$-cell v such that $v < \sigma$ and $f(\sigma) < f(v)$, so σ cannot be in the image of V. Also, there is no $(p+1)$-cell τ such that $\sigma < \tau$ and $f(\tau) < f(\sigma)$, so $V(\sigma)$ must be equal to 0.

Now assume that $V(\sigma)$=0 and σ is not in the image of V. Since σ is not in the image of V, there is no $(p-1)$-cell v such that $v < \sigma$ and $f(\sigma) < f(v)$. Also, there is no $(p+1)$-cell τ such that $\sigma < \tau$ and $f(\tau) < f(\sigma)$ since $V(\sigma)$=0. Hence, σ must be a critical cell.

□

Theorem 2.1.3. $\Phi \circ \partial = \partial \circ \Phi$.

Proof. We know that $\Phi = 1 + \partial \circ V + V \circ \partial$. So,

$\Phi \circ \partial = \partial + \partial \circ V \circ \partial + V \circ \partial \circ \partial = \partial + \partial \circ V \circ \partial$ (since $\partial \circ \partial = 0$.)

$\partial \circ \Phi = \partial + \partial \circ \partial \circ V + \partial \circ V \circ \partial = \partial + \partial \circ\ V \circ \partial$ (since $\partial \circ \partial = 0$.)

□

For the next theorem, let $\sigma_1, \sigma_2, ..., \sigma_r$ denote some oriented p-cells of M. Write

$$\Phi(\sigma_i) = \sum_j a_{ij}\sigma_j$$

Theorem 2.1.4. *(1) a_{ii}=0 or a_{ii}=1 for all i. In addition to this, a_{ii}=1 if and only if σ_i is a critical cell.*

(2) If i≠j, then $a_{ij} \in \mathbb{Z}$. Also, if i≠j, and $a_{ij} \neq 0$, then $f(\sigma_j) < f(\sigma_i)$.

Proof. Let us prove these two statements together. First, we know that both V and ∂ map integer coefficient chains to integer coefficient chains by their definition, so this automatically implies that $a_{ij} \in \mathbb{Z}$ for all i, j. On the other hand, if we take an oriented p-cell σ, then σ must satisfy exactly one of the following:
(i) σ is critical,
(ii) $\pm\sigma$ is in the image of V,
(iii) $V(\sigma) \neq 0$.
Now, let us examine these three cases separately.
Case 1: Assume that σ is a critical p-cell. Then,

$\Phi(\sigma_i) = \sigma + \partial(V(\sigma)) + V(\partial(\sigma)) = \sigma + V(\partial(\sigma)),$ since σ is critical implies that $V(\sigma) = 0$.

On the other hand, $\partial(\sigma) = \sum_{v^{(p-1)} < \sigma} \langle \partial\sigma, v \rangle v$. Thus,

$$V(\partial(\sigma)) = \sum_{v^{(p-1)} < \sigma} \langle \partial\sigma, v \rangle V(v).$$

Now, we know that σ is a critical cell, so $f(v^{(p-1)}) < f(\sigma)$ for all $v^{(p-1)} < \sigma$. If, v is a $(p-1)$-dimensional face of σ, then there are two possibilities for $v^{(p-1)}$.

(i): $V(v^{(p-1)}) = 0$.

(ii): $V(v^{(p-1)}) = \widetilde{\sigma}$, and this implies that

$$f(\widetilde{\sigma}) < f(v^{(p-1)}) < f(\sigma).$$

Hence, $\Phi(\sigma) = \sigma + \sum a_{\widetilde{\sigma}} \widetilde{\sigma}$, and $a_{\widetilde{\sigma}} \neq 0$ implies that $f(\widetilde{\sigma}) < f(\sigma)$.

Case 2: Assume that σ is p-cell such that $\pm\sigma \in \operatorname{im}(V)$. Then, there is an unique $(p-1)$-dimensional cell v such that $V(v^{(p-1)}) = \pm\sigma$ such that

$$\langle \partial\sigma, v \rangle V(v) = -\sigma$$

by the definition of V. Now, if $\widetilde{v}$ is another $(p-1)$-dimensional face of σ, then $V(\widetilde{v}) = 0$ or $V(\widetilde{v}) = \widetilde{\sigma}$. If $V(\widetilde{v}) = \widetilde{\sigma}$, then $f(\widetilde{\sigma}) < f(\widetilde{v}) < f(\sigma)$ by the definition of V. Now, let us look at $\Phi(\sigma)$:

$$\Phi(\sigma) = \sigma + \partial(V(\sigma)) + V(\partial(\sigma)) = \sigma + V(\partial(\sigma)) = \sigma + \sum_{v^{(p-1)} < \sigma} \langle \partial\sigma, v \rangle V(v).$$

Second equality holds because we know that $\sigma \in im(V)$, and V^2=0. Now, if we combine all of them, we will get $\Phi(\sigma) = \sum_{\sigma \neq \widetilde{\sigma}^{(p)}} a_{\widetilde{\sigma}} \widetilde{\sigma}$, and $a_{\widetilde{\sigma}} \neq 0$ implies that $f(\widetilde{\sigma}) < f(\sigma)$.

Case 3: Assume that $V(\sigma) \neq 0$, then there exists a $(p+1)$-dimensional cell τ such that $V(\sigma) = \pm\tau$. So, $V(\sigma) = -\langle \partial\tau, \sigma \rangle \tau$. Now, let us look at the $\Phi(\sigma)$

$$\Phi(\sigma) = \sigma + \partial(V(\sigma)) + V(\partial(\sigma))$$

Now, let us look at the $\partial(V(\sigma))$ and $V(\partial(\sigma))$ separately. First, let us look at $\partial(V(\sigma))$.

$$\begin{aligned}
\partial(V(\sigma)) &= -\langle \partial\tau, \sigma \rangle \partial\tau \\
&= -\langle \partial\tau, \sigma \rangle^2 \sigma + \sum_{\sigma \neq \widetilde{\sigma} < \tau} -\langle \partial\tau, \sigma \rangle \langle \partial\tau, \widetilde{\sigma} \rangle \widetilde{\sigma} \\
&= -\sigma + \sum_{\sigma \neq \widetilde{\sigma} < \tau} -\langle \partial\tau, \sigma \rangle \langle \partial\tau, \widetilde{\sigma} \rangle \widetilde{\sigma} \\
&= -\sigma + \sum_{\sigma \neq \widetilde{\sigma}^{(p)}} a_{\widetilde{\sigma}} \widetilde{\sigma}.
\end{aligned}$$

Also, note that if $\widetilde{\sigma} < \tau$ such that $\widetilde{\sigma} \neq \sigma$, then $f(\widetilde{\sigma}) < f(\tau) < f(\sigma)$ since $V(\sigma) = \pm\tau$. Lastly, let us look at the $V(\partial(\sigma))$.

$$\begin{aligned}
V(\partial(\sigma)) &= \sum_{v < \sigma} \langle \partial\sigma, v \rangle V(v) \\
&= \sum_{\sigma \neq \widetilde{\sigma}^{(p)}} b_{\widetilde{\sigma}} \widetilde{\sigma}.
\end{aligned}$$

Now, we know that $\sigma \notin im(V)$, so $f(v) < f(\sigma)$ for all $v < \sigma$. Thus, if $V(v^{(p-1)}) = \widetilde{\sigma}$, then $f(\widetilde{\sigma}) < f(v) < f(\sigma)$. Now, let us combine all of these information, and calculate the $\Phi(\sigma)$.

$$\begin{aligned}
\Phi(\sigma) &= \sigma + \partial(V(\sigma)) + V(\partial(\sigma)) \\
&= \sigma + \left(-\sigma + \sum_{\sigma \neq \widetilde{\sigma} < \tau} -\langle \partial\tau, \sigma \rangle \langle \partial\tau, \widetilde{\sigma} \rangle \widetilde{\sigma} \right) + \left(\sum_{v < \sigma} \langle \partial\sigma, v \rangle V(v) \right) \\
&= \sigma + \left(-\sigma + \sum_{\sigma \neq \widetilde{\sigma}^{(p)}} a_{\widetilde{\sigma}} \widetilde{\sigma} \right) + \left(\sum_{\sigma \neq \widetilde{\sigma}^{(p)}} b_{\widetilde{\sigma}} \widetilde{\sigma} \right) \\
&= \sum_{\sigma \neq \widetilde{\sigma}^{(p)}} c_{\widetilde{\sigma}} \widetilde{\sigma}
\end{aligned}$$

where $c_{\widetilde{\sigma}} \neq 0$ implies that $f(\widetilde{\sigma}) < f(\sigma)$. The results of these 3 cases complete the proof of the theorem.

□

Now, let C_p^{Φ} be the set of Φ-invariant p-chains of M, which is

$$C_p^{\Phi}(M,\mathbb{Z}) = \{c \in C_p(M,\mathbb{Z}) \mid \Phi(c) = c\}.$$

Notice that if c is a Φ-invariant p-chain, then

$$\Phi(\partial(c)) = \partial(\Phi(c)) \text{ by Theorem 2.1.3}$$
$$\Phi(\partial(c)) = \partial(c) \text{ since } c \text{ is a } \Phi\text{-invariant } p\text{-chain.}$$

Thus, as we can see here, the boundary operator ∂ sends elements of C_p^{Φ} to elements of C_{p-1}^{Φ}. Hence, we have the following differential complex

$$\mathscr{C}^{\Phi} : 0 \xrightarrow{\partial} C_n^{\Phi}(M,\mathbb{Z}) \xrightarrow{\partial} C_{n-1}^{\Phi}(M,\mathbb{Z}) \xrightarrow{\partial} \dots \xrightarrow{\partial} C_0^{\Phi}(M,\mathbb{Z}) \xrightarrow{\partial} 0.$$

Now, our aim is to show that the homologies of the complexes $\mathcal{C}^{\Phi}$ and M are isomorphic to each other.

Lemma 2.1.5. Let c be Φ-invariant p-chain of M over $\mathbb{Z}$ such that

$$c = \sum_{\sigma \in K_p} a_\sigma \sigma.$$

If σ^* is equal to any maximizer of $\{f(\sigma) | a_\sigma \neq 0\}$, then σ^* must be a critical cell of (the discrete Morse function) f.

Proof. As we know, c is a Φ-invariant p-chain of M, and $c = \sum_{\sigma \in K_p} a_\sigma \sigma$, so

$$c = \Phi(c) = \Phi\left(\sum_{\sigma \in K_p} a_\sigma \sigma\right) = \sum_{\sigma \in K_p} a_\sigma \Phi(\sigma)$$

since Φ is linear. Thus,

$$a_{\sigma^*} = \langle c, \sigma^* \rangle = \langle \sum_{\sigma \in K_p} a_\sigma \Phi(\sigma), \sigma^* \rangle = \sum_{\sigma \in K_p} a_\sigma \langle \Phi(\sigma), \sigma^* \rangle.$$

Now, we can say that $\langle \Phi(\sigma), \sigma^* \rangle = 0$ whenever $\sigma \neq \sigma^*$, and $f(\sigma) \leq f(\sigma^*)$ by the second statement of the Theorem 2.1.4. However, we know that σ^* is equal to any maximizer of $\{f(\sigma)|a_\sigma \neq 0\}$ by our assumption, and this implies that $f(\sigma) \leq f(\sigma^*)$ whenever $a_\sigma \neq 0$, so $\langle \Phi(\sigma), \sigma^* \rangle = 0$ whenever $a_\sigma \neq 0$. Thus,

$$0 \neq a_{\sigma^*} = \sum_{\sigma \in K_p} a_\sigma \langle \Phi(\sigma), \sigma^* \rangle = a_{\sigma^*} \langle \Phi(\sigma^*), \sigma^* \rangle \; a_\sigma \neq 0 \text{ by the assumption.}$$

Hence, $\langle \Phi(\sigma^*), \sigma^* \rangle \neq 0$, and this implies that σ^* is a critical cell by the first statement of the Theorem 2.1.4.

□

Theorem 2.1.6. *For N large enough, $\Phi^N = \Phi^{N+1} = ... = \Phi^\infty$.*

Proof. Let σ be an arbitrary p-cell for some p. In order to prove the theorem, we have to show that there exists $N \in \mathbb{N}$ such that

$$\Phi^N(\sigma) = \Phi^{N+1}(\sigma) = ... = \Phi^\infty(\sigma)$$

for this arbitrary σ. For this proof, let r_σ denote the number of p-cells whose values under the discrete morse function is strictly less than the value of σ. That is,

$$r_\sigma = \#\{\widetilde{\sigma} \in K | f(\widetilde{\sigma}) < f(\sigma)\}.$$

We will use induction on this number. For the base step let $r_\sigma = 0$. This means that there is no $\widetilde{\sigma} \in K$ such that $f(\widetilde{\sigma}) < f(\sigma)$, so Theorem 2.1.4 tells us that $\Phi(\sigma) = \sigma$ whenever σ is a critical cell, and $\Phi(\sigma) = 0$ otherwise. In both cases, we can see that $\Phi(\sigma)$ is Φ-invariant, thus $\Phi(\sigma) = \Phi^\infty(\sigma)$. Now, assume that we can find such an N for all $\alpha \in K$ if $r_\alpha \leq n$ and let $r_\sigma = n + 1$. After this point, let us separate the proof into two cases. Before that let $N_{\widetilde{\sigma}}$ denote the smallest natural number so that $\Phi(\widetilde{\sigma})$ is Φ-invariant.

Case 1: Assume that σ is not a critical cell, then Theorem 2.1.4 implies that

$$\Phi(\sigma) = \sum_{f(\widetilde{\sigma}) < f(\sigma)} a_{\widetilde{\sigma}} \widetilde{\sigma}.$$

If $a_{\widetilde{\sigma}} \neq 0$ in this summation, then $r_{\widetilde{\sigma}} \leq n$ since $r_\sigma = n + 1$ and $f(\widetilde{\sigma}) < (\sigma)$, so there exists $N_{\widetilde{\sigma}}$ such that $\Phi^{N_{\widetilde{\sigma}}}(\widetilde{\sigma})$ is Φ-invariant by our induction assumption. Moreover, there must be finitely many cells which has a non-zero coefficient in the above sum by the definition of Φ, so if we say $N = max\{N_{\widetilde{\sigma}} \mid a_{\widetilde{\sigma}} \neq 0\}$, then $\Phi^{N+1}(\sigma)$ will be

Φ-invariant which means $\Phi^{N+1}(\sigma) = \Phi^{\infty}(\sigma)$.
Case 2: This time assume that σ is a critical p-cell, and let $c = V(\partial(\sigma))$. Since σ is a critical cell, $V(\sigma) = 0$, so

$$\begin{aligned}\Phi &= (1 + V\partial + \partial V)(\sigma) \\ &= \sigma + V(\partial(\sigma)) + \partial(V(\sigma)) \\ &= \sigma + V(\partial(\sigma) \quad \text{since } V(\sigma) = 0 \\ &= \sigma + c.\end{aligned}$$

So,

$$\Phi^m(\sigma) = \sigma + c + \Phi(c) + ... + \Phi^{m-1}(c).$$

The last equality tells us that $\Phi^N(\sigma)$ is Φ-invariant if and only if $\Phi^N(c) = 0$ for all N. So, it is enough to find an N such that $\Phi^N(c) = 0$.

Now, if we look at the proof of Theorem 2.1.4, we can see that c is the linear combination of the p-cells $\widetilde{\sigma}$ such that $f(\widetilde{\sigma}) < f(\sigma)$, thus if we apply the same proof as we did in case 1, then we can see that there exists $\widetilde{N}$ such that $\Phi^{\widetilde{N}}(c)$ is Φ-invariant by induction. On the other hand, we know that $c = V(\partial(\sigma))$, so $c \in Im(V)$, and $Im(V)$ is Φ-invariant because

$$\begin{aligned}\Phi V &= (1 + \partial V + V\partial)V \\ &= V + \partial V^2 + V\partial V \\ &= V + V\partial V \quad \text{by Theorem 2.1.2} \\ &= V(1 + \partial V)\end{aligned}$$

Now, we know that $Im(V)$ is Φ-invariant, and $c \in Im(V)$, so $\Phi^n(c) \in Im(V)$ for all $n \in \mathbb{N}$, so $\Phi^{\widetilde{N}}(c) \in Im(V)$. Moreover, $Im(V)$ is orthogonal to the critical cells by Theorem 2.1.2.

Claim: $\Phi^{\widetilde{N}} = 0$.
Proof of the Claim: Assume not. So, if we write as $\Phi^{\widetilde{N}}(c) = \sum_{\widetilde{\sigma} \in K_p} a_{\widetilde{\sigma}}\widetilde{\sigma}$, then at least one of the coefficients is non-zero. Also, let σ^* is the maximizer of $\{f(\sigma) \mid a_\sigma \neq 0\}$, then we know that σ^* is a critical p-cell by Lemma 2.1.5 since $\Phi^{\widetilde{N}}(c)$ is a Φ-invariant

p-chain. So,

$$\begin{aligned} 0 &= \langle \Phi^{\widetilde{N}}(c), \sigma^* \rangle \quad \text{since } \Phi^{\widetilde{N}}(c) \in Im(V), \text{ and } \sigma^* \text{ is critical} \\ &= a_{\sigma^*}. \end{aligned}$$

However, we know that $a_{\sigma^*} \neq 0$ by the definition of σ^*. Thus, this gives us a contradiction. Hence, $\Phi^{\widetilde{N}}(c) = 0$.

As a result of this claim, we can say that $\Phi^{\widetilde{N}}(\sigma)$ is Φ-invariant as we mentioned above, and this finishes the proof.

□

Now, Theorem 2.1.6 tells us that for every chain c, we can find an N large enough so that

$$\Phi^N(c) = \Phi^{N+1}(c) = \Phi^{N+2}(c) = ...$$

So, this means that for every chain c, we can obtain a Φ-invariant chain $\Phi^N(c)$ for some N large enough, and we will denote this Φ-invariant chain as $\Phi^\infty(c)$ as we mentioned before. At the moment, for each p we have maps

$$\Phi^\infty : C_p(M, \mathbb{Z}) \to C_p^\Phi(M, \mathbb{Z})$$
$$i : C_p^\Phi \hookrightarrow C_p(M, \mathbb{Z})$$

where i is the inclusion map. Notice that $\Phi^\infty \circ i$ is actually the identity map on $C_p^\Phi(M, \mathbb{Z})$. By the way, these maps are will have an important role for the proof of the next theorem.

Theorem 2.1.7.

$$H_p(\mathscr{C}^\Phi) \cong H_p(M, \mathbb{Z})$$

for all p.

Proof. For the proof of this theorem, we will use the induced maps on homology of the maps i and Φ^∞. But firstly consider the following commutative diagram:

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & C_n(M,\mathbb{Z}) & \xrightarrow{\partial} & C_{n-1}(M,\mathbb{Z}) & \xrightarrow{\partial} & ... & \xrightarrow{\partial} & C_0(M,\mathbb{Z}) & \longrightarrow & 0 \\ & & {\scriptstyle\Phi^\infty}\big\downarrow \big\uparrow{\scriptstyle i} & & {\scriptstyle\Phi^\infty}\big\downarrow \big\uparrow{\scriptstyle i} & & & & {\scriptstyle\Phi^\infty}\big\downarrow \big\uparrow{\scriptstyle i} & & \\ 0 & \longrightarrow & C_n^\Phi(M,\mathbb{Z}) & \xrightarrow{\partial} & C_{n-1}^\Phi(M,\mathbb{Z}) & \xrightarrow{\partial} & ... & \xrightarrow{\partial} & C_0^\Phi(M,\mathbb{Z}) & \longrightarrow & 0 \end{array}$$

This diagram is commutative because we know that Φ^∞=Φ^N for some large enough N, and we know that $\partial\Phi = \Phi\partial$ by Theorem 2.1.3. Thus, these two imply that $\Phi^\infty\partial = \partial\Phi^\infty$. Also, we know that i is the inclusion map, so $\partial \circ i = i \circ \partial$ is obvious. Now, denote the induced maps on homology of the maps Φ^∞ and i as follows:

$$\Phi^\infty_* : H_*(M, \mathbb{Z}) \to H_*(C^\Phi_*)$$
$$i_* : H_*(C^\Phi_*) \to H_*(M, \mathbb{Z}).$$

In order to prove the theorem, we will show that these two induced maps are inverses of each other because we already know that these maps are homomorphisms. So, if we can show that these maps are inverses of each other, then this will imply that these maps are isomorphisms, and this will complete our proof. Firstly, we know that $\Phi^\infty \circ i = 1$, since i is the inclusion map, so

$$1 = (\Phi^\infty \circ i)_* = \Phi^\infty_* \circ i_*.$$

Lastly, we have to show that $i_* \circ \Phi^\infty_* = 1$. In order to show this, we will find an operator

$$L : C_*(M, \mathbb{Z}) \to C_{*+1}(M, \mathbb{Z})$$

such that

$$1 - i \circ \Phi^\infty = \partial L + L\partial.$$

At the moment, the natural question is why this equality is okay with us. The reason is that if we take a closed form, then it will be 0 under ∂. So, it will also be 0 under $L\partial$, and this closed form will be an exact form under ∂L. Thus, at the end of the day, any closed form will be an exact form under $\partial L + L\partial$ which means this map will be the zero map on homology. Now, let us find such an operator. We know that $\Phi^\infty = \Phi^N$ for some N large enough, and i is the inclusion map, so

$$\begin{aligned}
1 - i \circ \Phi^\infty &= 1 - \Phi^\infty \quad \text{since } i \text{ is the inclusion map} \\
&= 1 - \Phi^N \\
&= (1 - \Phi)(1 + \Phi + ... + \Phi^{N-1}) \\
&= (1 - 1 - \partial V - V\partial) \\
&= (-\partial V - V\partial)(1 + \Phi + ... + \Phi^{N-1}) \\
&= -\partial V(1 + \Phi + ... + \Phi^{N-1}) - V\partial(1 + \Phi + ... + \Phi^{N-1}) \\
&= \partial[-V(1 + \Phi + ... + \Phi^{N-1})] + [-V(1 + \Phi + ... + \Phi^{N-1})]\partial
\end{aligned}$$

Notice that the last equality comes from Theorem 2.1.3. After this, if we choose $L = -V(1 + \Phi + ... + \Phi^{N-1})$, then we are done.

$\square$

After this point, we will construct a chain complex whose chains will be spanned by critical cells, and we will call this complex the Morse complex. Moreover, our aim is to show that this complex is isomorphic to $\mathscr{C}^{\Phi}$.

Firstly, let us denote the chains which are generated by p-th critical cells as follows:

$$\mathscr{M}_p := \{ \sum_{\sigma \in K_p} a_\sigma \sigma \mid a_\sigma \in \mathbb{Z} \text{ and } a_\sigma \neq 0 \implies \sigma \text{ critical } \}.$$

We know that Φ^∞ is a map from $C_p(M, \mathbb{Z})$ to $C_p^\infty(M, \mathbb{Z})$ for all p, and $\mathscr{M}_p \subseteq C_p(M, \mathbb{Z})$, so if we take the restriction of Φ^∞ to $\mathscr{M}_p$, then we will get a map

$$\Phi_p^\infty : \mathscr{M}_p \to C_p^\Phi(M, \mathbb{Z}).$$

At this moment, we want to show that Φ_p^∞ is an isomorphism, but firstly, let us prove the following lemma.

Lemma 2.1.8. If σ and $\widetilde{\sigma}$ are critical p-cells, then

- $\langle \Phi^\infty(\sigma), \widetilde{\sigma} \rangle = 0$ if $\sigma \neq \widetilde{\sigma}$
- $\langle \Phi^\infty(\sigma), \widetilde{\sigma} \rangle = 1$ if $\sigma = \widetilde{\sigma}$

Proof. Firstly, assume that $\sigma \neq \widetilde{\sigma}$. Let $c_V(\partial\sigma)$, then

$$\Phi^\infty(\sigma) = \sigma + c + \Phi(c) + \cdots + \Phi^N(c) \text{ for some } N$$

by the proof of Theorem 2.1.6. Also, we know that $Im(V)$ is a Φ-invariant by the proof of Theorem 2.1.6, and $c \in Im(V)$ by its definition, so this implies that $\Phi^n(c) \in Im(V)$ for all n. Thus, $\langle c, \widetilde{\sigma} \rangle = 0$ and $\langle \Phi^n(c), \widetilde{\sigma} \rangle = 0$ for all n. Hence,

$$\begin{aligned} \langle \Phi^\infty(\sigma), \widetilde{\sigma} \rangle &= \langle \sigma + c + \Phi(c) + \cdots + \Phi^N(c), \widetilde{\sigma} \rangle \\ &= \langle \sigma, \widetilde{\sigma} \rangle + \langle c, \widetilde{\sigma} \rangle + \cdots + \langle \Phi^n(\sigma), \widetilde{\sigma} \rangle \\ &= \langle \sigma, \widetilde{\sigma} \rangle \\ &= 0. \end{aligned}$$

Now, assume that $\sigma = \widetilde{\sigma}$. The process is same with the preceding one except for the last equality. This time $\langle \sigma, \widetilde{\sigma} \rangle = 1$ by our assumption.

□

Theorem 2.1.9. $\Phi_p^\infty : \mathscr{M}_p \to C_p^\Phi(M, \mathbb{Z})$ *is an isomorphism for all p.*

Proof. It is enough to prove this for an arbitrary p. Firstly, let us show that this map is surjective.

Let $c \in C_p^\Phi(M, \mathbb{Z})$, and let $d = \sum_{\sigma \in A_p} \langle c, \sigma \rangle \sigma$, then $d \in \mathscr{M}_p$. If σ is a critical p-cell, then notice that

$$\begin{aligned}
\langle \Phi^\infty(d), \sigma \rangle &= \langle \Phi^\infty(\sum_{\sigma \in A_p} \langle c, \sigma \rangle \sigma), \sigma \rangle \\
&= \langle \sum_{\sigma \in A_p} \langle c, \sigma \rangle \Phi^\infty(\sigma), \sigma \rangle \\
&= \sum_{\sigma \in A_p} \langle c, \sigma \rangle \langle \Phi^\infty(\sigma), \sigma \rangle \\
&= \langle c, \sigma \rangle
\end{aligned}$$

by Lemma 2.1.8. Thus, $\langle \Phi^\infty, \sigma \rangle - \langle c, \sigma \rangle = 0$, so $\langle \Phi^\infty(d) - c, \sigma \rangle = 0$ for all critical p-cell σ.

Claim: $\Phi^\infty(d) - c = 0$.

Proof of the Claim: We know that $\Phi^\infty(d) - c$ is a Φ-invariant p-chain, since $\Phi^\infty(d) \in C_p^\Phi(M, \mathbb{Z})$, and $c \in C_p^\Phi(M, \mathbb{Z})$. Assume that $\Phi^\infty(d) - c \neq 0$, so $\Phi^\infty(d) - c = \sum_{\sigma \in K_p} a_\sigma \sigma$. If we say σ^* is the maximizer of $\{f(\sigma) \mid a_\sigma \neq 0\}$, then σ^* will be critical by Lemma 2.1.5, but we know that

$$\begin{aligned}
0 &= \langle \Phi^\infty(d) - c, \sigma^* \rangle \\
&= \langle \sum_{\sigma \in K_p} a_\sigma \sigma, \sigma^* \rangle
\end{aligned}$$

which gives us a contradiction by the definition of σ^*. Thus, $\Phi^\infty(d) - c = 0$, and this finishes the proof of the claim.

Hence, $\Phi^\infty(d) = c$, so $\Phi_p^\infty : \mathscr{M}_p \to C_p^\Phi(M, \mathbb{Z})$ is a surjection.

Now, let us show that Φ_p^∞ is an injection. Let $c \in \mathscr{M}_p$ such that $\Phi_p^\infty(c) = 0$. Since $c \in \mathscr{M}_p$, $c = \sum\limits_{\sigma \in A_p} a_\sigma \sigma$, so if σ is a critical p-cell, then

$$
\begin{aligned}
0 &= \langle 0, \sigma \rangle \\
&= \langle \Phi_p^\infty(c), \sigma \rangle \\
&= \langle \sum_{\sigma \in A_p} a_\sigma \Phi_p^\infty(\sigma), \sigma \rangle \\
&= \sum_{\sigma \in A_p} a_\sigma \langle \Phi_p^\infty(\sigma), \sigma \rangle \\
&= a_\sigma
\end{aligned}
$$

by Lemma 2.1.8. So, $c = 0$. Hence, Φ_p^∞ is an injection.

□

In order to define a chain complex, we need to define a boundary map. For this we will use the following map:

$$
\begin{aligned}
\widetilde{\partial}_p : \mathscr{M}_p &\to \mathscr{M}_{p-1} \\
c &\mapsto \sum_{\sigma \in A_p} \langle \widetilde{\partial}_p(c), \sigma \rangle \sigma
\end{aligned}
$$

where $\langle \widetilde{\partial}_p(c), \sigma \rangle = \langle \partial_p \Phi_p^\infty(c), \sigma \rangle$.

Theorem 2.1.10. $\Phi_{p-1}^\infty(\widetilde{\partial}_p(c)) = \partial_p(\Phi_p^\infty(c))$.

Proof. If $c \in C_p^\Phi(M, \mathbb{Z})$, then

$$
\begin{aligned}
\Phi_{p-1}^\infty(\widetilde{\partial}(c)) &= \Phi_{p-1}^\infty \left(\sum_{\sigma \in A_p} \langle \widetilde{\partial}_p(c), \sigma \rangle \sigma \right) \\
&= \Phi_{p-1}^\infty \left(\sum_{\sigma \in A_p} \langle \partial_{p-1} \Phi_p^\infty(c), \sigma \rangle \sigma \right) \\
&= \Phi_{p-1}^\infty \left(\sum_{\sigma \in A_p} \langle \Phi_{p-1}^\infty \partial_p(c), \sigma \rangle \sigma \right) \quad \text{by Theorem 2.1.3} \\
&= \Phi_{p-1}^\infty(\partial_p(c)).
\end{aligned}
$$

We know that $\Phi^{\infty}_{p-1}(\partial_p(c)) \in C^{\Phi}_{p-1}(M,\mathbb{Z})$, so the surjectivity part of the proof of the Theorem 2.1.9 implies the last equality. Also, $\Phi^{\infty}_{p-1}(\partial_p(c)) = \partial_p(\Phi^{\infty}_p(c))$ by Theorem 2.1.3. Thus, $\Phi^{\infty}_{p-1}(\widetilde{\partial}_p(c)) = \partial_p(\Phi^{\infty}_p(c))$.

□

Moreover, we know that Φ^{∞}_p is an isomorphism for all p. Thus, $\widetilde{\partial}_p = (\Phi^{\infty}_{p-1})^{-1}\partial_p\Phi^{\infty}_p$, thus $\widetilde{\partial}_{p-1} \circ \widetilde{\partial}_p = (\Phi^{\infty}_{p-2})^{-1} \circ \partial_{p-1} \circ \Phi^{\infty}_{p-1} \circ (\Phi^{\infty}_{p-1})^{-1} \circ \partial_p \circ \Phi^{\infty}_p$

$$\begin{aligned}\widetilde{\partial}_{p-1} \circ \widetilde{\partial}_p &= (\Phi^{\infty}_{p-2})^{-1} \circ \partial_{p-1} \circ \Phi^{\infty}_{p-1} \circ (\Phi^{\infty}_{p-1})^{-1} \circ \partial_p \circ \Phi^{\infty}_p \\ &= \Phi^{\infty}_{p-2} \circ \partial p - 1 \circ \partial_p \circ \Phi^{\infty}_p \\ &= 0\end{aligned}$$

since $\partial_{p-1} \circ \partial_p = 0$. So,

$$\mathscr{M} : 0 \to \mathscr{M}_n \xrightarrow{\widetilde{\partial}_n} \mathscr{M}_{n-1} \xrightarrow{\widetilde{\partial}_{n-1}} \dots \xrightarrow{\widetilde{\partial}_1} \mathscr{M}_0 \to 0$$

is a chain complex and it is called the Morse complex. Notice that the Morse complex is isomorphic to $\mathscr{C}^{\Phi}$ because $\mathscr{M}_p \cong C^{\infty}_p(M,\mathbb{Z})$ for all p, and the last theorem tells us that the following diagram commutes

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \mathscr{M}_n & \xrightarrow{\widetilde{\partial}_n} & \mathscr{M}_{n-1} & \xrightarrow{\widetilde{\partial}_{n-1}} & \dots & \xrightarrow{\widetilde{\partial}_1} & \mathscr{M}_0 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \Phi^{\infty}_n} & & \downarrow{\scriptstyle \Phi^{\infty}_{n-1}} & & & & \downarrow{\scriptstyle \Phi^{\infty}_0} & & \\ 0 & \longrightarrow & C^{\Phi}_n(M,\mathbb{Z}) & \xrightarrow{\partial_n} & C^{\Phi}_{n-1}(M,\mathbb{Z}) & \xrightarrow{\partial_{n-1}} & \dots & \xrightarrow{\partial_1} & C^{\Phi}_0(M,\mathbb{Z}) & \longrightarrow & 0 \end{array}$$

Moreover, we know that $H_p(C^{\Phi}_*) \cong H_p(M,\mathbb{Z})$ for all p, and the Morse complex is isomorphic to chain complex $\mathscr{C}^{\Phi}$, so $H_*(\mathscr{M}) \cong H_*(M,\mathbb{Z})$. Also, if $\mathbf{F}$ is a field, then

$$\begin{aligned}H_*(\mathscr{M} \otimes \mathbf{F}) &\cong H_*(\mathscr{M} \otimes \mathbf{F}) \oplus Tor(H_{*-1}(\mathscr{M}),\mathbf{F}) \\ &\cong H_*((M) \otimes \mathbf{F}) \text{ since } \mathbf{F} \text{ is a field} \\ &\cong H_*(M,\mathbb{Z}) \otimes \mathbf{F} \\ &\cong (H_*(M,\mathbb{Z}) \otimes \mathbf{F}) \oplus Tor(H_{*-1}(M,\mathbb{Z}),\mathbf{F}) \text{ since } \mathbf{F} \text{ is a field} \\ &\cong H_*(M,\mathbf{F})\end{aligned}$$

by Universal Coefficient Theorem. Note that the first isomorphism also follows from Universal Coefficient Theorem.

2.2 Multiparameter Persistent Homology

In this background section, we will give a basic definition of multiparameter persistent homology which is one of the main tools of the topological data analysis. The theory of multidimensional persistent homology was first given in [6]. However, we will give the categorical definition of the basic concepts in this background. Thus, we will use [2] as our source.

Definition 5. A subset of a poset $(P, \leq)$ is called an interval if $x, z \in I$, and $x \leq y \leq z$ implies $y \in I$.

Definition 6. Let $(P, \leq)$ be a poset. A filtration is a functor $F : P \to$ **Top** (where **Top** is the category of topological spaces whose objects are topological spaces, and whose morphisms are continuous functions, and P is the poset category with its relation) which satisfies:

- $F(x) \subseteq F(y)$ whenever $x \leq y$.
- $F(x \to y) : F(x) \to F(y)$ is the inclusion map for $x \leq y$.

In the above definition, if we take $P = T_1 \times T_2 \times \cdots \times T_n$ where T_i's are totally ordered sets, then F is called a multiparameter filtration or an n-parameter filtration. Moreover, if we take $n = 2$, then we will usually call it a bifiltration.

Example 2.2.1. In the following figure, we can see an example of a 2-parameter filtration (also known as a bifiltration) which is actually a diagram, where maps are inclusion maps of topological spaces.

$$
\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\uparrow & & \uparrow & & \uparrow & & \\
F_{0,2} & \hookrightarrow & F_{1,2} & \hookrightarrow & F_{2,2} & \hookrightarrow & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \\
F_{0,1} & \hookrightarrow & F_{1,1} & \hookrightarrow & F_{2,1} & \hookrightarrow & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \\
F_{0,0} & \hookrightarrow & F_{1,0} & \hookrightarrow & F_{2,0} & \hookrightarrow & \cdots
\end{array}
$$

Figure 2.2: An example from [2].

Note that, if $(P, \leq)$ is a poset category, and $F : P \to \mathscr{C}$ is a functor, then we will use F_x, and $F_{x,y}$ as our notations instead of $F(x)$, and $F(x \to y)$ from now on, respectively.
Now we will give the definition of a sublevel filtration, and the following construction of the sublevel filtration was first given in [12].

Definition 7. If $(P, \leq)$ is a poset, A is a topological space, and $\alpha : A \to P$ is a function, then the functor $S^{\uparrow}(\alpha) : P \to \mathbf{Top}$ such that $S^{\uparrow}(\alpha)_p = \{a \in A \mid \alpha(a) \leq p\}$ is called the sublevel filtration of α.

Remark 2.2.1. In the definition of sublevel filtration, the function α may not be continuous.

Definition 8. A filtration F is called 1-critical if it is isomorphic to a sublevel filtration of a function α. Otherwise, it is called multi-critical [1, 7].

Definition 9. Let $(P, \leq)$ be a partial ordered set, and let $\mathbf{Vect}$ be the category of vector spaces over a fixed field $\mathbf{F}$. Then a (P-indexed) persistence module is a functor $M : P \to \mathbf{Vect}$.

In the above definition, if we take $P = T_1 \times T_2 \times \cdots \times T_n$ where T_i's are totally ordered sets, then we will call it a multiparameter persistence module, and if $n = 2$, then it is called a bipersistence module.

Example 2.2.2. In the following figure, we can see an example of a 2-parameter persistence module (also known as a bipersistence module) which is actually a diagram of vector spaces over the fixed field $\mathbf{F}$.

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\uparrow & & \uparrow & & \uparrow & & \\
M_{(0,2)} & \longrightarrow & M_{(1,2)} & \longrightarrow & M_{(2,2)} & \longrightarrow & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \\
M_{(0,1)} & \longrightarrow & M_{(1,1)} & \longrightarrow & M_{(2,1)} & \longrightarrow & \cdots \\
\uparrow & & \uparrow & & \uparrow & & \\
M_{(0,0)} & \longrightarrow & M_{(1,0)} & \longrightarrow & M_{(2,0)} & \longrightarrow & \cdots
\end{array}$$

Figure 2.3: An example from [2].

Definition 10. A persistence module M is called indecomposable if it cannot be written as a direct sum of two non-zero persistence modules.

Definition 11. If I is an interval, and P is a poset, then the functor $\mathbf{F}_I$ which is defined by

$$(\mathbf{F}_I)_x = \begin{cases} \mathbf{F} & \text{if } x \in I \\ 0 & \text{if otherwise,} \end{cases} \qquad (\mathbf{F}_I)_{x,y} = \begin{cases} id_{\mathbf{F}} & \text{if } x \leq y \in I \\ 0 & \text{if otherwise.} \end{cases} \tag{2.1}$$

is called an interval module.

Note that interval modules are indecomposable. For the proof, you can see [4].

Now, if we take total a order set instead of any arbitrary poset in the definition of a persistence module, then we will obtain a special persistence module which is called a 1-parameter persistence module. The reason why these persistence modules are special is because we have an invariant which is called the barcode.

Definition 12. A set which includes a multiset of intervals is called a barcode.

Remark 2.2.2. In the above definition, a multiset means that the set can contain an interval several times.

Definition 13. A persistence module $M : P \to \mathbf{Vec}$ is said to be pointwise finite dimensional if $\dim(M_p)$ is finite for all $p \in P$.

Theorem 2.2.1. *(Structure Theorem of Persistence Modules) If M is a 1-parameter persistence module which is pointwise finite dimensional, then it can be written as the direct sum of the interval persistence modules $\mathbf{F}_I$ such that $I \in \mathcal{B}$ for a unique $\mathcal{B}$ up to isomorphism, i.e.*

$$M \cong \bigoplus_{I \in \mathcal{B}} \mathbf{F}_I. \tag{2.2}$$

We will not give the proof of this theorem, since we will not use it in the thesis. However, the reader can find it in [16] for the case $T = \mathbb{Z}$, [8] for all T with a countable basis, and [3] for all totally ordered sets T.

2.3 Discrete Morse Theory and Multifiltrations

In the previous chapters, we have seen the necessary background about discrete Morse theory and multiparameter persistent homology. Now, we can combine them to use the (discrete) Morse complex in order to get some information about a persistence module which is obtained by applying the homology functor to a filtration. This background section is based on the article [13].

Definition 14. Let X be a cell complex, $\{X^{\mathbf{u}}\}_{\mathbf{u}\in\mathbb{N}^n}$ be a 1-critical n-parameter filtration which is obtained from X, f be a discrete Morse function on X, V be the discrete gradient vector field associated with f, M be the Morse complex which is associated with V, and let $\{M^{\mathbf{u}}\}_{\mathbf{u}\in\mathbb{N}^n}$ be an n-parameter filtration of M. We call $\{M^{\mathbf{u}}\}_{\mathbf{u}\in\mathbb{N}^n}$ the induced filtration of $\{X^{\mathbf{u}}\}_{\mathbf{u}\in\mathbb{N}^n}$ if, for all $\sigma \in M$ and all $\mathbf{u} \in \mathbb{N}^n$, $\sigma \in X^{\mathbf{u}}$ if and only if $\sigma \in M^{\mathbf{u}}$.

Remark 2.3.1. Let X be a cell complex, V be a discrete gradient vector field on X, and M be a Morse complex associated with V. Then, we assume that, once and for all the remainder of the thesis, if we have a fixed 1-critical n-parameter filtration $\{X^{\mathbf{u}}\}_{\mathbf{u}\in\mathbb{N}^n}$ of X, then we use the n-parameter filtration $\{M^{\mathbf{u}}\}_{\mathbf{u}\in\mathbb{N}^n}$ of M as the induced filtration of $\{X^{\mathbf{u}}\}_{\mathbf{u}\in\mathbb{N}^n}$.

Definition 15. Let X be a cell complex, $\{X^{\mathbf{u}}\}_{\mathbf{u}\in\mathbb{N}^n}$ be a 1-critical n-parameter filtration of X, V be a discrete gradient vector field on X. Then, we say that V is **consistent** with the filtration $\{X^{\mathbf{u}}\}_{\mathbf{u}\in\mathbb{N}^n}$ if, for all $\sigma, \tau \in X$ such that $V(\sigma) = -\langle \partial\tau, \sigma\rangle\tau$, where ∂ is the boundary map of X, and for all $\mathbf{u} \in \mathbb{N}^n$, $\sigma \in X^{\mathbf{u}}$ if and only if $\tau \in X^{\mathbf{u}}$.

Let X be a cell complex, $\{X^{\mathbf{u}}\}_{\mathbf{u}\in\mathbb{N}^n}$ be a 1-critical n-parameter filtration which is obtained from X, f be a discrete Morse function on X, V be the discrete gradient vector field associated with f, M be the Morse complex which is associated with V, and let $\{M^{\mathbf{u}}\}_{\mathbf{u}\in\mathbb{N}^n}$ be an n-parameter filtration of M. We know that homology groups of X and M are isomorphic by Section 2.1. However, if $\mathbf{u} \in \mathbb{N}^n$, then the homology groups of $X^{\mathbf{u}}$ and $M^{\mathbf{u}}$ may not be isomorphic in general. Because, if we look at the restriction of f to $X^{\mathbf{u}}$, it will be again a discrete Morse function for $X^{\mathbf{u}}$, but its Morse complex may not be $M^{\mathbf{u}}$. This means that its homology groups may not be equal to the homology groups of $M^{\mathbf{u}}$. However, if V is consistent with the filtration

$\{X^{\mathbf{u}}\}_{\mathbf{u}\in\mathbb{N}^n}$, and if we look at the restriction of f to $X^{\mathbf{u}}$, then this time its Morse complex which is associated with V will be $M^{\mathbf{u}}$. Thus, if V is consistent with the filtration $\{X^{\mathbf{u}}\}_{\mathbf{u}\in\mathbb{N}^n}$, then the homology groups of $X^{\mathbf{u}}$ are isomorphic to homology groups of $M^{\mathbf{u}}$ for all $\mathbf{u} \in \mathbb{N}^n$. Consider the following example:

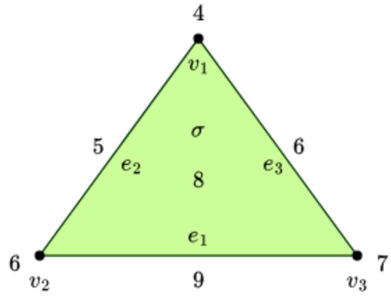

Figure 2.4: A simplicial complex X with a discrete Morse function f on it.

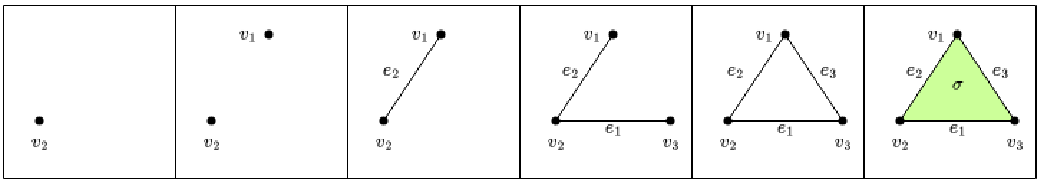

Figure 2.5: A 1-parameter filtration of the simplicial complex 2.4

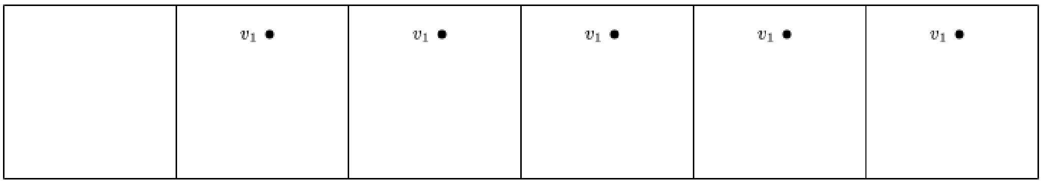

Figure 2.6: A 1-parameter filtration of the Morse complex M which is associated with the discrete Morse function f on 2.4

Example 2.3.1. Let $\mathbb{R}$ be our fixed field for this example. Let X be the simplicial complex in the Figure 2.4 with the discrete Morse function f which is defined in the same figure. Now, we have a Morse complex M associated to f, and notice that M has only one cell which is v_1. We can see a 1-parameter filtration of X in the Figure 2.5, and there is a 1-parameter filtration of M in the Figure 2.6 which is the induced 1-parameter filtration of 2.5, but if we look at the fifth subcomplex in both filtrations, then we can see that their first homology groups are not the same. The first homology group of the complex ,which is placed in the fifth grade of the filtration 2.5, is $\mathbb{R}$, but the first homology group of the complex, which is placed in the fifth grade of the filtration 2.6, is 0. As we explained above, the reason is that V_f is not consistent with the filtration 2.5. As we can see from 2.4, $V(e_1) = -\langle \partial\sigma, \sigma \rangle \sigma$, where ∂ is the boundary map of X and the sign $\langle \partial\sigma, \sigma \rangle$ is determined by a fixed orientation (see section 2.1), but e_1 is in the fifth grade of 2.5 and σ is not in that grade. Thus, V_f is not consistent with the filtration 2.5.

Lemma 2.3.1. Let X be a cell complex, $\{X^{\mathbf{u}}\}_{\mathbf{u}\in\mathbb{N}^n}$ be a 1-critical n-parameter filtration which is obtained by X, f be a discrete Morse function on X, V be the discrete gradient vector field associated with f and consistent with $\{X^{\mathbf{u}}\}_{\mathbf{u}\in\mathbb{N}^n}$, M be the Morse complex which is associated with V, and let $\{M^{\mathbf{u}}\}_{\mathbf{u}\in\mathbb{N}^n}$ be an n-parameter filtration of M. If $\mathbf{u}, \mathbf{v} \in \mathbb{N}^n$ such that $\mathbf{u} \leq \mathbf{v}$ then there exists a map $\Phi : X \to M$, which induces an isomorphism on homology for all degrees, such that the following diagram commutes

$$\begin{array}{ccc} X^{\mathbf{u}} & \xhookrightarrow{i^X_{\mathbf{u},\mathbf{v}}} & X^{\mathbf{v}} \\ \downarrow{\scriptstyle \Phi_{\mathbf{u}}} & & \downarrow{\scriptstyle \Phi_{\mathbf{v}}} \\ M^{\mathbf{u}} & \xhookrightarrow{i^M_{\mathbf{u},\mathbf{v}}} & M^{\mathbf{v}} \end{array} \tag{2.3}$$

where $\Phi_{\mathbf{u}}$ is the restriction of Φ to $X^{\mathbf{u}}$, $\Phi_{\mathbf{v}}$ is the restriction of Φ to X^V, and vertical maps denote the inclusion maps.

Proof. By Section 2.1, we know that there exists a quasi-isomorphism $\Phi : X \to M$, and this map depends on the discrete Morse function f. Also, the restriction of f to $X^{\mathbf{u}}$ is a discrete Morse function on $X^{\mathbf{u}}$ for all $\mathbf{u} \in \mathbb{N}^n$, and the corresponding Morse complex of $X^{\mathbf{u}}$ with respect to this restriction map is $M^{\mathbf{u}}$ for all $\mathbf{u} \in \mathbb{N}^n$, since V is consistent with the $\{X^{\mathbf{u}}\}_{\mathbf{u}\in\mathbb{N}^n}$. Thus, $\Phi_{\mathbf{u}}$ is map from $X^{\mathbf{u}}$ to $M^{\mathbf{u}}$ for all $\mathbf{u} \in \mathbb{N}^n$, and the induced map $(\Phi_{\mathbf{u}})_* : H_q(X^{\mathbf{u}}) \to H_q(M^{\mathbf{u}})$ is an isomorphism for all $\mathbf{u} \in \mathbb{N}^n$, for all q. Moreover, this implies that if $\mathbf{u} \leq \mathbf{v}$, then $\Phi_{\mathbf{u}}$ is also restriction of $\Phi_{\mathbf{v}}$, and we know that $i^X_{\mathbf{u},\mathbf{v}}$ and $i^M_{\mathbf{u},\mathbf{v}}$ are inclusion maps, so the given diagram commutes.

□

Theorem 2.3.2. *Let X be a cell complex, $\{X^{\mathbf{u}}\}_{\mathbf{u}\in\mathbb{N}^n}$ be a 1-critical n-parameter filtration which is obtained from X, f be a discrete Morse function on X, V be the discrete gradient vector field associated with f and consistent with $\{X^{\mathbf{u}}\}_{\mathbf{u}\in\mathbb{N}^n}$,*

M be the Morse complex which is associated with V, and let $\{M^{\mathbf{u}}\}_{\mathbf{u}\in\mathbb{N}^n}$ be an n-parameter filtration of M. Then, persistence modules $\{H_q(X^{\mathbf{u}}), \iota_q^{\mathbf{u},\mathbf{v}}\}_{\mathbf{u}\leq\mathbf{v}\in\mathbb{N}^n}$ and $\{H_q(M^{\mathbf{u}}), \iota_q^{\mathbf{u},\mathbf{v}}\}_{\mathbf{u}\leq\mathbf{v}\in\mathbb{N}^n}$, where $\iota_q^{\mathbf{u},\mathbf{v}}$ are the induced inclusion maps for all $\mathbf{u},\mathbf{v}\in\mathbb{N}^n$, are isomorphic.

Proof. Let $\mathbf{u}\leq\mathbf{v}\in\mathbb{N}^n$, then we know that there exists $\Phi_{\mathbf{u}}$ which induces isomorphisms on the q-th homology groups of $X^{\mathbf{u}}$ and $M^{\mathbf{u}}$ for all $\mathbf{u}\in\mathbb{N}^n$ for all q by Lemma 2.3.1. Moreover, the diagram 2.3 commutes for all $\mathbf{u}\leq\mathbf{v}\in\mathbb{N}^n$ by Lemma 2.3.1. This implies that

$$\begin{array}{ccc} H_q(X^{\mathbf{u}}) & \xhookrightarrow{\iota_q^{\mathbf{u},\mathbf{v}}} & H_q(X^{\mathbf{v}}) \\ \downarrow{\scriptstyle \Phi_{\mathbf{u}}} & & \downarrow{\scriptstyle \Phi_{\mathbf{v}}} \\ H_q(M^{\mathbf{u}}) & \xhookrightarrow{\iota_q^{\mathbf{u},\mathbf{v}}} & H_q(M^{\mathbf{v}}) \end{array} \tag{2.4}$$

commutes, since

$$\begin{aligned} (i^M_{\mathbf{u},\mathbf{v}})_* \circ (\Phi_{\mathbf{u}})_* &= (i^M_{\mathbf{u},\mathbf{v}} \circ \Phi_{\mathbf{u}})_* \\ &= (\Phi_{\mathbf{v}} \circ i^X_{\mathbf{u},\mathbf{v}})_* \\ &= (\Phi_{\mathbf{v}})_* \circ (i^X_{\mathbf{u},\mathbf{v}})_*, \end{aligned}$$

and we know that $(\Phi_{\mathbf{u}})_*(H_q(X^{\mathbf{u}})) \cong H_q(M^{\mathbf{u}})$ for all $\mathbf{u}\in\mathbb{N}^n$. Hence, these two imply that $\{H_q(X^{\mathbf{u}}), \iota_q^{\mathbf{u},\mathbf{v}}\}_{\mathbf{u}\leq\mathbf{v}\in\mathbb{N}^n}$ and $\{H_q(M^{\mathbf{u}}), \iota_q^{\mathbf{u},\mathbf{v}}\}_{\mathbf{u}\leq\mathbf{v}\in\mathbb{N}^n}$ are isomorphic n-parameter persistence modules.

□

2.4 Koszul Complex

Before we start this background section, note that we will use the book [15] as our source in this section. In this background, $\mathbf{F}$ denotes a fixed field, and $S = \mathbf{F}[x_1, \ldots, x_n]$ will denote the polynomial ring over $\mathbf{F}$. A product $x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n}$ will be called a monomial if $(a_1, a_2, \ldots, a_n) \in \mathbb{N}^n$ which means $(a_1, a_2, \ldots, a_n)$ is a vector of non-negative integers. The monomials $x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n}$ will be denoted by $\mathbf{x}^{\mathbf{a}}$ for $(a_1, a_2, \ldots, a_n) = \mathbf{a}$. Moreover, we know that S is a vector space over $\mathbf{F}$, so we can see it as a direct sum of its vector subspaces as follows:

$$S = \bigoplus_{\mathbf{a} \in \mathbb{N}^n} S_{\mathbf{a}}$$

such that $S_{\mathbf{a}} = \mathbf{F}\{\mathbf{x}^{\mathbf{a}}\}$. Notice that if $\mathbf{a}, \mathbf{b} \in \mathbb{N}^n$, then the product of the subspaces $S_{\mathbf{a}}$ and $S_{\mathbf{b}}$ will give us the subspace $S_{\mathbf{a}+\mathbf{b}}$, so we will say that $S = \mathbf{F}[x_1, \ldots, x_n]$ is an $\mathbb{N}^n$-graded $\mathbf{F}$-algebra.

Definition 16. If $\mathbf{a} \in \{0, 1\}^n$, then the monomial $\mathbf{x}^{\mathbf{a}}$ is said to be square-free.

In this section, our aim is to define the Koszul complex, and we will see that it is actually the minimal free resolution of $\mathbf{F} = S/\langle x_1, \ldots, x_n \rangle$. We will use free graded modules, and reduced chain complexes of simplicial complexes, so firstly let us give some definitions.

Definition 17. An S-module M is an $\mathbb{N}^n$-graded S-module if it satisfies the following conditions:

- $M = \bigoplus_{\mathbf{b} \in \mathbb{N}^n} M_{\mathbf{b}}$,
- $\mathbf{x}^{\mathbf{a}}.M_{\mathbf{b}} \subseteq M_{\mathbf{a}+\mathbf{b}}$ for all $\mathbf{a}, \mathbf{b} \in \mathbb{N}^n$.

Note that for the definition of an (abstract) simplicial complex, we will use the standart definition. However we should be careful about $\emptyset$ (empty set). Because, as a simplex its dimension is -1, and the simplicial complex $\{\emptyset\}$ which is called the irrelevant complex is not the same as $\{\}$ which is called the void complex.

Definition 18. The (reduced) chain complex of a simplicial complex of Δ on $\{1, \ldots, n\}$ over $\mathbf{F}$ is the complex $\widetilde{C}_*(\Delta; \mathbf{F})$:

$$0 \longrightarrow \mathbf{F}^{F_{n-1}(\Delta)} \xrightarrow{\partial_{n-1}} \mathbf{F}^{F_{n-2}(\Delta)} \longrightarrow \cdots \longrightarrow \mathbf{F}^{F_0(\Delta)} \xrightarrow{\partial_0} \mathbf{F}^{F_{-1}(\Delta)} \longrightarrow 0$$

where $F_i(\Delta) = \{\sigma \in \Delta \mid |\sigma| = i+1\}$, and $\mathbf{F}^{F_i(\Delta)}$ is the vector space over $\mathbf{F}$ whose basis is $\{e_\sigma \mid \sigma \in F_i(\Delta)\}$ such that $e_\sigma = \sum_{j\in\sigma} e_j$ where e_j's are the standart basis elements of $\mathbb{N}^n$. Moreover, if $\sigma \in F_i(\Delta)$, then

$$\partial_i(e_\sigma) = \sum_{j\in\sigma} sign(j,\sigma) e_{\sigma\setminus\{j\}}$$

where sign$(j,\sigma) = (-1)^{r-1}$ whenever the index of j is equal to r in the set σ.

Note 3. Sometimes we use sign(j, e_σ) instead of sign(j,σ).

Definition 19. A chain complex of free S-modules

$$F_* : 0 \longrightarrow F_l \xrightarrow{\Phi_l} F_{l-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{\Phi_1} F_0 \longrightarrow 0$$

is called a free resolution of a module M if coker$(\Phi_1) = M$, and $H_i(F_*) = 0$ for all $i > 0$.

Normally, we know that a free S-module of finite rank is isomorphic to S^r for some $r > 0$. However, we are using $\mathbb{N}^n$-graded S-modules, so an $\mathbb{N}^n$-graded free S-module F will be as follows:

$$F \cong S(-\mathbf{a_1}) \oplus S(-\mathbf{a_2}) \oplus \cdots \oplus S(-\mathbf{a_n}) \tag{2.5}$$

where $S(-\mathbf{a})$ denotes the free graded S-module generated in degree $\mathbf{a}$ for any $\mathbf{a} \in \mathbb{N}^n$, so we can see that $S(-\mathbf{a}) \cong \langle \mathbf{x^a} \rangle$ as an $\mathbb{N}^n$-graded S-module. Also, we will denote the basis element of $S(-\mathbf{a})$ by $1_\mathbf{a}$. Moreover, if F_i's are $\mathbb{N}^n$-graded free S-modules in the above definition, then the maps between these graded free modules must be degree preserving which means that $\Phi_i((F_i)_\mathbf{a}) \subseteq (F_{i-1})_\mathbf{a}$ for any $\mathbf{a} \in \mathbb{N}^n$.

Theorem 2.4.1. *(Hilbert Syzygy Theorem) [10] If $S = \mathbf{F}[x_1, \ldots, x_n]$ is a polynomial ring over $\mathbf{F}$, and M is a module over S, then M has a free resolution whose length is less than or equal to n.*

Before passing to the next definition, let us use the symbol "$\leq$" to denote a partial order relation on $\mathbb{N}^n$ as follows: If $\mathbf{a} = (a_1, \ldots, a_n), \mathbf{b} = (b_1, \ldots, b_n) \in \mathbb{N}^n$, then $\mathbf{a} \leq \mathbf{b}$ if and only if $a_i \leq b_i$ for all $i \in \{1, \ldots, n\}$.

Definition 20. A monomial matrix is a matrix whose entries are scalars $\lambda_{pq} \in \mathbf{F}$, and its columns and rows are labeled by its source degrees $\mathbf{a}_p$, and its target degrees $\mathbf{a}_q$, respectively. Moreover, $\lambda_{pq} = 0$ unless $\mathbf{a}_q \leq \mathbf{a}_p$.

Remark 2.4.1. The general monomial matrix represents a map that looks like

$$\bigoplus_p S(-\mathbf{a}_p) \xrightarrow{\varphi} \bigoplus_q S(-\mathbf{a}_q)$$

where
$$\varphi = \begin{array}{cc} & \begin{array}{ccccc} \bullet & \bullet & \mathbf{a_p} & \bullet & \bullet \end{array} \\ \begin{array}{c} \bullet \\ \bullet \\ \mathbf{a_q} \\ \bullet \\ \bullet \end{array} & \left(\begin{array}{ccccc} & & & & \\ & & & & \\ & & \lambda_{pq} & & \\ & & & & \\ & & & & \end{array} \right) \end{array}.$$

Note that the entry λ_{pq} on the above monomial matrix means that this matrix sends the basis element of $S(-\mathbf{a}_p)$ to an element of $S(-\mathbf{a}_q)$ whose coefficient is λ_{pq}, and whose monomial is $\mathbf{x}^{\mathbf{a}_p - \mathbf{a}_q}$ times the basis element of $S(-\mathbf{a}_q)$. This means that we send an element from degree $\mathbf{a}_p$ to degree $\mathbf{a}_p$, so monomial matrix is degree-preserving. Moreover, the exponent of $\lambda^{\mathbf{a}_p - \mathbf{a}_q}$ is non-negative thanks to the requirement in the definition of monomial matrices. As an additional note, sometimes we will use $\mathbf{x}^{\mathbf{a}}$ instead of $\mathbf{a}$ for the labelling of a monomial matrix.

Definition 21. A monomial matrix is called minimal if $\lambda_{pq} = 0$ when $\mathbf{a}_q = \mathbf{a}_p$. Also, we call a chain complex minimal if its maps can be written as minimal monomial matrices.

The above definition tells us that if a homomorphism of a resolution can be written as minimal monomial matrices, then we will call it a minimal resolution.

Now, we are ready to give the main definition of this section.

Definition 22. Let Δ^n be the simplicial complex which consists of all the subsets of $\{1, \ldots, n\}$, and let $\widetilde{C}_*(\Delta^n, S)$ be the (reduced) chain complex of Δ^n over S. Now, label the columns and rows of the boundary matrices of the (reduced) chain complex of Δ^n with $\mathbf{x}^\sigma$ for the corresponding basis elements e_σ. After renumbering the homological degree of $\widetilde{C}_*(\Delta^n, S)$ so that $\emptyset$ sits in homological degree 0, the resulting chain complex is called the Koszul complex of free n-graded S-modules, and is denoted by $\mathbf{K}_*$.

Definition 23. The support of the vector $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{N}^n$ is

$$\{i \in \{1, 2, \ldots, n\} \mid a_i \neq 0\}.$$

Remark 2.4.2. In this background section, we will denote the set $\{1, 2, \ldots, n\}$ by [n] from now on.

Example 2.4.1. Now, let us give an example of the Koszul complex when $n = 3$.

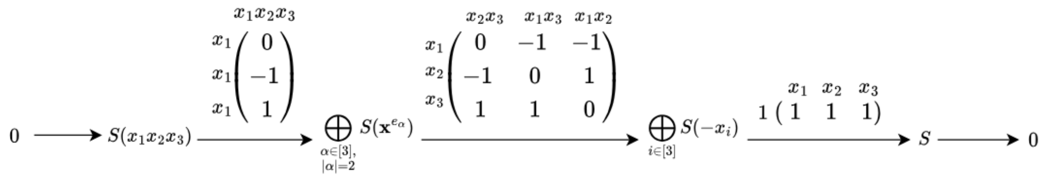

Figure 2.7: Koszul complex when $n = 3$ where $[3] = \{1, 2, 3\}$.

Lemma 2.4.2. Let $\mathbf{K}_*$ be the Koszul complex of free n-graded S-modules, and $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{N}^n$, then $(\mathbf{K}_*)_\mathbf{a}$ is isomorphic to $\widetilde{C}_{*-1}(\sigma; \mathbf{F})$ where σ is the support of $\mathbf{a}$.

Proof. Observe that $(\mathbf{K_i})_\mathbf{a} = \bigoplus_{\substack{\alpha\subseteq[n], \\ |\alpha|=i}} S(-e_\alpha)_\mathbf{a}$, and notice that if α is not a subset of σ, then $S(-e_\alpha)_{e_\sigma} = 0$, and this implies that $S(-e_\alpha)_\mathbf{a} = 0$, so

$$\begin{aligned}
(\mathbf{K_i})_\mathbf{a} &= \bigoplus_{\substack{\alpha\subseteq\sigma, \\ |\alpha|=i}} S(-e_\alpha)_\mathbf{a} \\
&= \bigoplus_{\substack{\alpha\subseteq\sigma, \\ |\alpha|=i}} \mathbf{F}\{\mathbf{x}^{\mathbf{a}-e_\alpha}.1_{e_\alpha}\} \\
&= \bigoplus_{\substack{\alpha\subseteq\sigma, \\ |\alpha|=i}} \mathbf{F}\{\mathbf{x}^\mathbf{a}\} \\
&= \mathbf{F}^{F_{i-1}(\sigma)} \\
&= \widetilde{C}_{i-1}(\sigma; \mathbf{F}).
\end{aligned}$$

Also, let φ_i be the i-th monomial matrix of $\mathbf{K}_*$, and let $(\varphi_i)_\mathbf{a} := \widetilde{\partial}_i$, then notice that

$$\widetilde{\partial}_i(\mathbf{x}^{\mathbf{a}-e_\alpha}.1_{e_\alpha}) = \sum_{j\in\alpha} sign(j, \alpha)\mathbf{x}^{\mathbf{a}-e_\alpha}\mathbf{x}^{e_j}.1_{e_\alpha - e_j}$$

and

$$\partial_{i-1}(e_\alpha) = \sum_{j\in\alpha} sign(j,\alpha)e_{\alpha\setminus\{j\}}.$$

So, if $(f_*)_{\mathbf{a}} : (\mathbf{K}_*)_{\mathbf{a}} \longrightarrow \widetilde{C}_{*-1}(\sigma, \mathbf{F})$ where it sends $x^{\mathbf{a}-e_\alpha}.1_{e_\alpha}$ to e_α for all α, then we can easily see that $(f_*)_{\mathbf{a}}$ is a chain isomorphism by the above observation.

□

Theorem 2.4.3. *The Koszul complex* $\mathbf{K}_*$ *of free* n*-graded* S*-modules is the minimal free resolution of* $\mathbf{F} = S/\langle x_1, \dots, x_n\rangle$ *where* $\langle x_1, \dots, x_n\rangle$ *is a maximal ideal of* S*.*

Proof. We have to show that $\mathbf{K}_*$ satisfies three conditions in order to see that it is actually the minimal resolution of $\mathbf{F} = S/\langle x_1, \dots, x_n\rangle$. Firstly, $H_0(\mathbf{K}_*)$ must be $\mathbf{F} = S/\langle x_1, \dots, x_n\rangle$, but we know that $\mathbf{K_1} = \bigoplus_{j\in[n]} S(-e_j)$, and $\mathbf{K_0} = S$, and the monomial matrix between $\mathbf{K_1}$ and $\mathbf{K_0}$ send 1_{e_j} to $\mathbf{x}^{e_j}.1 = x_j$. Thus, $H_0(\mathbf{K}_*) = S/\langle x_1, \dots, x_n\rangle = \mathbf{F}$. Secondly, if $\mathbf{a} \in \mathbb{N}^n$, and σ is the support of $\mathbf{a}$, then we know that $(\mathbf{K}_*)_{\mathbf{a}} \cong (\widetilde{C}_{*-1}(\sigma;\mathbf{F}))_{\mathbf{a}}$ by Lemma 2.4.2, so $H_i((\mathbf{K}_*)_{\mathbf{a}}) \cong H_i((\widetilde{C}_{*-1}(\sigma,\mathbf{F})))$ for all i, but we know that $H_i((\widetilde{C}_{*-1}(\sigma,\mathbf{F}))) = 0$ for all $i > 0$, since σ is contractible because it is a simplex. Thus, second condition also holds. These two results tell us that $\mathbf{K}_*$ is actually a free resolution of $\mathbf{F} = S/\langle x_1, \dots, x_n\rangle$. Lastly, we know that source degrees and target degrees of any monomial matrix of $\mathbf{K}_*$ are different, so it is trivially minimal. Hence, $\mathbf{K}_*$ is the minimal free resolution of $\mathbf{F} = S/\langle x_1, \dots, x_n\rangle$.

□

Definition 24. Let M be an n-graded S-module which is finitely generated, and $\mathcal{F}_*$ be the free minimal resolution of M, and $F_i = \bigoplus_{\mathbf{a}\in\mathbb{N}^n} S(-\mathbf{a})^{\beta_{i,\mathbf{a}}}$, then we call $\beta_{i,\mathbf{a}} = \beta_{i,\mathbf{a}}(M)$ the i-th Betti number of M in degree $\mathbf{a}$.

Note that this is one way to find the $\mathbb{N}^n$-graded Betti numbers of a finitely generated n-graded S-module, but we can also find them with using the Tor functor. Now, let us try to find Betti numbers in that way, but we have to give some extra definitions, and use tensor products for this.

If M and N are two $\mathbb{N}^n$-graded S-modules, then $M \otimes N$ is also an $\mathbb{N}^n$-graded S-module, and $(M \otimes N)_{\mathbf{c}}$ is generated by the set

$$\{m_{\mathbf{a}} \otimes n_{\mathbf{b}} \mid m_{\mathbf{a}} \in M_{\mathbf{a}}, n_{\mathbf{b}} \in N_{\mathbf{b}}, \mathbf{a}+\mathbf{b} = \mathbf{c}\}.$$

Definition 25. If M is an n-graded S-module, then the n-graded S-module $S(-\mathbf{a})\otimes_S M$ is called the $\mathbb{N}^n$-graded translate of M by $\mathbf{a}$, and it is denoted by $M(-\mathbf{a})$.

Proposition 2.4.4. If M is an $\mathbb{N}^n$-graded S-module, then $M(-\mathbf{a})_{\mathbf{b}} \cong M_{\mathbf{b}-\mathbf{a}}$.

Proof. Let $\alpha \otimes \beta \in M(-\mathbf{a})_{\mathbf{b}} = S(-\mathbf{a}) \otimes M$ where $\alpha \in S(-\mathbf{a})_{\mathbf{u}}, \beta \in M_{\mathbf{v}}$ such that $\mathbf{u} + \mathbf{v} = \mathbf{b}$. Now, $\alpha = k.\mathbf{x}^{\mathbf{u}-\mathbf{a}}.1_{\mathbf{a}}$ such that $k \in \mathbf{F}$ since $\alpha \in S(-\mathbf{a})$, so $\alpha \otimes \beta = k.\mathbf{x}^{\mathbf{u}-\mathbf{a}}.1_{\mathbf{a}} \otimes \beta = 1_{\mathbf{a}} \otimes k.\mathbf{x}^{\mathbf{u}-\mathbf{a}}.\beta$ where $k.\mathbf{x}^{\mathbf{u}-\mathbf{a}}.\beta \in M_{\mathbf{b}-\mathbf{a}}$. Thus, $M(-\mathbf{a})_{\mathbf{b}}$ is generated by the set $\{1_{\mathbf{a}} \otimes m \mid m \in M_{\mathbf{b}-\mathbf{a}}\}$. Thus, $M(-\mathbf{a})_{\mathbf{b}} \cong 1_{\mathbf{a}} \otimes M_{\mathbf{b}-\mathbf{a}}$, and we know that $1_{\mathbf{a}} \otimes M_{\mathbf{b}-\mathbf{a}} \cong M_{\mathbf{b}-\mathbf{a}}$. Hence, $M(-\mathbf{a})_{\mathbf{b}} \cong M_{\mathbf{b}-\mathbf{a}}$. □

Corollary 2.4.5. $S(-\mathbf{a}) \otimes \mathbf{F} = \mathbf{F}(-\mathbf{a})$ will give us a copy of the field $\mathbf{F}$ at the grade $\mathbf{a}$.

Proof. We know that $\mathbf{F} = S/\langle x_1, \dots, x_n\rangle$, and

$$(\mathbf{F}(-\mathbf{a}))_{\mathbf{b}} = \mathbf{F}_{\mathbf{b}-\mathbf{a}} = (S/\langle x_1, \dots, x_n\rangle)_{\mathbf{b}-\mathbf{a}}$$

by the Theorem 2.4.3. Also,

$$(S/\langle x_1, \dots, x_n\rangle)_{\mathbf{b}-\mathbf{a}} = (S)_{\mathbf{b}-\mathbf{a}}/(\langle x_1, \dots, x_n\rangle)_{\mathbf{b}-\mathbf{a}}.$$

Moreover, we know that $\langle x_1, \dots, x_n\rangle$ is a maximal ideal of S, so

$$(S)_{\mathbf{c}} = (\langle x_1, \dots, x_n\rangle)_{\mathbf{c}} \text{ for all } \mathbf{c} \in S \setminus \{\mathbf{0}\}.$$

In addition to this, $(S)_{\mathbf{0}} = \mathbf{F}$, and $(\langle x_1, \dots, x_n\rangle)_{\mathbf{0}} = 0$. Thus,

$$(S/\langle x_1, \dots, x_n\rangle)_{\mathbf{b}-\mathbf{a}} = (S)_{\mathbf{b}-\mathbf{a}}/(\langle x_1, \dots, x_n\rangle)_{\mathbf{b}-\mathbf{a}}$$

is equal to $\mathbf{F}$ if $\mathbf{b} = \mathbf{a}$, and it is equal to $\mathbf{0}$ otherwise. Hence, $\mathbf{F}_{\mathbf{b}-\mathbf{a}}$ gives us a copy of the field $\mathbf{F}$ at the grade $\mathbf{a}$.

□

As a reminder, if we apply the functor $\_\_ \otimes N$ to a free resolution of M, and take its i-th homology, then the resulting module will be $Tor_i^S(M, N)$. Moreover, we can also find the same module by applying the functor $M \otimes \_\_$ to a free resolution of N, and taking its i-th homology. For more information on Tor, see [17]. Now, we are ready to give the following theorem.

Theorem 2.4.6. *If M is an $\mathbb{N}^n$ graded S-module, then*

$$\beta_{i,\mathbf{a}}(M) = dim_{\mathbf{F}}(Tor_i^S(\mathbf{F}, M)_{\mathbf{a}}).$$

Proof. Let $\mathcal{F}_*$ be the minimal free resolution of M, and $\varphi_i : \bigoplus_{\mathbf{a}\in\mathbb{N}^n} S(-\mathbf{a})^{\beta_{i,\mathbf{a}}} \longrightarrow \bigoplus_{\mathbf{a}\in\mathbb{N}^n} S(-\mathbf{a})^{\beta_{i-1,\mathbf{a}}}$ be the i-th monomial matrix of $\mathcal{F}_*$. Let λ_{pq} denote an entry of this matrix whose source degree is $\mathbf{a}_p$ and target degree is $\mathbf{a}_q$. We know that $\lambda_{pq} = 0$ whenever $\mathbf{a}_p \leq \mathbf{a}_q$ since φ_i is a monomial matrix. Also, φ_i must be a minimal monomial matrix since $\mathcal{F}_*$ is a minimal free resolution of M by the definition, and this implies that $\lambda_{pq} = 0$ whenever $\mathbf{a}_p = \mathbf{a}_q$. In addition to these, we know that if $\mathbf{a}_q < \mathbf{a}_p$, then φ_i sends 1_p to $\mathbf{x}^{p-q}.1_q$ by the definition of a monomial matrix. Now, if we tensor $\mathcal{F}_*$ with $\mathbf{F} = S/\langle x_1, \ldots, x_n\rangle$, then we can see that

$$\begin{aligned} \varphi_i \otimes id(1_p \otimes k) &= \mathbf{x}^{p-q}.1_q \otimes k \\ &= 1_q \otimes \mathbf{x}^{p-q}k \\ &= 1_q \otimes 0 \\ &= 1_q \otimes 0 \\ &= 0 \end{aligned}$$

for all $k \in \mathbf{F} = S/\langle x_1, \ldots, x_n\rangle$. This implies that $\varphi_i \otimes id = 0$ for all i. Thus,

$$\begin{aligned} Tor_i^S(\mathbf{F}, M) &= \bigoplus_{\mathbf{a}\in\mathbb{N}^n} S(-\mathbf{a})^{\beta_{i,\mathbf{a}}} \otimes_S \mathbf{F} \\ &= \bigoplus_{\mathbf{a}\in\mathbb{N}^n} (S(-\mathbf{a})^{\beta_{i,\mathbf{a}}} \otimes \mathbf{F}) \\ &\cong \bigoplus_{\mathbf{a}\in\mathbb{N}^n} \mathbf{F}(-\mathbf{a})^{\beta_{i,\mathbf{a}}}. \end{aligned}$$

Hence, $dim_{\mathbf{F}}(Tor_i^S(\mathbf{F}, M)_{\mathbf{a}}) = dim_{\mathbf{F}}(\mathbf{F}(-\mathbf{a})^{\beta_{i,\mathbf{a}}}) = \beta_{i,\mathbf{a}}$ by Corollary 2.4.5.

□

Definition 26. Let M be an n-graded S-module, and $\mathbf{K}_*$ be the Koszul complex of free n-graded S-modules, then we call $(M \otimes_S \mathbf{K}_*)_{\mathbf{a}}$ the Koszul complex of M at the grade $\mathbf{a} \in \mathbb{N}^n$, and it is denoted by $\mathbf{K}_*(x_1, \ldots, x_n; M)(\mathbf{a})$.

Now, we know that $\beta_{i,\mathbf{a}}(M) = dim_{\mathbf{F}} Tor_i^S(\mathbf{F}, M)_{\mathbf{a}}$ for an $\mathbb{N}^n$-graded S-module, but we also know that we can calculate $Tor_i^S(\mathbf{F}, M)$ by applying the minimal free resolution of $\mathbf{F} = S/\langle x_1, \ldots, x_n\rangle$ to $\_\_ \otimes M$ as we mentioned above, and we know that the minimal free resolution of $\mathbf{F} = S/\langle x_1, \ldots, x_n \rangle$ is the Koszul complex of S by Theorem 2.4.3. Thus,

$$\begin{aligned} Tor_i^S(\mathbf{F}, M)_{\mathbf{a}} &= H_i((\mathbf{K}_* \otimes_S M)_{\mathbf{a}}) \\ &\cong H_i((M \otimes_S \mathbf{K}_*)_{\mathbf{a}}) \\ &\cong H_i(\mathbf{K}_*(x_1, \ldots, x_n; M)(\mathbf{a})). \end{aligned}$$

Hence,

$$\begin{aligned} \beta_{i,\mathbf{a}}(M) &= dim_{\mathbf{F}}(Tor_i^S(\mathbf{F}, M)_{\mathbf{a}}) \\ &= dim_{\mathbf{F}}(H_i(\mathbf{K}_*(x_1, \ldots, x_n; M)(\mathbf{a})). \end{aligned}$$

On the other hand, if M is an $\mathbb{N}^n$-graded S-module, and $\mathbf{K}_*(x_1, \ldots, x_n; M)(\mathbf{a})$ its Koszul complex at the grade $\mathbf{a} \in \mathbb{N}^n$, then we know that $\mathbf{K_i}(x_1, \ldots, x_n; M)(\mathbf{a}) = \bigoplus_{\substack{\alpha \in [n], \\ |\alpha| = i}} (S(-e_\alpha) \otimes M)_{\mathbf{a}}$, and we know that $(S(-e_\alpha) \otimes M)_{\mathbf{a}} \cong M_{\mathbf{a}-e_\alpha}$ for all $\alpha \in [n]$ by Proposition 2.4.4, so $\mathbf{K_i}(x_1, \ldots, x_n; M)(\mathbf{a}) \cong \bigoplus_{\substack{\alpha \subseteq [n], \\ |\alpha| = i}} M_{\mathbf{a}-e_\alpha}$ for all $i \in [n]$. Now, we will define a new chain complex, and we will show that this chain complex will be isomorphic to $\mathbf{K}_*(x_1, \ldots, x_n; M)(\mathbf{a})$, but firstly let us define a map d_i^M. If $z \in M_{\mathbf{a}-e_\alpha}$ such that $\mathbf{a} \in \mathbb{N}^n, \alpha \subseteq [n], |\alpha| = i$, then set

$$d_i^M(z) = \sum_{j \in \alpha} sign(j, e_\alpha)\mathbf{x}^j.z \in \bigoplus_{\substack{\alpha \subseteq [n], \\ |\alpha| = i}} \left(\bigoplus_{j \in \alpha} M_{\mathbf{a}-e_\alpha+e_j} \right) = \bigoplus_{\substack{\alpha \subseteq [n], \\ |\alpha| = i-1}} M_{\mathbf{a}-e_\alpha}, \qquad (2.6)$$

then just extend d_i linearly to a map

$$d_i^M : \bigoplus_{\substack{\alpha \subseteq [n], \\ |\alpha| = i}} M_{\mathbf{a}-e_\alpha} \longrightarrow \bigoplus_{\substack{\alpha \subseteq [n], \\ |\alpha| = i-1}} M_{\mathbf{a}-e_\alpha}.$$

Proposition 2.4.7. $d_{i-1}^M \circ d_i^M = 0$.

Proof. Since we obtain d_i^M by taking linear extensions of 2.6, it is enough to take an element from a summand of $\bigoplus_{\substack{\alpha \subseteq [n], \\ |\alpha| = i}} M_{\mathbf{a}-e_\alpha}$, and show that $d_{i-1}^M \circ d_i^M$ sends it to 0. Now, let $z \in M_{\mathbf{a}-e_\alpha}$ such that $\alpha \subseteq [n], |\alpha| = i$, and let $\alpha = \{j_1 < j_2 < \cdots < j_{i-1} < j_i\}$, then

$$
\begin{aligned}
d_{i-1}^M(d_i^M(z)) &= d_{i-1}^M\left(\sum_{j_l\in\alpha} sign(j_l,e_\alpha)\mathbf{x}^{j_l}.z\right)\\
&= \sum_{j_l\in\alpha} sign(j_l,e_\alpha)d_{i-1}^M(\mathbf{x}^{j_l}.z)\\
&= \sum_{j_l\in\alpha}\left(sign(j_l,e_\alpha)\sum_{j_k\in\alpha\setminus\{j_l\}} sign(j_k,e_{\alpha\setminus\{j_l\}})\right)\mathbf{x}^{j_k}(\mathbf{x}^{j_l}.z)\\
&= \sum_{j_k<j_l} sign(j_k,e_{\alpha\setminus\{j_l\}})sign(j_l,e_\alpha)(\mathbf{x}^{j_k}\mathbf{x}^{j_l}.z)\\
&+ \sum_{j_k>j_l} sign(j_k,e_{\alpha\setminus\{j_l\}})sign(j_l,e_\alpha)(\mathbf{x}^{j_k}\mathbf{x}^{j_l}.z)\\
&= \sum_{j_k<j_l}(-1)^{k-1}(-1)^{l-1}(\mathbf{x}^{j_k}\mathbf{x}^{j_l}.z)\\
&+ \sum_{j_k>j_l}(-1)^{k-1-1}(-1)^{l-1}(\mathbf{x}^{j_k}\mathbf{x}^{j_l}.z)\\
&= 0.
\end{aligned}
$$

□

At the moment, we know that $d_{i-1}^M \circ d_i^M = 0$ by the previous proposition, so we have the following chain complex:

$$\mathscr{K}(n,M)(\mathbf{a}) : 0 \longrightarrow K_n \xrightarrow{d_n^M} K_{n-1} \xrightarrow{d_{n-1}^M} \dots \xrightarrow{d_2^M} K_1 \xrightarrow{d_1^M} K_0 \longrightarrow 0. \tag{2.7}$$

where $K_i = \bigoplus_{\substack{\alpha\subseteq[n],\\ |\alpha|=i}} M_{\mathbf{a}}$.

Theorem 2.4.8. *If M is an N^n-graded S-module, then*

$$\mathbf{K}_*(x_1,\dots,x_n;M)_{\mathbf{a}} \cong \mathscr{K}(n,M)(\mathbf{a}) \tag{2.8}$$

for all $\mathbf{a}\in\mathbb{N}^n$.

Proof. Let $\mathbf{a}\in\mathbb{N}^n$, let φ_i denote the i-th chain map of $\mathbf{K}_*$ for all $i\in\{1,\dots,n\}$, then the i-th chain map of $\mathbf{K}_*(x_1,\dots,x_n;M)_{\mathbf{a}}$ will be

$$(\varphi_i)_{\mathbf{a}}\otimes id : \left(\bigoplus_{\substack{\alpha\subseteq[n],\\ |\alpha|=i}} S(-e_\alpha)\otimes M\right)_{\mathbf{a}} \longrightarrow \left(\bigoplus_{\substack{\alpha\subseteq[n],\\ |\alpha|=i-1}} S(-e_\alpha)\otimes M\right)_{\mathbf{a}}.$$

By the proof of Proposition 2.4.4, we know that $(S(-e_\alpha) \otimes M)_{\mathbf{a}}$ is generated by the set $\{1_{e_\alpha} \otimes m \mid m \in M_{\mathbf{a}-e_\alpha}\}$. Observe that if $m \in M_{\mathbf{a}-e_\alpha}$ such that $\alpha \subseteq n, |\alpha| = i$, then

$$
\begin{aligned}
(\varphi_i)_{\mathbf{a}} \otimes id(1_{e_\alpha} \otimes m) &= \sum_{j\in\alpha} sign(j, e_\alpha)\mathbf{x}^j 1_{e_\alpha - e_j} \otimes m \\
&= \sum_{j\in\alpha} (sign(j, e_\alpha)\mathbf{x}^j 1_{e_\alpha - e_j} \otimes m) \\
&= \sum_{j\in\alpha} (1_{e_{\alpha\setminus\{j\}}} \otimes sign(j, e_\alpha)\mathbf{x}^j.m).
\end{aligned}
$$

Now, let $\Phi : \bigoplus_{\substack{\alpha\subseteq[n],\\ |\alpha|=i}} (S(-e_\alpha) \otimes M)_{\mathbf{a}} \longrightarrow \bigoplus_{\substack{\alpha\subseteq[n],\\ |\alpha|=i}} (M_{\mathbf{a}-e_\alpha})$ where $\Phi_i(1_{e_\alpha} \otimes m) = m$ for all $m \in M_{\mathbf{a}-e_\alpha}$. This map is an isomorphism by Proposition 2.4.4. Moreover,

$$
\begin{aligned}
\Phi_{i-1}(((\varphi_i)_{\mathbf{a}} \otimes id)(1_{e_\alpha} \otimes m)) &= \Phi_{i-1}\left(\sum_{j\in\alpha} 1_{e_{\alpha\setminus\{j\}}} \otimes sign(j, e_\alpha)\mathbf{x}^j.m\right) \\
&= \sum_{j\in\alpha} sign(j, e_\alpha)\mathbf{x}^j.m \\
&= d_i^M(m) \\
&= d_i^M(\Phi_i(1_{e_\alpha} \otimes m))
\end{aligned}
$$

which implies $\Phi_{i-1}((\varphi_i)_{\mathbf{a}} \otimes id) = d_i^M(\Phi_i)$. Hence,

$$\mathbf{K}_*(x_1, \ldots, x_n; M)(\mathbf{a}) \cong \mathscr{K}(n, M)(\mathbf{a})$$

. □

CHAPTER 3

KOSZUL COMPLEXES AND PERSISTENCE MODULES

In the first section of this chapter, we will give an equivalent definition for the Betti tables of a persistence module by using the Koszul complex defined in 2.4. In the second section, we will iteratively construct the Koszul complex of a persistence module by using mapping cones. The source for this chapter is [13].

3.1 The Koszul Complex of a Persistence Module

Before starting our main chapter, let us fix some notations in this thesis. We will denote the set $\{1, \ldots, n\}$ by [n]. For any $\alpha \subseteq [n]$, $e_\alpha := \sum_{j \in \alpha} e_j$ where e_i's denote the i-th standard basis of $\mathbb{N}^n$ which means the i-th coordinate of e_i is 1, and its other coordinates are 0. In addition to these, we use the symbol "$\leq$" to denote a partial order relation on $\mathbb{N}^n$ as follows: If $\mathbf{a} = (a_1, \ldots, a_n), \mathbf{b} = (b_1, \ldots, b_n) \in \mathbb{N}^n$, then $\mathbf{a} \leq \mathbf{b}$ if and only if $a_i \leq b_i$ for all $i \in [n]$.

As in the background sections, we use $\mathbf{F}$ as a fixed field. Let X be a finite cell complex. Also, we assume that, once and for all the remainder of this thesis, any n-parameter filtration of X is a 1-critical filtration (see Definition 8).

Definition 27. Let σ be a cell of the cell complex X. If $\sigma \in X^{\mathbf{u}} - \bigcup_{j=1}^{n} X^{\mathbf{u}-e_j}$, then we call $\mathbf{u}$ the **entrance grade** of σ in the filtration.

Notation 3.1.1. The symbol $\wedge$ will denote the greatest lower bound, and $\vee$ will denote the least upper bound.

Remark 3.1.1. The assumption of one-criticality of an n-filtration $\{X^{\mathbf{u}}\}_{\mathbf{u} \in \mathbb{N}^n}$ is the same as assuming that each cell of X has a unique entrance grade in the n-filtration

$\{X^{\mathbf{u}}\}_{\mathbf{u}\in\mathbb{N}^n}$. If $\{U_j\}_{j=1,\ldots,k}$ is a finite set of n-parameter filtration grades such that $\mathbf{u}_j = (u_{j,1}, \ldots, u_{j,n})$ for all j, then $\bigcap_{j=1}^{k} X^{\mathbf{u}_j} = X^{\wedge\{\mathbf{u}_j\}_j} = X^{(min\{u_{j,1}\}_j,\ldots,min\{u_{j,n}\}_j)}$, and this tells us that $\bigcap_{j\in\alpha} X^{\mathbf{u}-e_j} = X^{\mathbf{u}-e_\alpha}$ where $\alpha \subseteq [n]$.

In topological data analysis, persistence modules are one of the main tools, and it is common to obtain an n-parameter persistence module from an n-filtration by applying the homology functor. If $\{X^{\mathbf{u}}\}_{\mathbf{u}\in\mathbb{N}^n}$ is an n-filtration, then $\{H_q(X^{\mathbf{u}})\}_{\mathbf{u}\in\mathbb{N}^n}$ will be an n-parameter persistence module with the set of linear transformations $\{\iota_q^{\mathbf{u},\mathbf{v}}\}_{\mathbf{u}\le\mathbf{v}\in\mathbb{N}^n}$ where $\iota_q^{\mathbf{u},\mathbf{v}}$'s are linear transformations which are induced by the inclusion maps $X^{\mathbf{u}} \hookrightarrow X^{\mathbf{v}}$ for all $\mathbf{u} \le \mathbf{v}$.

In this work, we will study the n-parameter persistence modules which are obtained from the 1-critical filtrations by applying the homology functor, and we will be interested in their Betti tables.

Now, let $S = \mathbf{F}[x_1, \ldots, x_n]$,and let $\{H_q(X^{\mathbf{u}}); \iota_q^{\mathbf{u},\mathbf{v}}\}$ be the n-parameter persistence module which is obtained by applying the q-th homology functor to the n-filtration $\{X^{\mathbf{u}}\}_{\mathbf{u}\in\mathbb{N}^n}$. We know that there is a 1-1 correspondence between n-graded S-modules, and n-parameter persistence modules by [6]. Thus, we can see our n-parameter persistence module $\{H_q(X^{\mathbf{u}}); \iota_q^{\mathbf{u},\mathbf{v}}\}$ as the n-graded S-module $V_q = \bigoplus_{\mathbf{u}\in\mathbb{N}^n} H_q(X^{\mathbf{u}})$ with the following action $x_i.z = \iota_q^{\mathbf{u},\mathbf{u}+e_i}(z)$. Thus, we can define the Betti tables of $\{H_q(X^{\mathbf{u}}); \iota_q^{\mathbf{u},\mathbf{v}}\}$ by using its Koszul complex (see Section 2.4) as a function $\xi_i^q : \mathbb{N}^n \to \mathbb{N}$ such that

$$\begin{aligned}\xi_i^q(\mathbf{u}) &= dim_{\mathbf{F}}(Tor_i^S(V_q, \mathbf{F})_{\mathbf{u}}) \\ &= dim_{\mathbf{F}}(H_i(\mathbf{K}_*(x_1, \ldots, x_n; V_q)(\mathbf{u})) \\ &= dim_{\mathbf{F}}(H_i(\mathscr{K}_*(n, V_q)(\mathbf{u}))).\end{aligned}$$

From Section 2.4, we know that $\mathbf{K}_*(x_1, \ldots, x_n; V_q)(\mathbf{u})$ is isomorphic to $\mathscr{K}(n, V_q)(\mathbf{u})$, so we can use

$$\mathbf{K_i}(x_1, \ldots, x_n; V_q)(\mathbf{u}) = \bigoplus_{\substack{\alpha\subseteq[n],\\ |\alpha|=i}} H_q(X^{\mathbf{u}-e_\alpha}),$$

and $d_i^{V_q}$ as the i-th chain maps of $\mathbf{K_i}$. Remember from the Section 2.4 that $d_i^{V_q}$ is the i-th chain map of $\mathscr{K}_*(n, V_q)(\mathbf{u})$, and $\mathscr{K}_i(n, V_q)(\mathbf{u}) = \bigoplus_{\substack{\alpha \subseteq [n], \\ |\alpha| = i}} H_q(X^{\mathbf{u}-e_\alpha})$. As an additional note, we will use d_i instead of $d_i^{V_q}$ for the sake of notation in the remaining part of the thesis.

Now, let us give an example when $S = \mathbf{F}[x_1]$, and $S = \mathbf{F}[x_1, x_2]$.

Example 3.1.1. Firstly, let $\{X^{\mathbf{u}}\}_{\mathbf{u} \in \mathbb{N}}$ be a 1-parameter filtration, and let V_q be the 1-parameter persistence module which is obtained by applying the q-th homology functor to the $\{X^{\mathbf{u}}\}_{\mathbf{u} \in \mathbb{N}}$, so $V_q = \bigoplus_{\mathbf{u} \in \mathbb{N}} H_q(X^{\mathbf{u}})$, then

$$\mathbf{K}_*(x_1; V_q)(\mathbf{u}) : 0 \longrightarrow V_q^{\mathbf{u}-1} \xrightarrow{d_1 = \iota_q^{\mathbf{u}-1,\mathbf{u}}} V_q^{\mathbf{u}} \longrightarrow 0.$$

Hence,

$$\begin{aligned} \xi_0^q(\mathbf{u}) &= dim_{\mathbf{F}}(V_q^{\mathbf{u}} / im(d_1)) \\ &= dim_{\mathbf{F}}(coker(d_1)). \end{aligned}$$

This time let $\{X^{\mathbf{u}}\}_{\mathbf{u} \in \mathbb{N}^2}$ be a 2-parameter filtration, and V_q be the persistence module which is obtained by applying the q-th homology functor to the $\{X^{\mathbf{u}}\}_{\mathbf{u} \in \mathbb{N}^2}$, so $V_q = \bigoplus_{\mathbf{u} \in \mathbb{N}^2} H_q(X^{\mathbf{u}})$, then

$$\mathbf{K}_*(x_1, x_2; V_q)(\mathbf{u}) : 0 \longrightarrow V_q^{\mathbf{u}-e_1-e_2} \xrightarrow{d_2} V_q^{\mathbf{u}-e_1} \oplus V_q^{\mathbf{u}-e_2} \xrightarrow{d_1} V_q^{\mathbf{u}} \longrightarrow 0$$

where

$$\begin{aligned} d_1 &= [sign(1, e_1)\iota_q^{\mathbf{u}-e_1,\mathbf{u}}, sign(1, e_2)\iota_q^{\mathbf{u}-e_2,\mathbf{u}}] \\ &= [\iota_q^{\mathbf{u}-e_1,\mathbf{u}}, \iota_q^{\mathbf{u}-e_2,\mathbf{u}}], \end{aligned}$$

and

$$\begin{aligned} d_2 &= \begin{bmatrix} sign(2, e_1+e_2)\iota_q^{\mathbf{u}-e_1-e_2,\mathbf{u}-e_1} \\ sign(1, e_1+e_2)\iota_q^{\mathbf{u}-e_1-e_2,\mathbf{u}-e_2} \end{bmatrix} \\ &= \begin{bmatrix} -\iota_q^{\mathbf{u}-e_1-e_2,\mathbf{u}-e_1} \\ \iota_q^{\mathbf{u}-e_1-e_2,\mathbf{u}-e_2} \end{bmatrix}. \end{aligned}$$

So,

$$\begin{aligned} \xi_0^q(\mathbf{u}) &= dim_{\mathbf{F}}(V_q^{\mathbf{u}} / im(d_1)) \\ &= dim_{\mathbf{F}}(coker(d_1)) \end{aligned}$$

$$\xi_1^q(\mathbf{u}) = dim_{\mathbf{F}}(ker(d_1)/im(d_2))$$

$$\xi_2^q(\mathbf{u}) = dim_{\mathbf{F}}(ker(d_2)).$$

Now, let us give another example in order to better understand the concept of Betti tables.

Example 3.1.2. [14] Assume that we have the following 2-parameter filtration 3.1 of a cell complex K.

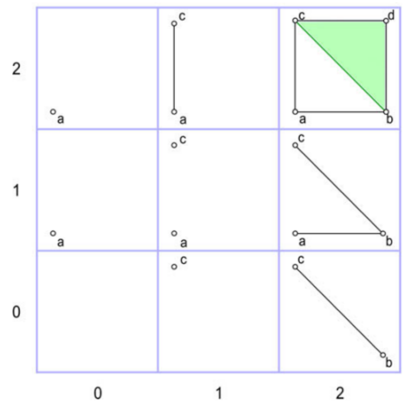

Figure 3.1: A 2-parameter filtered cell complex K.

Now, if we apply 0-th homology functor to this filtration, we know that we will obtain a persistence module which is the following

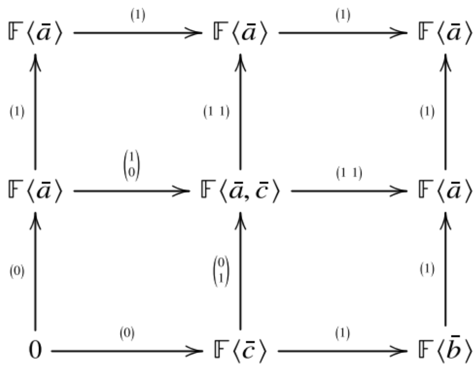

Figure 3.2: Persistence module which is obtained by applying the 0-th homology functor to 3.1

If we look at the grades of the persistence module 3.2, we can see that we denote our $\mathbb{F}$-vector spaces with their generators. Moreover, linear transformations which are induced by inclusion maps are denoted by matrices. Now, let us examine the Betti tables of this persistence module.

0	0	0
1	0	0
0	1	0

ξ_0

0	1	0
0	0	1
0	0	0

ξ_1

0	0	1
0	0	0
0	0	0

ξ_2

Figure 3.3: Betti tables of the persistence module in 3.2

The non-zero entries in the table of ξ_0 in the Figure 3.3 gives us the grades of the filtration in which there are birth of the new homological classes. In the table of ξ_1 of Figure 3.3, the non-zero entries denote the grades of the filtration in which there are death of some homological classes. Lastly, the non-zero of table of ξ_2 in the Figure 3.3 denotes the connections between the unrelated deaths of former homological classes.

Remark 3.1.2. Let $V = \{V^{\mathbf{u}}, \varphi^{\mathbf{u},\mathbf{v}}\}_{\mathbf{u}\leq\mathbf{v}\in\mathbb{N}^n}$, $W = \{W^{\mathbf{u}}, \psi^{\mathbf{u},\mathbf{v}}\}_{\mathbf{u}\leq\mathbf{v}\in\mathbb{N}^n}$ be two n-parameter persistence modules. If $\Phi = \{\Phi^{\mathbf{u}} : V^{\mathbf{u}} \to W^{\mathbf{u}}\}_{\mathbf{u}\in\mathbb{N}^n}$ is a natural transformation between these two n-parameter persistence modules, then Φ induces a chain map between $\mathbf{K}_*(x_1, \ldots, x_n; V)$, and $\mathbf{K}_*(x_1, \ldots, x_n; W)$ where the map between $\mathbf{K}_\mathbf{i}(x_1, \ldots, x_n; V)$, and $\mathbf{K}_\mathbf{i}(x_1, \ldots, x_n; W)$ is $\bigoplus_{\substack{\alpha\subseteq[n],\\ |\alpha|=i}} \Phi^{\mathbf{u}-e_\alpha}$. It can be easily shown that if Φ is an isomorphism, then the induced map will also be an isomorphism between $\mathbf{K}_*(x_1, \ldots, x_n; V)$, and $\mathbf{K}_*(x_1, \ldots, x_n; W)$.

3.2 Explicit Construction of the Koszul Complex via Mapping Cones

In this section, our aim is to explicitly construct the Koszul complex of a persistence module V_q by using mapping cones. In order to see the construction of the Koszul complex by using mapping cones in a classical way, we refer the reader to [9] and [5].

Definition 28. Let $f_* : B_* \to C_*$ be a chain map. The **mapping cone** of f is a chain complex whose degree i-th part is $B_{i-1} \oplus C_i$, and whose i-th differential map is as follows:

$$\delta_i : B_{i-1} \oplus C_i \longrightarrow B_{i-2} \oplus C_{i-1}$$
$$(b, c) \longmapsto (-\partial^B_{i-1}(b), \partial^C_i(c) + f_{i-1}(b)).$$

Notation 3.2.1. The mapping cone of a chain map f is denoted by $Cone(f)_*$.

Let X be a cell complex, and let $\mathscr{F} := \{X^{\mathbf{u}}\}_{\mathbf{u}\in\mathbb{N}^n}$ be its n-filtration. If we apply the homology functor to $\mathscr{F}$, we will obtain an n-parameter persistence module which is $\bigoplus_{\mathbf{u}\in\mathbb{N}^n} H_q(X^{\mathbf{u}})$. Let us denote this n-parameter persistent homology module by V_q. We know that the i-th Koszul complex of V_q at the grade $\mathbf{u} \in \mathbb{N}^n$ $\mathbf{K_i}(x_1, \ldots, x_n; V_q)(\mathbf{u})$ is $\bigoplus_{\substack{\alpha\subseteq[n],\\ |\alpha|=i}} H_q(X^{\mathbf{u}-e_\alpha})$ for all i. Observe that we can determine $\mathbf{K}_*(x_1, \ldots, x_n; V_q)(\mathbf{u})$ by using a smaller n-parameter filtration which is $\mathscr{F}^{\mathbf{u}} := \{X^{\mathbf{u}-e_\alpha}\}_{\alpha\subseteq[n]}$. Notice that this new n-parameter persistence module $\mathscr{F}^{\mathbf{u}}$ includes 2^n subcomplexes of $\mathscr{F}$. Moreover, if we fix an $j \in [n]$, then $X^{\mathbf{u}-e_\alpha-e_j} \subseteq X^{\mathbf{u}-e_\alpha}$ will give us a 1-parameter filtration for all $\alpha \in [n] \setminus \{j\}$. Thus, we can obtain 2^{n-1} 1-filtrations by partitioning $\mathscr{F}^{\mathbf{u}}$. Beyond that, if we fix an $J := \{j_1, \ldots, j_t\} \subseteq [n]$, then each of $\{X^{\mathbf{u}-e_\alpha-e_\gamma}\}_{\gamma\subseteq J}$ will be a t-parameter filtration for each $\alpha \subseteq [n] \setminus J$. Notice that $[n] \setminus J$ has 2^{n-t} subsets, so we can obtain 2^{n-t} t-parameter filtrations from $\mathscr{F}^{\mathbf{u}}$ by partitioning it. Also, we can go further, and we can obtain a $(t-1)$-parameter filtration from two t-parameter filtrations $\{X^{\mathbf{u}-e_\alpha-e_\gamma}\}_{\gamma\subseteq J}$, and $\{X^{\mathbf{u}-e_\alpha-e_k-e_\gamma}\}_{\gamma\subseteq J}$ by taking the union of them where $k \in [n] \setminus J$, and $\alpha \subseteq [n] \setminus (J \cup \{k\})$. At this moment, our aim is to show that the Koszul complex of the $(t+1)$-parameter filtration which is obtained by taking the union of two t-parameter filtrations is the mapping cone of a map between the Koszul complexes of these two t-parameter filtrations.

For the remaining of this chapter, let us fix $J := \{j_1, \ldots, j_t\}$ for some $t \leq n$, $k \in [n] \setminus J$, $\alpha \in [n] \setminus (J \cup \{k\})$, and $w = \mathbf{u} - e_\alpha$ for some $\mathbf{u} \in \mathbb{N}^n$.

Lemma 3.2.2. The following diagram

$$\begin{array}{ccc} \mathbf{K_i}(x_{j_1}, \ldots, x_{j_t}; V_q)(\mathbf{w} - e_k) & \xrightarrow{d_i} & \mathbf{K_{i-1}}(x_{j_1}, \ldots, x_{j_t}; V_q)(\mathbf{w} - e_k) \\ \downarrow & & \downarrow \\ \mathbf{K_i}(x_{j_1}, \ldots, x_{j_t}; V_q)(\mathbf{w}) & \xrightarrow{d_i} & \mathbf{K_{i-1}}(x_{j_1}, \ldots, x_{j_t}; V_q)(\mathbf{w}) \end{array}$$

where vertical arrows are the maps induced by the inclusion maps in direction e_k commutes for all i.

Proof. We know that

$$\mathbf{K_i}(x_{j_1}, \ldots, x_{j_t})(\mathbf{w} - e_k) = \bigoplus_{\substack{\alpha \subseteq [n], \\ |\alpha| = i}} H_q(X^{\mathbf{w} - e_k - e_\gamma})$$

which is a direct sum of vector spaces, so it is enough to show that the above diagram commutes for all direct summands of $\mathbf{K_i}(x_{j_1}, \ldots, x_{j_t})(\mathbf{w} - e_k)$. Thus, let us show that the above diagram commutes for $H_q(X^{\mathbf{w} - e_k - e_\gamma}) \subseteq \mathbf{K_i}(x_{j_1}, \ldots, x_{j_t})(\mathbf{w} - e_k)$ such that $\gamma \subseteq J$, $|\gamma| = i$. Let $z \in H_q(X^{\mathbf{w} - e_k - e_\gamma})$, then

$$\begin{aligned} d_i(\iota^{\mathbf{w} - e_k - e_\gamma, \mathbf{w} - e_\gamma}(z)) &= \sum_{j \in \gamma} sign(j, e_\gamma) x_j . \iota^{\mathbf{w} - e_k - e_\gamma, \mathbf{w} - e_\gamma}(z) \\ &= \sum_{j \in \gamma} sign(j, e_\gamma) \iota^{\mathbf{w} - e_\gamma, \mathbf{w} - e_\gamma - e_j}(\iota^{\mathbf{w} - e_k - e_\gamma, \mathbf{w} - e_\gamma}(z)) \\ &= \sum_{j \in \gamma} sign(j, e_\gamma) \iota^{\mathbf{w} - e_k - e_\gamma, \mathbf{w} - e_\gamma - e_j}(z). \end{aligned}$$

On the other hand, observe that

$$\begin{aligned} d_i(z) &= \sum_{j \in \gamma} sign(j, e_\gamma) x_j . z \\ &= \sum_{j \in \gamma} sign(j, e_\gamma) \iota^{\mathbf{w} - e_k - e_\gamma, \mathbf{w} - e_k - e_\gamma - e_j}(z). \end{aligned}$$

Now, if we apply the inclusion map in direction e_k to $d_i(z)$, then we obtain

$$\sum_{j \in \gamma} sign(j, e_\gamma) \iota^{\mathbf{w} - e_k - e_\gamma - e_j, \mathbf{w} - e_\gamma - e_j}(\iota^{\mathbf{w} - e_k - e_\gamma, \mathbf{w} - e_k - e_\gamma - e_j}(z))$$

which is equal to

$$d_i(\iota^{\mathbf{w}-e_k-e_\gamma,\mathbf{w}-e_\gamma}(z)) = \sum_{j\in\gamma} sign(j,e_\gamma)\iota^{\mathbf{w}-e_k-e_\gamma,\mathbf{w}-e_\gamma-e_j}(z).$$

Hence, the given diagram commutes.

□

Notation 3.2.3. Denote the chain map which is induced by inclusions in the direction e_k between $\mathbf{K}_*(x_{j_1},\ldots,x_{j_t};V_q)(\mathbf{w}-e_k)$ and $\mathbf{K}_*(x_{j_1},\ldots,x_{j_t};V_q)(\mathbf{w})$ by

$$f^k(x_{j_1},\ldots,x_{j_t};V_q)(\mathbf{w}-e_k).$$

Notice that

$$f_i^k(x_{j_1},\ldots,x_{j_t};V_q)(\mathbf{w}-e_k) : \bigoplus_{\substack{\gamma\subseteq J,\\ |\gamma|=i}} H_q(X^{\mathbf{w}-e_k-e_\gamma}) \to \bigoplus_{\substack{\gamma\subseteq J,\\ |\gamma|=i}} H_q(X^{\mathbf{w}-e_\gamma})$$

such that

$$f_i^k(x_{j_1},\ldots,x_{j_t};V_q)(\mathbf{w}-e_k)(a) = \sum_{\substack{\gamma\subseteq[n],\\ |\gamma|=i}} \iota^{\mathbf{w}-e_k-e_\gamma,\mathbf{w}-e_\gamma}(a_\gamma)$$

for all $a \in \bigoplus_{\substack{\gamma\subseteq J,\\ |\gamma|=i}} H_q(X^{\mathbf{w}-e_k-e_\gamma})$ where a_γ is the component of a which comes from the direct summand $H_q(X^{\mathbf{w}-e_k-e_\gamma})$.

Theorem 3.2.4. *The chain complexes $Cone(f^k(x_{j_1},\ldots,x_{j_t};V_q)(\mathbf{w}-e_k))_*$ and $\mathbf{K}_*(x_k,x_{j_1},\ldots,x_{j_t};V_q)(\mathbf{w})$ are isomorphic.*

Proof. Firstly, let us prove that each of the chain groups are isomorphic

$$Cone(f^k(x_{j_1},\ldots,x_{j_t};V_q)(\mathbf{w}-e_k))_i \cong \mathbf{K}_\mathbf{i}(x_k,x_{j_1},\ldots,x_{j_t};V_q)(\mathbf{w}).$$

Let $i \in \{0,1,\ldots,n\}$, then

$$\begin{aligned}
\mathbf{K}_\mathbf{i}(x_k,x_{j_1},\ldots,x_{j_t};V_q)(\mathbf{w}) &= \bigoplus_{\substack{\gamma\subseteq\ J\cup\{k\},\\ |\gamma|=i}} H_q(X^{\mathbf{w}-e_\gamma}) \\
&\cong \left(\bigoplus_{\substack{\gamma\subseteq J,\\ |\gamma|=i}} H_q(X^{\mathbf{w}-e_\gamma})\right) \oplus \left(\bigoplus_{\substack{\gamma\subseteq J,\\ |\gamma|=i-1}} H_q(X^{\mathbf{w}-e_k-e_\gamma})\right) \\
&= Cone(f^k(x_{j_1},\ldots,x_{j_t};V_q)(\mathbf{w}-e_k))_i
\end{aligned}$$

by the definition of a mapping cone.

Notice that the second isomorphism holds in the above proof because

$$\{\gamma \mid \gamma \subseteq J \cup \{k\}, |\gamma| = i\} = \{\gamma \mid \gamma \subseteq J, |\gamma| = i\} \cup \{\gamma \cup \{k\} \mid \gamma \subseteq J, |\gamma| = i-1\}.$$

Now, let us use $Cone((f^k)(\mathbf{w}-e_k))_i$ instead of $Cone(f^k(x_{j_1},\ldots,x_{j_t};V_q)(\mathbf{w}-e_k))_i$, and $Cone((f^k)(\mathbf{w}-e_k))_{i-1}$ instead of $Cone(f^k(x_{j_1},\ldots,x_{j_t};V_q)(\mathbf{w}-e_k))_{i-1}$ in the remaining part of this proof. Let us show that the following diagram is commutative.

$$\begin{array}{ccc} Cone((f^k)(\mathbf{w}-e_k))_i & \xrightarrow{\delta_i} & Cone((f^k)(\mathbf{w}-e_k))_{i-1} \\ \downarrow\cong & & \downarrow\cong \\ \mathbf{K_i}(x_k, x_{j_1},\ldots,x_{j_t};V_q)(\mathbf{w}) & \xrightarrow{d_i} & \mathbf{K}_{i-1}(x_k, x_{j_1},\ldots,x_{j_t};V_q)(\mathbf{w}) \end{array}$$

Firstly, we remember that

$$Cone((f^k)(\mathbf{w}-e_k))_i = \mathbf{K_{i-1}}(x_{j_1},\ldots,x_{j_t};V_q)(\mathbf{w}-e_k) \oplus \mathbf{K_i}(x_{j_1},\ldots,x_{j_t};V_q)(\mathbf{w}),$$

and let $(a,b) \in Cone((f^k)(\mathbf{w}-e_k))_i$ where

$$a \in \mathbf{K_{i-1}}(x_{j_1},\ldots,x_{j_t};V_q)(\mathbf{w}-e_k) = \bigoplus_{\substack{\gamma\subseteq J,\\ |\gamma|=i-1}} H_q(X^{\mathbf{w}-e_k-e_\gamma})$$

$$b \in \mathbf{K_i}(x_{j_1},\ldots,x_{j_t};V_q)(\mathbf{w}) = \bigoplus_{\substack{\gamma\subseteq J,\\ |\gamma|=i}} H_q(X^{\mathbf{w}-e_\gamma}).$$

Let a_γ denote the component of a that belongs to the direct summand $H_q(X^{\mathbf{w}-e_k-e_\gamma})$ for all γ, and b_γ denote the component of b which belongs to the direct summand $H_q(X^{\mathbf{w}-e_\gamma})$ for all γ. Thus,

$$\delta_i(a,b) = (-d_{i-1}(a), d_i(b) + f^k_{i-1}(a))$$

where

$$d_{i-1}(a) = \sum_{\substack{\gamma\subseteq J,\\ |\gamma|=i-1}} \sum_{j\in\gamma} sign(j,\gamma)\iota^{\mathbf{w}-e_k-e_\gamma,\mathbf{w}-e_k-e_\gamma+e_j}(a_\gamma)$$

$$d_i(b) = \sum_{\substack{\gamma\subseteq J,\\ |\gamma|=i}} \sum_{j\in\gamma} sign(j,\gamma)\iota^{\mathbf{w}-e_\gamma,\mathbf{w}-e_\gamma+e_j}(b_\gamma)$$

$$f^k_{i-1}(a) = \sum_{\substack{\gamma\subseteq J,\\ |\gamma|=i-1}} \iota^{\mathbf{w}-e_k-e_\gamma,\mathbf{w}-e_\gamma}(a_\gamma).$$

We know that

$$Cone((f^k)(\mathbf{w}-e_k))_{i-1} = \bigoplus_{\substack{\gamma\subseteq J,\\ |\gamma|=i-2}} H_q(X^{\mathbf{w}-e_k-e_\gamma}) \oplus \bigoplus_{\substack{\gamma\subseteq J,\\ |\gamma|=i-1}} H_q(X^{\mathbf{w}-e_\gamma})$$

where

$$\bigoplus_{\substack{\gamma\subseteq J,\\ |\gamma|=i-2}} H_q(X^{\mathbf{w}-e_k-e_\gamma}) = \mathbf{K_{i-1}}(x_{j_1},\ldots,x_{j_t};V_q)(\mathbf{w}-e_k)$$
$$\bigoplus_{\substack{\gamma\subseteq J,\\ |\gamma|=i-1}} H_q(X^{\mathbf{w}-e_\gamma}) = \mathbf{K_i}(x_{j_1},\ldots,x_{j_t};V_q)(\mathbf{w}).$$

Thus, we can see that

$$d_{i-1}(a) \in \mathbf{K_{i-1}}(x_{j_1},\ldots,x_{j_t};V_q)(\mathbf{w}-e_k)$$
$$d_i(b) \in \mathbf{K_i}(x_{j_1},\ldots,x_{j_t};V_q)(\mathbf{w})$$
$$f^k_{i-1}(a) \in \mathbf{K_i}(x_{j_1},\ldots,x_{j_t};V_q)(\mathbf{w}).$$

On the other hand,

$$d_i(a,b) = d_i(a,0) + d_i(0,b).$$

Firstly,

$$d_i(a,0) = \sum_{\substack{\gamma\subseteq J,\\ |\gamma|=i-1}} \left(\sum_{j\in\gamma\cup\{k\}} sign(j,\gamma\cup\{k\})\iota^{\mathbf{w}-e_k-e_\gamma,\mathbf{w}-e_k-e_\gamma+e_j}(a_\gamma) \right)$$

and $\sum_{j\in\gamma\cup\{k\}} sign(j,\gamma\cup\{k\})\iota^{\mathbf{w}-e_k-e_\gamma,\mathbf{w}-e_k-e_\gamma+e_j}(a_\gamma)$ is equal to

$$\sum_{j\in\gamma} sign(j,\gamma\cup\{k\})\iota^{\mathbf{w}-e_k-e_\gamma,\mathbf{w}-e_k-e_\gamma+e_j}(a_\gamma) + sign(k,\gamma\cup\{k\})\iota^{\mathbf{w}-e_k-e_\gamma,\mathbf{w}-e_\gamma}(a_\gamma)$$

so

$$
\begin{aligned}
d_{i-1}(a,0) &= \sum_{\substack{\gamma\subseteq J,\\ |\gamma|=i-1}} \sum_{j\in\gamma} sign(j,\gamma\cup\{k\})\iota^{\mathbf{w}-e_k-e_\gamma,\mathbf{w}-e_k-e_\gamma+e_j}(a_\gamma)\\
&+ \sum_{\substack{\gamma\subseteq J,\\ |\gamma|=i-1}} sign(k,\gamma\cup\{k\})\iota^{\mathbf{w}-e_k-e_\gamma,\mathbf{w}-e_\gamma}(a_\gamma)\\
&= \sum_{\substack{\gamma\subseteq J,\\ |\gamma|=i-1}} \sum_{j\in\gamma} -sign(j,\gamma)\iota^{\mathbf{w}-e_k-e_\gamma,\mathbf{w}-e_k-e_\gamma+e_j}(a_\gamma)\\
&+ \sum_{\substack{\gamma\subseteq J,\\ |\gamma|=i-1}} \iota^{\mathbf{w}-e_k-e_\gamma,\mathbf{w}-e_\gamma}(a_\gamma)\\
&= (d_{i-1}(a), f^k_{i-1}(a)).
\end{aligned}
$$

The second equality holds since x_k is the first coordinate in the given Koszul complexes, so k will be the first element of the set $\gamma\cup\{k\}$.

Secondly,

$$
\begin{aligned}
d_i(0,b) &= \sum_{\substack{\gamma\subseteq J,\\ |\gamma|=i}} \sum_{j\in\gamma} sign(j,\gamma)\iota^{\mathbf{w}-e_\gamma,\mathbf{w}-e_\gamma+e_j}(b_\gamma)\\
&= (0, d_i(b)),
\end{aligned}
$$

and these two imply that the above diagram commutes, and we are done.

□

Remark 3.2.1. Notice that the chain complex $\mathbf{K}_*(x_1,\dots,x_i,\dots,x_j,\dots,x_n;V_q)$ is isomorphic to the chain complex $\mathbf{K}_*(x_1,\dots,x_j,\dots,x_i,\dots,x_n;V_q)$ for all $i,j\in\{1,\dots,n\}$ since $A\oplus B\cong B\oplus A$ for all vector spaces A and B, and by the definition of the Koszul complex of a persistent homology module (see Section 3.1).

By Remark 3.2.1, we can iteratively obtain $\mathbf{K}_*(x_1,\dots,x_n;V_q)(\mathbf{u})$ by starting with $\mathbf{K}_*(x_i;V_q)(\mathbf{u}-e_{[n]}+e_i)$ for some $i\in\{1,\dots,n\}$. In the next chapter, acyclicity of a mapping cone will be very important for us because we will obtain several results by proving the acyclicity of some specific mapping cones. Thus, the next proposition is important for us to give an equivalent condition to acyclicity of a mapping cone.

Proposition 3.2.5. [17] Let $f: A_*\to B_*$ be a chain map. Then, the mapping cone of f, $Cone(f_*)$, is acyclic (which means $Cone(f_*)$ is an exact chain complex) if and only if the chain map f is a quasi-isomophism.

Proof. In order to prove this proposition, let us look at the following sequence

$$0 \longrightarrow B \xrightarrow{\iota} Cone(f) \xrightarrow{j} A[-1] \longrightarrow 0. \tag{3.1}$$

where $\iota(y) = (0, y)$ for all $y \in B$, $j(x, y) = -x$ for all $(x, y) \in Cone(f)$ and $A[-1]_n = A_{n-1}$ for all $n \in \mathbb{N}$.

Claim: The above sequence 3.1 is a short exact sequence.

Proof of the Claim: We know that ι is the inclusion map, so it is an injection. Also, we can easily see that j is a surjection by its definition. Moreover,

$$\begin{aligned} im(\iota) &= \{(0, y) \mid y \in B\} \\ &= ker(j). \end{aligned}$$

So, the above sequence 3.1 is a short exact sequence.

Thus, there exists a long exact sequence

$$... \longrightarrow H_{n+1}(Cone(f)) \xrightarrow{j_*} H_n(A) \xrightarrow{\partial_n} H_n(B) \xrightarrow{\iota_*} H_n(Cone(f)) \longrightarrow ...$$

where $\partial_n([x]) = [\iota_{n-1}^{-1} d_n^{Cone(f)} j_{n-1}^{-1}(x)]$ for all $[x] \in H_n(A)$, and ι_* and j_* are induced maps on homology of ι and j, respectively. Note that we know that j is not injective, but $j_{n-1}^{-1}(x)$ just gives us a representative from the set $\{(x, y) \in Cone(f) \mid j(x, y) = x\}$, and we know that ∂_n is well-defined by homological algebra (see [17]). Now, observe that

$$\begin{aligned} \partial_n([x]) &= [\iota_{n-1}^{-1} d_n^{Cone(f)} j_{n-1}^{-1}(x)] \\ &= [\iota_{n-1}^{-1} d_n^{Cone(f)}(-x, 0)] \\ &= [\iota_{n-1}^{-1}(d_{n-1}^A(x), f(x))] \\ &= [f(x)] \\ &= (f_*)_n([x]). \end{aligned}$$

Thus, $\partial_n = (f_*)_n$ for all $n \in \mathbb{N}$. Now,

($\Rightarrow$) Assume that $Cone(f)$ is acyclic. This implies that, $H_{n+1}(Cone(f)) = 0$ and

$H_n(Cone(f)) = 0$, so $\partial_n = (f_*)_n$ is an isomorphism for all $n \in \mathbb{N}$ by the above long exact sequence.
($\Leftarrow$) Assume that $(f_*)_n$ is an isomorphism for all $n \in \mathbb{N}$, then this implies that

$$ker(\iota_*)_n = im(f_*)_n = H_n(D) \tag{3.2}$$

for all $n \in \mathbb{N}$. Also, $im(j_*)_n = ker(f_*)_n = 0$ for all $n \in \mathbb{N}$, so $ker(j_*)_n = H_{n+1}(Cone(f))$ for all $n \in \mathbb{N}$. However, $ker(j_*)_n = im(\iota_*)_{n+1}$ for all $n \in \mathbb{N}$. Thus, $im(\iota_*)_n = H_n(Cone(f))$ for all $n \in \mathbb{N}$. Also, 3.2 implies that $im(\iota_*)_n = 0$ for all $n \in \mathbb{N}$. Thus, $H_n(Cone(f)) = 0$ for all $n \in \mathbb{N}$. Hence, $Cone(f)$ is acyclic. □

Corollary 3.2.6. Let A_* and be B_* be acyclic chain complexes. If $f : A_* \to B_*$ is a chain map, then $Cone(f)_*$ is acyclic.

Proof. We know that A_* and B_* are acyclic chain complexes by our assumption, so $H_q(A_*) = H_q(B_*) = 0$ for all $q \in \mathbb{N}$. This implies that the chain map f is a quasi-isomorphism. Hence, $Cone(f_*)$ is acyclic by Proposition 3.2.5. □

CHAPTER 4

ENTRANCE GRADES OF CRITICAL CELLS AND SUPPORT OF BETTI TABLES

In this chapter, our aim is to delimit the support of Betti tables of an n-parameter persistence module which is obtained by a 1-critical n-parameter filtration. We will show this result in the last theorem of this chapter. The followings are based on fourth section of the article [13].

As we did in the former chapters, let us fix the tools which we are going to use. Let X be a cell complex, and $\{X^{\mathbf{u}}\}_{\mathbf{u}\in\mathbb{N}^n}$ be the n-parameter filtration of X, and let V_q denote the n-parameter persistence module which is obtained by applying the q-th homology functor to $\{X^{\mathbf{u}}\}_{\mathbf{u}\in\mathbb{N}^n}$ for all q. In this chapter, we will frequently use the results which we have shown in the discrete Morse theory section 2.3. Thus, let us fix V as our discrete gradient vector field which is consistent with the filtration, so we have a Morse complex M which comes with the discrete gradient vector field V. Also, notice that one of our assumptions is that the n-parameter filtration $\{X^{\mathbf{u}}\}_{\mathbf{u}\in\mathbb{N}^n}$ is **exhaustive** which means that $X = \bigcup_{\mathbf{u}\in\mathbb{N}^n} X^{\mathbf{u}}$, and this implies that $X_q = \bigcup_{\mathbf{u}\in\mathbb{N}^n} X_q^{\mathbf{u}}$ for all q. Thus, we can also say that $M_q = \bigcup_{\mathbf{u}\in\mathbb{N}^n} M_q^{\mathbf{u}}$.

Notation 4.0.1. Let A be a non-empty subset of the set of cells of X. We denote the set of the entrance grades of the cells of A as $\mathcal{G}(A) \subseteq \mathbb{N}^n$, that is

$$\mathcal{G}(A) = \{\mathbf{u} \in \mathbb{N}^n \mid \alpha \in X^{\mathbf{u}} \setminus \left(\bigcup_{j=1}^{n} X^{\mathbf{u}-e_j}\right) \quad \text{for some } \alpha \in A\}$$

Notation 4.0.2. Let G be a non-empty subset of the poset $(\mathbb{N}^n, \leq)$. In relation to the least upper bound of the set $\mathbb{N}^n$, we denote the closure of G by $\overline{G}$, that is

$$\overline{G} = \{\bigvee B \mid B \subseteq G, B \neq \emptyset\}$$

where $\bigvee B$ is the least upper bound of the set B in $(\mathbb{N}^n, \leq)$.

In addition to these notations, we will denote the support of the Betti table $\xi_i^q : \mathbb{N}^n \to \mathbb{N}$ by $supp\, \xi_i^q$, that is

$$supp\, \xi_i^q = \{\mathbf{u} \in \mathbb{N}^n \mid \xi_i^q(\mathbf{u}) \neq 0\}.$$

Now, let us give the first proposition of this chapter which has two important corollaries, and these will help us when we prove the last theorem. Note that we will give the proof of the following proposition by using contraposition.

Proposition 4.0.3. Let $\mathbf{u} \in \mathbb{N}^n$, and A be the subset of the set of the cells of X. Then, the following statements are equivalent.

(1) $\mathbf{u} \notin \overline{\mathcal{G}(A)}$,
(2) $(X^{\mathbf{u}-e_{\alpha_j}} \setminus X^{\mathbf{u}-e_{\alpha_j}-e_j}) \cap A = \emptyset$ for some $j \in [n]$ for all $\alpha \subseteq [n] \setminus \{j\}$.

Proof. (2 $\Rightarrow$ 1) Assume that $\mathbf{u} \in \overline{\mathcal{G}(A)}$, then $\mathbf{u} \in \mathcal{G}(A)$ or $\mathbf{u} \in \overline{\mathcal{G}(A)} \setminus \mathcal{G}(A)$. If $\mathbf{u} \in \mathcal{G}(A)$, then there exists $\sigma \in A$ such that $\sigma \in X^{\mathbf{u}} \setminus \bigcup_{j=1}^{n} X^{e_j}$, so $\sigma \in X^{\mathbf{u}} \setminus X^{e_j}$ for all $j \in [n]$. Thus, if we choose $\alpha_j = \emptyset \subseteq [n] \setminus \{j\}$ for all $j \in [n]$, then $(X^{\mathbf{u}-e_{\alpha_j}} \setminus X^{\mathbf{u}-e_{\alpha_j}-e_j}) \cap A \neq \emptyset$, so we are done. If $\mathbf{u} \in \overline{\mathcal{G}(A)} \setminus \mathcal{G}(A)$, then $\mathbf{u} = \bigvee B$ for some non-empty $B \subseteq \mathcal{G}(A)$ such that $|B| \geq 2$. Say $B = \{\mathbf{v_1}, \ldots, \mathbf{v_r}\}$ such that $r \geq 2$. Since $\mathbf{u}$ is the least upper bound of B, $\mathbf{u} - e_j$ cannot be an upper bound of B for any $j \in [n]$. So, there exists $\mathbf{v_{\ell(j)}} \in B$ such that $\mathbf{v_{\ell(j)}} \nleq \mathbf{u} - e_j$ for all $j \in [n]$. Also, we know that $B \subseteq \mathcal{G}(A)$, so $\mathbf{v_{\ell(j)}} \in \mathcal{G}(A)$ for all $j \in [n]$, thus there exists a $\sigma_{\ell(j)} \in A$ such that the entrance grade of $\sigma_{\ell(j)}$ is $\mathbf{v_{\ell(j)}}$ for all $j \in [n]$, and this implies that $\sigma_{\ell(j)} \in X^{\mathbf{v_{\ell(j)}}} \subseteq X^{\mathbf{u}}$ for all $j \in [n]$. On the other hand, we know that each cell has a unique entrance grade since our n-parameter filtration is 1-critical, and $\mathbf{v_{\ell(j)}} \nleq \mathbf{u} - e_j$, so $\sigma_{\ell(j)} \notin X^{\mathbf{u}-e_j}$ for all $j \in [n]$. Thus, $\sigma_{\ell(j)} \in (X^{\mathbf{u} \setminus X^{\mathbf{u}-e_j}}) \cap A$, so again if we take $\alpha_j = \emptyset$ for all $j \in [n]$, then $(X^{\mathbf{u}-e_{\alpha_j}} \setminus X^{\mathbf{u}-e_{\alpha_j}-e_j}) \cap A \neq \emptyset$ for all $j \in [n]$.

(1 $\Rightarrow$ 2) Assume that for all $j \in [n]$, there exists $\alpha_j \in [n] \setminus \{j\}$ such that $(X^{\mathbf{u}-e_{\alpha_j}} \setminus X^{\mathbf{u}-e_{\alpha_j}-e_j}) \cap A \neq \emptyset$. Thus, there exists $\sigma_j \in (X^{\mathbf{u}-e_{\alpha_j}} \setminus X^{\mathbf{u}-e_{\alpha_j}-e_j}) \cap A$ for all $j \in [n]$. Let $\mathbf{v(j)}$ be the entrance grade of σ_j for all $j \in [n]$, and let $\mathbf{v} = \bigvee\{\mathbf{v(n)}, \ldots, \mathbf{v(n)}\}$. Notice that $\mathbf{v(j)} \leq \mathbf{u} - e_{\alpha_j} \leq \mathbf{u}$ since $\sigma_j \in X^{\mathbf{u}-e_{\alpha_j}}$ for all $j \in [n]$. Thus, $\mathbf{v} \leq \mathbf{u}$

since $\mathbf{v} = \bigvee\{\mathbf{v(n)}, \ldots, \mathbf{v(n)}\}$, and $\mathbf{v(j)} \leq \mathbf{u}$ for all $j \in [n]$. Now, let us show that $\mathbf{v} = \mathbf{u}$. Assume that $\mathbf{u} \not\leq \mathbf{v}$, then there exists $j \in [n]$ such that $\mathbf{v} \leq \mathbf{u} - e_j$ since $\mathbf{u} \not\leq \mathbf{v}$ and $\mathbf{v} \leq \mathbf{u}$. Also, we know that $\mathbf{v} = \bigvee\{\mathbf{v(n)}, \ldots, \mathbf{v(n)}\}$, so $\mathbf{v(j)} \leq \mathbf{u} - e_j$, but then $\sigma_j \in X^{\mathbf{u}-e_j}$ since $\sigma_j \in X^{\mathbf{v(j)}}$. Moreover, we also know that $\sigma_j \in X^{\mathbf{u}-e_{\alpha_j}}$, but $\alpha_j \in [n] \setminus \{j\}$, so $\mathbf{u} - e_{\alpha_j}$ and $\mathbf{u} - e_j$ are not comparable in $(\mathbb{N}^n, \leq)$. Also, we know that our n-parameter filtration is 1-critical, so the entrance grade of a cell must be unique, and this implies that $\sigma_j \in X^{\wedge(\mathbf{u}-e_{\alpha_j}, \mathbf{u}-e_j)} = X^{\mathbf{u}-e_{\alpha_j}-e_j}$ which is a contradiction. Thus, $\mathbf{u} \leq \mathbf{v}$, and we also know that $\mathbf{v} \leq \mathbf{u}$, so $\mathbf{u} = \mathbf{v}$. Hence, $\mathbf{u} \in \overline{\mathcal{G}(A)}$ since $\mathbf{u} = \mathbf{v} = \bigvee\{\mathbf{v(n)}, \ldots, \mathbf{v(n)}\}$, and $\mathbf{v(j)} \in \mathcal{G}(A)$ for all $j \in [n]$.

□

Corollary 4.0.4. Let $\mathbf{u} \in \mathbb{N}^n$. Then, $\mathbf{u} \notin \overline{\mathbf{G}(M_q)}$ if and only if there exists $j \in [n]$ such that $M_q^{\mathbf{u}-e_{\alpha_j}-e_j} = M_q^{\mathbf{u}-e_{\alpha_j}}$ for all $\alpha_j \in [n] \setminus \{j\}$.

Proof. ($\Rightarrow$) Let $\mathbf{u} \notin \overline{\mathbf{G}(M_q)}$, then there exists $j \in [n]$ such that for all $\alpha_j \subseteq [n] \setminus \{j\}$ the following equality holds

$$(X^{\mathbf{u}-e_{\alpha_j}} \setminus X^{\mathbf{u}-e_{\alpha_j}-e_j}) \cap M_q = \emptyset$$

by Proposition 4.0.3, and this implies that

$$M_q^{\mathbf{u}-e_{\alpha_j}} \setminus M_q^{\mathbf{u}-e_{\alpha_j}-e_j} = \emptyset,$$

so

$$M_q^{\mathbf{u}-e_{\alpha_j}} = M_q^{\mathbf{u}-e_{\alpha_j}-e_j}.$$

($\Leftarrow$) Assume that the second statement holds which implies there exists j such that

$$M_q^{\mathbf{u}-e_{\alpha_j}} \setminus M_q^{\mathbf{u}-e_{\alpha_j}-e_j} = \emptyset,$$

for all $\alpha_j \subseteq [n] \setminus \{j\}$, so

$$(X^{\mathbf{u}-e_{\alpha_j}} \setminus X^{\mathbf{u}-e_{\alpha_j}-e_j}) \cap M_q = \emptyset,$$

thus $\mathbf{u} \notin \overline{\mathcal{G}(M_q)}$ by Proposition 4.0.3.

□

In addition to this corollary, we can also obtain some information on the maps of $\{H_q(X^{\mathbf{u}}), \iota_q^{\mathbf{u},\mathbf{v}}\}$ and $\{H_{q-1}(X^{\mathbf{u}}), \iota_{q-1}^{\mathbf{u},\mathbf{v}}\}$ when we take $A = M_q$ in Proposition 4.0.3.

Corollary 4.0.5. If $\mathbf{u} \notin \overline{\mathcal{G}(M_q)}$, then there exists $j \in [n]$ such that for all $\alpha_j \in [n] \setminus \{j\}$ the inclusion map from $X^{\mathbf{u}-e_{\alpha_j}-e_j}$ to $X^{\mathbf{u}-e_{\alpha_j}}$ induces a surjection

$$\iota_q^{\mathbf{u}-e_{\alpha_j}-e_j,\mathbf{u}-e_{\alpha_j}} : H_q(X^{\mathbf{u}-e_{\alpha_j}-e_j}) \to H_q(X^{\mathbf{u}-e_{\alpha_j}}),$$

and an injection

$$\iota_{q-1}^{\mathbf{u}-e_{\alpha_j}-e_j,\mathbf{u}-e_{\alpha_j}} : H_{q-1}(X^{\mathbf{u}-e_{\alpha_j}-e_j}) \to H_{q-1}(X^{\mathbf{u}-e_{\alpha_j}}).$$

Proof. If $\mathbf{u} \notin \overline{\mathcal{G}(M_q)}$, then this implies that there exists $j \in [n]$ such that for all subsets $\alpha_j \in [n] \setminus \{j\}$,

$$M_q^{\mathbf{u}-e_{\alpha_j}-e_j} = M_q^{\mathbf{u}-e_{\alpha_j}}$$

by corollary 4.0.4. Thus, the relative homology

$$H_q(M^{\mathbf{u}-e_{\alpha_j}-e_j}, M^{\mathbf{u}-e_{\alpha_j}}) = 0$$

for all $\alpha_j \in [n] \setminus \{j\}$. Also, the diagram between the long exact sequences of the relative homology of $(X^{\mathbf{u}-e_{\alpha_j}-e_j}, X^{\mathbf{u}-e_{\alpha_j}})$ and $(M^{\mathbf{u}-e_{\alpha_j}-e_j}, M^{\mathbf{u}-e_{\alpha_j}})$ commutes by Lemma 2.3.1, and Theorem 2.3.2 implies the isomorphism between homology groups in the same diagram:

$$\begin{array}{ccccccccc}
H_q(X^{\mathbf{w}_j-e_j}) & \to & H_q(X^{\mathbf{w}_j}) & \to & H_q(X^{\mathbf{w}_j-e_j}, X^{\mathbf{w}_j}) & \to & H_{q-1}(X^{\mathbf{w}_j-e_j}) & \to & H_{q-1}(X^{\mathbf{w}_j}) \\
\downarrow\cong & & \downarrow\cong & & \downarrow & & \downarrow\cong & & \downarrow\cong \\
H_q(M^{\mathbf{w}_j-e_j}) & \to & H_q(M^{\mathbf{w}_j}) & \to & H_q(M^{\mathbf{w}_j-e_j}, M^{\mathbf{w}_j}) & \to & H_{q-1}(M^{\mathbf{w}_j-e_j}) & \to & H_{q-1}(M^{\mathbf{w}_j})
\end{array}$$

where $\mathbf{w}_j$ denotes $\mathbf{u} - e_{\alpha_j}$. Then, Five Lemma implies that

$$H_q(X^{\mathbf{w}_j-e_j}, X^{\mathbf{w}_j}) \cong H_q(M^{\mathbf{w}_j-e_j}, M^{\mathbf{w}_j}).$$

Thus,

$$(X^{\mathbf{w}_j-e_j}, X^{\mathbf{w}_j}) = 0.$$

Thus, exactness of the long exact sequence of relative homology $(X^{\mathbf{u}-e_{\alpha_j}-e_j}, X^{\mathbf{u}-e_{\alpha_j}})$ implies that

$$\iota_q^{\mathbf{u}-e_{\alpha_j}-e_j,\mathbf{u}-e_{\alpha_j}} : H_q(X^{\mathbf{u}-e_{\alpha_j}-e_j}) \to H_q(X^{\mathbf{u}-e_{\alpha_j}}),$$

is a surjection, and

$$\iota_{q-1}^{\mathbf{u}-e_{\alpha_j}-e_j,\mathbf{u}-e_{\alpha_j}} : H_{q-1}(X^{\mathbf{u}-e_{\alpha_j}-e_j}) \to H_{q-1}(X^{\mathbf{u}-e_{\alpha_j}}).$$

is an injection.

□

Remark 4.0.1. If $\mathbf{u} \notin \overline{\mathcal{G}(M_q)} \cup \overline{\mathcal{G}(M_{q+1})}$, then $\mathbf{u} \notin \overline{\mathcal{G}(M_q)}$ and $\mathbf{u} \notin \overline{\mathcal{G}(M_{q+1})}$. On the other hand, $\mathbf{u} \notin \overline{\mathcal{G}(M_q)}$ implies that

(i) There exists $j \in [n]$ such that, $M_q^{\mathbf{u}-e_{\alpha_j}-e_j} = M_q^{\mathbf{u}-e_{\alpha_j}}$ for all $\alpha_j \subseteq [n] \setminus \{j\}$

by Corollary 4.0.4. On the other hand, $\mathbf{u} \notin \overline{\mathcal{G}(M_{q+1})}$ implies that

(ii) There exists $\ell \in [n]$ such that, $M_{q+1}^{\mathbf{u}-e_{\alpha_\ell}-e_\ell} = M_{q+1}^{\mathbf{u}-e_{\alpha_\ell}}$ for all $\alpha_\ell \subseteq [n] \setminus \{\ell\}$

by Corollary 4.0.4.

In Remark 4.0.1, we cannot say anything about the indices j and ℓ of the statements (i) and (ii), they may or may not be the same. However, both cases enable us to prove the acyclicity of the special Koszul complexes. We will see this in the following two lemmas.

Lemma 4.0.6. Let $\mathbf{u} \notin \overline{\mathcal{G}(M_q)} \cup \overline{\mathcal{G}(M_{q+1})}$, and j and ℓ be the indices in the properties (i) and (ii) of the Remark 4.0.1, respectively. If the indices j, ℓ are the same in the properties (i), (ii) of Remark 4.0.1, then the Koszul complex $\mathbf{K}_*(x_1, \ldots, x_n; V_q)(\mathbf{u})$ is acyclic.

Proof. By the proof of Corollary 4.0.5, we know that (i) implies that

$$\iota_{q}^{\mathbf{u}-e_{\alpha_j}-e_j,\mathbf{u}-e_{\alpha_j}} : H_{q}(X^{\mathbf{u}-e_{\alpha_j}-e_j}) \to H_{q}(X^{\mathbf{u}-e_{\alpha_j}})$$

is a surjection for all $\alpha_j \subseteq [n] \setminus \{j\}$. On the other hand, (ii) implies that

$$\iota_{q-1}^{\mathbf{u}-e_{\alpha_j}-e_j,\mathbf{u}-e_{\alpha_j}} : H_{q-1}(X^{\mathbf{u}-e_{\alpha_j}-e_j}) \to H_{q-1}(X^{\mathbf{u}-e_{\alpha_j}})$$

is an injection for all $\alpha_\ell \subseteq [n] \setminus \{\ell\}$ by the proof of the Corollary 4.0.5. However, $j = \ell$ by our assumption, and

$$\iota_{q}^{\mathbf{u}-e_{\alpha_j}-e_j,\mathbf{u}-e_{\alpha_j}} : H_{q}(X^{\mathbf{u}-e_{\alpha_j}-e_j}) \to H_{q}(X^{\mathbf{u}-e_{\alpha_j}})$$

is an isomorphism, for all $\alpha_j \subseteq [n] \setminus \{j\}$ whenever $j = \ell$. Thus, this isomorphism implies that the induced chain map in the direction e_j

$$f^j(\mathbf{u} - e_j) : \mathbf{K}_*(x_1, \ldots, \widehat{x_j}, \ldots, x_n; V_q)(\mathbf{u} - e_j) \to \mathbf{K}_*(x_1, \ldots, \widehat{x_j}, \ldots, x_n; V_q)(\mathbf{u}),$$

where $f^j(\mathbf{u} - e_j) = f^j(x_1, \ldots, \widehat{x_j}, \ldots, x_n; V_q)(\mathbf{u} - e_j)$, is an isomorphism by its definition (for the definition see 3.2). As a result of this, since $\mathbf{K}_*(x_1, \ldots, x_n; V_q)(\mathbf{u})$ is the mapping cone of the chain map $f^j(\mathbf{u} - e_j)$, $\mathbf{K}_*(x_1, \ldots, x_n; V_q)(\mathbf{u})$ is acyclic by Proposition 3.2.5.

□

Lemma 4.0.7. Let $\mathbf{u} \notin \overline{\mathcal{G}(M_q)} \cup \overline{\mathcal{G}(M_{q+1})}$, and j and ℓ be the indices in the properties (i) and (ii) in the Remark 4.0.1, respectively. If the indices j and ℓ are not the same in the properties (i) and (ii) in Remark 4.0.1, then the Koszul complex $\mathbf{K}_*(x_j, x_\ell; V_q)(\mathbf{u} - e_\alpha)$ is acyclic for all $\alpha \subseteq [n] \setminus \{j, \ell\}$.

Proof. Let $\alpha \subseteq [n] \setminus \{j, \ell\}$, and $\mathbf{w} = \mathbf{u} - e_\alpha$. We want to show that $\mathbf{K}_*(x_j, x_\ell; V_q)(\mathbf{w})$ is acyclic, but we know that the n-parameter persistence modules $V_q = \bigoplus_{\mathbf{u} \in \mathbb{N}^n} H_q(X^{\mathbf{u}})$ and $V'_q := \bigoplus_{\mathbf{u} \in \mathbb{N}^n} H_q(M^{\mathbf{u}})$ are isomorphic by Theorem 2.3.2, so $\mathbf{K}_*(x_j, x_\ell; V_q)(\mathbf{u})$ and $\mathbf{K}_*(x_j, x_\ell; V'_q)(\mathbf{u})$ are isomorphic for all $\mathbf{u} \in \mathbb{N}^n$ by the Section 3.1. Thus, it is enough to show that $\mathbf{K}_*(x_j, x_\ell; V'_q)(\mathbf{w})$ is acyclic. At the moment, we know that $\mathbf{K}_*(x_j, x_\ell; V'_q)(\mathbf{w})$ is the mapping cone of

$$f^\ell : \mathbf{K}_*(x_j; V'_q)(\mathbf{w} - e_\ell) \to \mathbf{K}_*(x_j; V'_q)(\mathbf{w})$$

by Theorem 3.2.4. Moreover,

$$\mathbf{K}_*(x_j; V'_q)(\mathbf{w} - e_\ell) : 0 \to H_q(M^{\mathbf{w} - e_\ell - e_j}) \xrightarrow{\iota_q^{\mathbf{w} - e_\ell - e_j, \mathbf{w} - e_\ell}} H_q(M^{\mathbf{w} - e_\ell}) \to 0,$$

and

$$\mathbf{K}_*(x_j; V'_q)(\mathbf{w}) : 0 \to H_q(M^{\mathbf{w} - e_j}) \xrightarrow{\iota_q^{\mathbf{w} - e_j, \mathbf{w}}} H_q(M^{\mathbf{w}}) \to 0.$$

Since $\alpha \cup \{\ell\}$ and α are subsets of $[n] \setminus \{j\}$, $\iota_q^{\mathbf{w} - e_\ell - e_j, \mathbf{w} - e_\ell}$ and $\iota_q^{\mathbf{w} - e_j, \mathbf{w}}$ are surjections by corollary 4.0.5. Thus, it is enough to show that

$$f' : ker(\iota_q^{\mathbf{w} - e_\ell - e_j, \mathbf{w} - e_\ell}) \to ker(\iota_q^{\mathbf{w} - e_j, \mathbf{w}})$$

is an isomorphism where we denote by f' the restriction of the map $f_1^\ell(x_j; V'_q)(\mathbf{w} - e_\ell) = \iota_q^{\mathbf{w} - e_\ell - e_j, \mathbf{w} - e_\ell}$ to the $ker(\iota_q^{\mathbf{w} - e_\ell - e_j, \mathbf{w} - e_\ell})$. We know that $\mathbf{u} \notin \overline{\mathcal{G}(M_{q+1})}$ by our assumption, so $f_1^\ell(x_j; V_q')(\mathbf{w} - e_\ell) = \iota_q^{\mathbf{w}-e_\ell-e_j,\mathbf{w}-e_\ell}$ is an injection by corollary 4.0.5, but we know that f' is its restriction, so f' is also an injection. On the other hand, we know that the domain and codomain of f' are finite dimensional vector spaces, so it is enough to show that the dimension of the domain and codomain are the same in order to show that f' is an isomorphism since it is already an injection map between finite dimensional vector spaces. For the sake of the proof, let us denote by $Z_q(M^{\mathbf{v}})$ the submodule which is generated by the cycles of $C_q(M^{\mathbf{v}})$, and denote by $B_q(M^{\mathbf{v}})$ the submodule which is generated by the boundaries of $C_q(M^{\mathbf{v}})$ for all $\mathbf{v} \in \mathbb{N}^n$. Now, we know that Remark 4.0.1 (i), and (ii) hold by our assumption. The first property (i) tells us that

$$
\begin{aligned}
M_q^{\mathbf{w}-e_\ell-e_j} &= M_q^{\mathbf{w}-e_\ell}\\
M_q^{\mathbf{w}-e_j} &= M_q^{\mathbf{w}}
\end{aligned}
$$

since $\mathbf{w} = \mathbf{u} - e_\alpha$ and $\alpha \subseteq [n] \setminus \{j, \ell\} \subseteq [n] \setminus \{j\}$. Thus,

$$
\begin{aligned}
Z_q(M^{\mathbf{w}-e_\ell-e_j}) &= Z_q(M^{\mathbf{w}-e_\ell})\\
Z_q(M^{\mathbf{w}-e_j}) &= Z_q(M^{\mathbf{w}}).
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
M_{q+1}^{\mathbf{w}-e_\ell-e_j} &= M_{q+1}^{\mathbf{w}-e_j}\\
M_{q+1}^{\mathbf{w}-e_\ell} &= M_{q+1}^{\mathbf{w}}
\end{aligned}
$$

since $\mathbf{w} = \mathbf{u} - e_\alpha$, and $\alpha \subseteq [n] \setminus \{j, \ell\} \subseteq [n] \setminus \{\ell\}$. Thus,

$$
\begin{aligned}
B_q(M^{\mathbf{w}-e_\ell-e_j}) &= B_q(M^{\mathbf{w}-e_j})\\
B_q(M^{\mathbf{w}-e_\ell}) &= B_q(M^{\mathbf{w}}).
\end{aligned}
$$

Thus,

$$\begin{aligned}
dim_{\mathbf{F}}(H_q(M^{\mathbf{w}-e_\ell})) &= dim_{\mathbf{F}}(Z_q(M^{\mathbf{w}-e_\ell})/B_q(M^{\mathbf{w}-e_\ell})) \\
&= dim_{\mathbf{F}}(Z_q(M^{\mathbf{w}-e_\ell})/B_q(M^{\mathbf{w}})) \\
&= dim_{\mathbf{F}}(Z_q(M^{\mathbf{w}-e_\ell})) - dim_{\mathbf{F}}(B_q(M^{\mathbf{w}}))) \\
dim_{\mathbf{F}}(H_q(M^{\mathbf{w}-e_\ell-e_j})) &= dim_{\mathbf{F}}(Z_q(M^{\mathbf{w}-e_j-e_\ell})/B_q(M^{\mathbf{w}-e_j-e_\ell})) \\
&= dim_{\mathbf{F}}(Z_q(M^{\mathbf{w}-e_\ell})/B_q(M^{\mathbf{w}-e_j})) \\
&= dim_{\mathbf{F}}(Z_q(M^{\mathbf{w}-e_\ell})) - dim_{\mathbf{F}}(B_q(M^{\mathbf{w}-e_j}))) \\
dim_{\mathbf{F}}(H_q(M^{\mathbf{w}})) &= dim_{\mathbf{F}}(Z_q(M^{\mathbf{w}})/B_q(M^{\mathbf{w}})) \\
&= dim_{\mathbf{F}}(Z_q(M^{\mathbf{w}})) - dim_{\mathbf{F}}(B_q(M^{\mathbf{w}})) \\
dim_{\mathbf{F}}(H_q(M^{\mathbf{w}-e_j})) &= dim_{\mathbf{F}}(Z_q(M^{\mathbf{w}-e_j})/B_q(M^{\mathbf{w}-e_j})) \\
&= dim_{\mathbf{F}}(Z_q(M^{\mathbf{w}})/B_q(M^{\mathbf{w}-e_j})) \\
&= dim_{\mathbf{F}}(Z_q(M^{\mathbf{w}})) - dim_{\mathbf{F}}(B_q(M^{\mathbf{w}-e_j})).
\end{aligned}$$

So, if we apply the Rank-Nullity theorem, then we will obtain

$$\begin{aligned}
dim_{\mathbf{F}}(ker(\iota_q^{\mathbf{w}-e_\ell-e_j,\mathbf{w}-e_\ell})) &= dim_{\mathbf{F}}(H_q(M^{\mathbf{w}-e_\ell-e_j})) - dim_{\mathbf{F}}(im(\iota_q^{\mathbf{w}-e_\ell-e_j,\mathbf{w}-e_\ell})) \\
&= dim_{\mathbf{F}}(H_q(M^{\mathbf{w}-e_\ell-e_j})) - dim_{\mathbf{F}}(H_q(\mathbf{w}-e_\ell)) \\
&= dim_{\mathbf{F}}(B_q(M^{\mathbf{w}})) - dim_{\mathbf{F}}(B_q(M^{\mathbf{w}-e_j})), \\
dim_{\mathbf{F}}(ker(\iota_q^{\mathbf{w}-e_j,\mathbf{w}})) &= dim_{\mathbf{F}}(H_q(M^{\mathbf{w}-e_j})) - dim_{\mathbf{F}}(im(\iota_q^{\mathbf{w}-e_j,\mathbf{w}})) \\
&= dim_{\mathbf{F}}(H_q(M^{\mathbf{w}-e_j})) - dim_{\mathbf{F}}(H_q(M^{\mathbf{w}})) \\
&= dim_{\mathbf{F}}(B_q(M^{\mathbf{w}})) - dim_{\mathbf{F}}(B_q(M^{\mathbf{w}-e_j})).
\end{aligned}$$

Thus,

$$dim_{\mathbf{F}}(ker(\iota_q^{\mathbf{w}-e_\ell-e_j,\mathbf{w}-e_\ell})) = dim_{\mathbf{F}}(ker(\iota_q^{\mathbf{w}-e_j,\mathbf{w}})).$$

Hence, f' is an isomorphism. □

At the moment, we have all of the necessary background, and information in order to prove our main theorem, so let us continue with it.

Theorem 4.0.8. *Let $\{X^{\mathbf{u}}\}_{\mathbf{u}\in\mathbb{N}^n}$ be an n-parameter filtration of the cell complex X such that it is exhaustive, V a discrete gradient vector field which is consistent with the n-parameter filtration $\{X^{\mathbf{u}}\}_{\mathbf{u}\in\mathbb{N}^n}$, and $\{M^{\mathbf{u}}\}_{\mathbf{u}\in\mathbb{N}^n}$ an n-parameter filtration of the* *cell complex M which is the Morse complex of X associated with V. Then,*

$$\bigcup_{i=0}^{n} supp\, \xi_i^q \subseteq \overline{\mathcal{G}(M_q)} \cup \overline{\mathcal{G}(M_{q+1})},$$

for all $q \in \mathbb{N}$. Moreover,

$$supp\, \xi_0^q \subseteq \overline{\mathcal{G}(M_q)},$$

and

$$supp\, \xi_n^q \subseteq \overline{\mathcal{G}(M_{q+1})},$$

for all $q \in \mathbb{N}$.

Proof. Let $q \in \mathbb{N}$. Firstly, let us show that $\bigcup_{i=0}^{n} supp\, \xi_i^q \subseteq \overline{\mathcal{G}(M_q)} \cup \overline{\mathcal{G}(M_{q+1})}$. Let $\mathbf{u} \notin \overline{\mathcal{G}(M_q)} \cup \overline{\mathcal{G}(M_{q+1})}$, then we know that the properties (i) and (ii) of the Remark 4.0.1 hold. Thus, we have indices j and ℓ which are coming from the properties (i) and (ii), respectively. At the moment, we have two cases.

Case 1: Assume that the indices j and ℓ are the same. Then, the Koszul complex $\mathbf{K}_*(x_1, \ldots, x_n; V_q)(\mathbf{u})$ is acyclic by Lemma 4.0.6, so

$$H_i(\mathbf{K}_*(x_1, \ldots, x_n; V_q)(\mathbf{u})) = 0$$

for all $i \in \{0, 1, \ldots, n\}$. Thus,

$$\xi_i^q(\mathbf{u}) = dim_{\mathbf{F}}(H_i(\mathbf{K}_*(x_1, \ldots, x_n; V_q)(\mathbf{u}))) = 0$$

for all $i \in \{0, 1, \ldots, n\}$. This implies that $\mathbf{u} \notin \bigcup_{i=0}^{n} supp\, \xi_i^q$. Hence,

$$\bigcup_{i=0}^{n} supp\, \xi_i^q \subseteq \overline{\mathcal{G}(M_q)} \cup \overline{\mathcal{G}(M_{q+1})}.$$

Case 2: Assume that the indices j and ℓ are different from each other. Then, we can say that $\mathbf{K}_*(x_j, x_\ell; V_q)(\mathbf{u} - e_\alpha)$ is acyclic for all $\alpha \subseteq [n] \setminus \{j, \ell\}$ by Lemma 4.0.7. Observe that if $k \in [n] \setminus \{j, \ell\}$, and $\alpha \subseteq [n] \setminus \{j, \ell, k\}$, then Theorem 3.2.4 implies that $\mathbf{K}_*(x_k, x_j, x_\ell; V_q)(\mathbf{u} - e_\alpha)$ is the mapping cone of the chain map

$$f^k(x_j, x_\ell; V_q)(\mathbf{u} - e_\alpha - e_k) : \mathbf{K}_*(x_j, x_\ell)(\mathbf{u} - e_\alpha - e_k) \to \mathbf{K}_*(x_j, x_\ell; V_q)(\mathbf{u} - e_\alpha),$$

and $\mathbf{K}_*(x_k, x_j, x_\ell; V_q)(\mathbf{u} - e_\alpha)$ is an acyclic chain complex by the Corollary 3.2.6. Also, we know that we can obtain the Koszul complex $\mathbf{K}_*(x_1, \ldots, x_n; V_q)(\mathbf{u})$ from the Koszul complexes $\mathbf{K}_*(X_j, x_\ell; V_q)(\mathbf{u} - e_\alpha)$ by using Theorem 3.2.4. In addition to this, we know that we get an acyclic Koszul complex at each step when we construct $\mathbf{K}_*(x_1, \ldots, x_n; V_q)(\mathbf{u})$ by the above observation, so $\mathbf{K}_*(x_1, \ldots, x_n; V_q)(\mathbf{u})$ is an acyclic chain complex. Thus,

$$H_i(\mathbf{K}_*(x_1, \ldots, x_n; V_q)(\mathbf{u})) = 0$$

for all $i \in \{0, 1, \ldots, n\}$. Thus,

$$\begin{aligned}\xi_i^q(\mathbf{u}) &= dim_{\mathbf{F}}(H_i(\mathbf{K}_*(x_1, \ldots, x_n; V_q)(\mathbf{u}))) \\ &= 0\end{aligned}$$

for all $i \in \{0, 1, \ldots, n\}$. This implies that $\mathbf{u} \notin \bigcup_{i=0}^n supp\, \xi_i^q$.

In both cases, we obtained that $\mathbf{u} \notin \bigcup_{i=0}^n supp\, \xi_i^q$ whenever $\mathbf{u} \notin \overline{\mathcal{G}(M_q)} \cup \overline{\mathcal{G}(M_{q+1})}$. Hence,

$$\bigcup_{i=0}^{n} supp\, \xi_i^q \subseteq \overline{\mathcal{G}(M_q)} \cup \overline{\mathcal{G}(M_{q+1})}.$$

Secondly, let us show that $\xi_0^q \subseteq \overline{\mathcal{G}(M_q)}$. Let $\mathbf{u} \notin \overline{\mathcal{G}(M_q)}$, then there exists $j \in [n]$ such that

$$\iota_q^{\mathbf{u}-e_j,\mathbf{u}} : H_q(X^{\mathbf{u}-e_j}) \to H_q(X^{\mathbf{u}})$$

is a surjection by Corollary 4.0.5. Also, we know that

$$\begin{aligned}\mathbf{K}_0(x_1, \ldots, x_n; V_q)(\mathbf{u}) &= H_q(X^{\mathbf{u}}) \\ \mathbf{K}_1(x_1, \ldots, x_n; V_q)(\mathbf{u}) &= \bigoplus_{j\in[n]} H_q(X^{\mathbf{u}-e_j}),\end{aligned}$$

and $\iota_q^{\mathbf{u}-e_j,\mathbf{u}}$ is the restriction of d_1 to $H_q(X^{\mathbf{u}-e_j})$ where d_i is the differential map of $\mathbf{K}_*(x_1, \ldots, x_n; V_q)(\mathbf{u})$ from $\mathbf{K}_i(x_1, \ldots, x_n; V_q)(\mathbf{u})$ to $\mathbf{K}_{i-1}(x_1, \ldots, x_n; V_q)(\mathbf{u})$ for all $i \in \{0, 1, \ldots, n\}$, so d_1 is also a surjection, thus

$$\begin{aligned}H_0(\mathbf{K}_*(x_1, \ldots, x_n; V_q)(\mathbf{u})) &= ker(d_0)/im(d_1) \\ &= H_q(X^{\mathbf{u}})/H_q(X^{\mathbf{u}}) \\ &= 0.\end{aligned}$$

Thus,

$$\begin{aligned}\xi_0^q(\mathbf{u}) &= dim_{\mathbf{F}}(H_0(\mathbf{K}_*(x_1, \ldots, x_n; V_q)(\mathbf{u}))) \\ &= 0,\end{aligned}$$

so $\mathbf{u} \notin supp\, \xi_0^q$. Hence,

$$supp\, \xi_0^q \subseteq \overline{\mathcal{G}(M_q)}.$$

Lastly, let us show that $supp\, \xi_n^q \subseteq \overline{\mathcal{G}(M_{q+1})}$. Let $\mathbf{u} \notin \overline{\mathcal{G}(M_{q+1})}$, then there exists $j \in [n]$ such that

$$d_n = \iota_q^{\mathbf{u}-e_{[n]\setminus\{j\}}-e_j, \mathbf{u}-e_{[n]\setminus\{j\}}} = \iota_q^{\mathbf{u}-e_{[n]}, \mathbf{u}-e_{[n]\setminus\{j\}}} : H_q(X^{\mathbf{u}-e_{[n]}}) \to H_q(X^{\mathbf{u}-e_{[n]\setminus\{j\}}})$$

is an injection by Corollary 4.0.5. However, we also know that

$$\begin{aligned} \mathbf{K}_n(x_1, \ldots, x_n; V_q)(\mathbf{u}) &= H_q(X^{\mathbf{u}-e_{[n]}}), \\ \mathbf{K}_{n-1}(x_1, \ldots, x_n; V_q)(\mathbf{u}) &= \bigoplus_{j \in [n]} H_q(X^{\mathbf{u}-e_{[n]\setminus\{j\}}}). \end{aligned}$$

Thus,

$$\begin{aligned} H_n(\mathbf{K}_*(x_1, \ldots, x_n; V_q)(\mathbf{u})) &= ker(d_n) \\ &= 0. \end{aligned}$$

As a result of this,

$$\begin{aligned} \xi_n^q(\mathbf{u}) &= dim_{\mathbf{F}}(H_n(\mathbf{K}_*(x_1, \ldots, x_n; V_q)(\mathbf{u}))) \\ &= 0. \end{aligned}$$

This implies that $\mathbf{u} \notin supp\, \xi_n^q$. Hence,

$$supp\, \xi_n^q \subseteq \overline{\mathcal{G}(M_{q+1})}.$$

□

CHAPTER 5

SUPPORT OF BETTI TABLES FOR BIFILTRATIONS

Our main result was Theorem 4.0.8 for this thesis. In this chapter, we will restrict ourselves to $n = 2$, and we will strengthen our results for this restrictive case. In this chapter, we will use two articles as our source. One of the article is "relative-perfectness of discrete gradient vector fields and multi-parameter persistent homology" [14], which is written by Claudia Landi and Sara Scaramuccia, to prove an inequality, which will be important for us to prove the main theorem of this chapter. The other article is our main source which is "on the support of Betti tables of multiparameter homology modules" [13], which is written by Claudia Landi and Andrea Guidolin.

5.1 Estimation of Betti Tables via Critical Cells

We will use the same tools which we have used in Chapter 4 in this chapter, so fix the structures as we did in the chapter 4. Also, note that we calculate all dimensions with respect to the field **F**.

Proposition 5.1.1. [14] Let $q \in \mathbb{N}$, $\mathbf{u} \in \mathbb{N}^n$, and $\iota_q^{\mathbf{u}}$ be the induced map which is obtained by applying the q-th homology functor to the inclusion map $\bigcup_{i=1}^{n} X^{\mathbf{u}-e_i} \hookrightarrow X^{\mathbf{u}}$, then

$$dim(H_q(X^{\mathbf{u}}, \bigcup_{i=1}^{n} X^{\mathbf{u}-e_i})) = dim(coker(\iota_q^{\mathbf{u}})) + dim(ker(\iota_{q-1}^{\mathbf{u}})).$$

Proof. In order to prove the equality, firstly let us write the relative long exact homological sequence of the pair $(X^{\mathbf{u}}, \bigcup_{i=1}^{n} X^{\mathbf{u}-e_i})$

$$\ldots \xrightarrow{\delta_{q+1}^{\mathbf{u}}} H_q(\bigcup_{i=1}^{n} X^{\mathbf{u}-e_i}) \xrightarrow{\iota_q^{\mathbf{u}}} H_q(X^{\mathbf{u}}) \xrightarrow{j_q^{\mathbf{u}}} H_q(X^{\mathbf{u}}, \bigcup_{i=1}^{n} X^{\mathbf{u}-e_i}) \xrightarrow{\delta_q^{\mathbf{u}}} \ldots$$

Now,

$$\begin{aligned}
dim(coker(\iota_q^{\mathbf{u}})) + dim(ker(\iota_{q-1}^{\mathbf{u}})) &= dim(H_q(X^{\mathbf{u}})/im(\iota_q^{\mathbf{u}})) + dim(ker(\iota_{q-1}^{\mathbf{u}})) \\
&= dim(dim(H_q(X^{\mathbf{u}})) - dim(im(\iota_q^{\mathbf{u}}) \\
&+ dim(ker(\iota_{q-1}^{\mathbf{u}})) \\
&= dim(H_q(X^{\mathbf{u}})) - dim(ker(j_q^{\mathbf{u}})) \\
&+ dim(im(\delta_q^{\mathbf{u}})) \\
&= dim(H_q(X^{\mathbf{u}})) - dim(ker(j_q^{\mathbf{u}})) \\
&+ dim(H_q(X^{\mathbf{u}}, \bigcup_{i=1}^{n} X^{\mathbf{u}-e_i})) - dim(ker(\delta_q^{\mathbf{u}})) \\
&= dim(ker(j_q^{\mathbf{u}})) + dim(H_q(X^{\mathbf{u}}, \bigcup_{i=1}^{n} X^{\mathbf{u}-e_i})) \\
&- dim(im(j_q^{\mathbf{u}})) \\
&= dim(H_q(X^{\mathbf{u}}, \bigcup_{i=1}^{n} X^{\mathbf{u}-e_i})).
\end{aligned}$$

After this point, we will fix $n = 2$ until the end of this section. Also, when we study with a fixed $\mathbf{u} \in \mathbb{N}^2$, we will set $\mathbf{x} = \mathbf{u} - e_1, \mathbf{y} = \mathbf{u} - e_2$ and $\mathbf{z} = \mathbf{u} - e_1 - e_2$.

□

Lemma 5.1.2. [14] If A, C and D are finite dimensional vector spaces and the following diagram

$$\begin{array}{ccc} C & \xrightarrow{\gamma} & D \\ & {\scriptstyle\mu}\searrow \quad \swarrow{\scriptstyle\iota} & \\ & A & \end{array}$$

is commutative, then the following equalities hold:

i) $dim(ker(\mu)) = dim(im(\gamma) \cap ker(\iota)) + dim(ker(\gamma))$,

ii) $dim(ker(\iota)) = dim(im(\gamma) \cap ker(\iota)) - dim(im(\gamma)) + dim(im(\gamma) + ker(\iota))$,

iii) $dim(coker(\iota)) = dim(coker(\mu)) - dim(D) + dim(im(\gamma) + ker(\iota))$.

Proof. (i) By the commutativity of the given diagram, we know that $\mu = \iota \circ \gamma$, so

$$\begin{aligned} dim(ker(\mu)) &= dim(ker(\iota \circ \gamma)) \\ &= dim(ker(\gamma)) + dim(im(\gamma) \cap ker(\iota)). \end{aligned}$$

(ii) This equality is a well-known linear algebra result, so we will skip the proof of this equality.

(iii)

$$\begin{aligned} dim(coker(\iota)) &= dim(A/im(\iota)) \\ &= dim(A) - dim(im(\iota)) \\ &= dim(A) - dim(D) - dim(ker(\iota)) \\ &= dim(A) - dim(im(\mu)) + dim(im(\mu)) - dim(D) + dim(ker(\iota)) \\ &= dim(A/im(\mu)) + dim(C) - dim(ker(\mu)) - dim(D) \\ &+ dim(ker(\iota)) \\ &= dim(coker(\mu)) + dim(ker(\gamma)) + dim(im(\gamma)) \\ &- dim(im(\gamma) \cap ker(\iota)) + dim(ker(\gamma)) - dim(D) + dim(ker(\iota)) \\ &= dim(coker(\mu)) - dim(D) + dim(im(\gamma) \cap ker(\iota)). \end{aligned}$$

□

After the following proposition, we will prove an inequality which will be a corollary of the proposition. This corollary will play a key role in the main theorem of this chapter as we mentioned at the beginning of this section.

Proposition 5.1.3. [14] If $q \in \mathbb{N}, \mathbf{u} \in \mathbb{N}^n, \alpha_q^{\mathbf{u}} : H_q(X^{\mathbf{x}}) \to H_q(X^{\mathbf{x}} \cup X^{\mathbf{y}}), \beta_q^{\mathbf{u}} : H_q(X^{\mathbf{y}}) \to H_q(X^{\mathbf{x}} \cup X^{\mathbf{y}}), \iota_q^{\mathbf{u}} : H_q(X^{\mathbf{x}} \cup X^{Y}) \to H_q(X^{\mathbf{u}})$ are the linear maps induced by the inclusions, then the following equalities hold:

(i) $dim(coker(\iota_q^{\mathbf{u}})) = \xi_0^q(\mathbf{u}) - dim(H_q(X^{\mathbf{x}} \cup X^{\mathbf{y}})) + dim(ker(\iota_q^{\mathbf{u}}) + im(\alpha_q^{\mathbf{u}} - \beta_q^{\mathbf{u}}))$,

(ii) $dim(ker(\iota_q^{\mathbf{u}})) = \xi_1^q(\mathbf{u}) + \xi_2^{q-1} - dim(H_q(X^{\mathbf{x}} \cup X^{\mathbf{y}})) + dim(ker(\iota_q^{\mathbf{u}}) + im(\alpha_q^{\mathbf{u}} - \beta_q^{\mathbf{u}}))$

where $\mathbf{x}, \mathbf{y}, \mathbf{z}$ are as we mentioned above.

Proof. We know that $H_q(X^{\mathbf{x}}) \oplus H_q(X^{\mathbf{y}}), H_q(X^{\mathbf{x}} \cup X^{\mathbf{y}})$ and $H_{q-1}(X^{\mathbf{u}})$ are finite dimensional vector spaces, and the following diagram

$$\begin{array}{ccc} H_q(X^{\mathbf{x}}) \oplus H_q(X^{\mathbf{y}}) & \xrightarrow{\alpha_q^{\mathbf{u}} - \beta_q^{\mathbf{u}}} & H_q(X^{\mathbf{x}} \cup X^{\mathbf{y}}) \\ & {\scriptstyle d_1^{q,\mathbf{u}}} \searrow \quad \swarrow {\scriptstyle \iota_q^{\mathbf{u}}} & \\ & H_{q-1}(X^{\mathbf{u}}) & \end{array},$$

where $d_1^{q,\mathbf{u}} = [\iota_q^{\mathbf{x},\mathbf{u}}, \iota_q^{\mathbf{y},\mathbf{u}}]$ for all $q \in \mathbb{N}$, is commutative by functoriality of the homology. Thus, we can apply the results which we obtained in Lemma 5.1.2. The first equality of the Lemma 5.1.2 implies that

$$dim(ker(d_1^{q,\mathbf{u}})) = dim(ker(\iota_q^{\mathbf{u}}) \cap im(\alpha_q^{\mathbf{u}} - \beta_q^{\mathbf{u}})) + dim(ker(\alpha_q^{\mathbf{u}} - \beta_q^{\mathbf{u}})), \tag{5.1}$$

the second equality of the Lemma 5.1.2 implies that

$$\begin{aligned} dim(ker(\iota_q^{\mathbf{u}})) = dim(im(\alpha_q^{\mathbf{u}} - \beta_q^{\mathbf{u}}) \cap ker(\iota_q^{\mathbf{u}})) - dim(im(\alpha_q^{\mathbf{u}} - \beta_q^{\mathbf{u}})) \\ + dim(im(\alpha_q^{\mathbf{u}} - \beta_q^{\mathbf{u}}) + ker(\iota_q^{\mathbf{u}})), \end{aligned} \tag{5.2}$$

and lastly the third equality of the Lemma 5.1.2 implies that

$$\begin{aligned} dim(coker(\iota_q^{\mathbf{u}})) = dim(coker(d_1^{q,\mathbf{u}})) - dim(H_q(X^{\mathbf{x}} \cup X^{\mathbf{y}})) \\ + dim(im(\alpha_q^{\mathbf{u}} - \beta_q^{\mathbf{u}}) + ker(\iota_q^{\mathbf{u}})). \end{aligned} \tag{5.3}$$

At the moment, we have three equalities, and this will play the key roles for the rest of the proof. By using the third equality 5.3, we can directly show that the first equality of the Proposition 5.1.3. We know that

$$dim(coker(d_1^{q,\mathbf{u}})) = \xi_0^q(\mathbf{u}) \text{ by the Example 3.1.1.}$$

Thus, if we replace $dim(coker(d_1^{q,\mathbf{u}}))$ with $\xi_0^q(\mathbf{u})$, then we will obtain the first equality of the Proposition 5.1.3.

For the second equality, we will use the triad $(X^{\mathbf{x}} \cup X^{\mathbf{y}}, X^{\mathbf{x}}, X^{\mathbf{y}})$. Considering the Mayer-Vietoris exact sequence of this triad, then we obtain the following long exact sequence

$$\ldots \to H_q(X^{\mathbf{z}}) \xrightarrow{d_2^{q,\mathbf{u}}} H_q(X^{\mathbf{x}}) \oplus H_q(X^{\mathbf{y}}) \xrightarrow{\alpha_q^{\mathbf{u}} - \beta_q^{\mathbf{u}}} H_q(X^{\mathbf{x}} \cup X^{\mathbf{y}}) \xrightarrow{\delta_q^{\mathbf{u}}} H_{q-1}(X^{\mathbf{z}}) \to \ldots$$

where $X^{\mathbf{z}} = X^{\mathbf{u}-e_1-e_2} = X^{\mathbf{u}-e_1} \cap X^{\mathbf{u}-e_2} = X^{\mathbf{x}} \cap X^{\mathbf{y}}$, and

$$d_2^{q,\mathbf{u}} = \begin{bmatrix} -\iota_q^{\mathbf{u}-e_1-e_2,\mathbf{u}-e_1} \\ \iota_q^{\mathbf{u}-e_1-e_2,\mathbf{u}-e_2} \end{bmatrix}$$

for all $q \in \mathbb{N}$. Now, the equality 5.1, the Example 3.1.1 and the above long exact sequence implies that

$$\begin{aligned}\xi_1^q(\mathbf{u}) &= dim(ker(d_1^{q,\mathbf{u}})) - dim(im(d_2^{q,\mathbf{u}})) \\ &= dim(ker(\alpha_q^{\mathbf{u}} - \beta_q^{\mathbf{u}})) + dim(ker(\iota_q^{\mathbf{u}}) \cap im(\alpha_q^{\mathbf{u}} - \beta_q^{\mathbf{u}})) \\ &- dim(im(d_2^{q,\mathbf{u}})) \text{ by the equality 5.1} \\ &= dim(ker(\iota_q^{\mathbf{u}}) \cap im(\alpha_q^{\mathbf{u}} - \beta_q^{\mathbf{u}}))\end{aligned} \tag{5.4}$$

by exactness. Moreover, the above long exact sequence, the Example 3.1.1 and the rank-nullity theorem implies that

$$\begin{aligned}\xi_2^{q-1}(\mathbf{u}) &= dim(ker(d_2^{q-1,\mathbf{u}})) \text{ by example 3.1.1} \\ &= dim(im(\delta_q^{\mathbf{u}})) \text{ by exactness} \\ &= dim(H_q(X^{\mathbf{x}} \cup X^{\mathbf{y}})) - dim(ker(\delta_q^{\mathbf{u}})) \text{ by rank-nullity theorem} \\ &= dim(H_q(X^{\mathbf{x}} \cup X^{\mathbf{y}})) - dim(im(\alpha_q^{\mathbf{u}} - \beta_q^{\mathbf{u}})) \text{ by exactness.}\end{aligned} \tag{5.5}$$

Lastly, if we apply the equalities which is obtained in 5.2, 5.4 and 5.5, then we get

$$\begin{aligned}dim(ker(\iota_q^{\mathbf{u}})) &= \xi_1^q(\mathbf{u}) - dim(im(\alpha_q^{\mathbf{u}} - \beta_q^{\mathbf{u}})) + dim(im(\alpha_q^{\mathbf{u}} - \beta_q^{\mathbf{u}}) + ker(\iota_q^{\mathbf{u}})) \\ &= \xi_1^q(\mathbf{u}) + \xi_2^{q-1}(\mathbf{u}) - dim(H_q(X^{\mathbf{x}} \cup X^{\mathbf{y}})) \\ &+ dim(im(\alpha_q^{\mathbf{u}} - \beta_q^{\mathbf{u}}) + ker(\iota_q^{\mathbf{u}})).\end{aligned}$$

□

At this moment, we can prove an important equality which will be used in the second section of this chapter to prove the main theorem of section 2.

Corollary 5.1.4. [14] For any $q \in \mathbb{N}$,

$$\xi_0^q(\mathbf{u}) + \xi_1^{q-1}(\mathbf{u}) - \xi_2^{q-1}(\mathbf{u}) \leq dim(H_q(X^{\mathbf{u}}, X^{\mathbf{x}} \cup X^{\mathbf{y}})) \leq \xi_0^q(\mathbf{u}) + \xi_1^{q-1}(\mathbf{u}) + \xi_2^{q-2}(\mathbf{u})$$

Proof. On the one hand,

$$
\begin{aligned}
dim(H_q(X^{\mathbf{u}}, X^{\mathbf{x}} \cup X^{\mathbf{y}})) &= dim(coker(\iota_q^{\mathbf{u}})) + dim(ker(\iota_{q-1}^{\mathbf{u}})) \text{ by Proposition 5.1.1} \\
&= \xi_0^q(\mathbf{u}) + \xi_1^{q-1}(\mathbf{u}) + \xi_2^{q-2}(\mathbf{u}) \\
&- dim(H_q(X^{\mathbf{x}} \cup X^{\mathbf{y}})) + dim(ker(\iota_q^{\mathbf{u}}) + im(\alpha_q^{\mathbf{u}} - \beta_q^{\mathbf{u}})) \\
&- dim(H_{q-1}(X^{\mathbf{x}} \cup X^{\mathbf{y}})) \\
&+ dim(ker(\iota_{q-1}^{\mathbf{u}}) + im(\alpha_{q-1}^{\mathbf{u}} - \beta_{q-1}^{\mathbf{u}})) \text{ by 5.1.3} \\
&\geq \xi_0^q(\mathbf{u}) + \xi_1^{q-1}(\mathbf{u}) + \xi_2^{q-2}(\mathbf{u}) \\
&- dim(H_q(X^{\mathbf{x}} \cup X^{Y})) + dim(im(\alpha_q^{\mathbf{u}} - \beta_q^{\mathbf{u}})) \\
&- dim(H_{q-1}(X^{\mathbf{x}} \cup X^{\mathbf{y}})) + dim(im(\alpha_{q-1}^{\mathbf{u}} - \beta_{q-1}^{\mathbf{u}})) \\
&= \xi_0^q(\mathbf{u}) + \xi_1^{q-1} - \xi_2^{q-1} \text{ by equation 5.5}
\end{aligned} \tag{5.6}
$$

Note that the inequality in the equation 5.6 holds since

$$im(\alpha_q^{\mathbf{u}} - \beta_q^{\mathbf{u}}) \subseteq ker(\iota_q^{\mathbf{u}}) + im(\alpha_q^{\mathbf{u}} - \beta_q^{\mathbf{u}})$$

for all $q \in \mathbb{N}$.

On the other hand, we know that

$$ker(\iota_q^{\mathbf{u}}) + im(\alpha_q^{\mathbf{u}} - \beta_q^{\mathbf{u}}) \subseteq H_q(X^{\mathbf{x}} \cup X^{\mathbf{y}})$$

for all $q \in \mathbb{N}$, thus

$$dim(H_q(X^{\mathbf{x}} \cup X^{\mathbf{y}})) \geq dim(ker(\iota_q^{\mathbf{u}}) + im(\alpha_q^{\mathbf{u}} - \beta_q^{\mathbf{u}}))$$

for all $q \in \mathbb{N}$, thus the second equality of the equation 5.6 gives us the right-hand side of the inequality. □

5.2 Homological Critical Grades

As we said at the beginning of the first section of this chapter, we will strengthen the main Theorem 4.0.8 of the thesis for $n = 2$. In order to do this, firstly let us define some sets.

Definition 29. [13] Let $\mathbf{u} \in \mathbb{N}^n$, $q \in \mathbb{N}$. We call $\mathbf{u}$ as a q-homological critical grade if $dim(H_q(X^{\mathbf{u}}, \bigcup_{i=1}^{n} X^{\mathbf{u}-e_i})) \neq 0$.

Notation 5.2.1. [13] We denote the set of q-homological critical grades as $\mathcal{C}_q(X)$, that is,

$$\mathcal{C}_q(X) := \{\mathbf{u} \in \mathbb{N}^n \mid dim(H_q(X^{\mathbf{u}}, \bigcup_{i=1}^{n} X^{\mathbf{u}-e_i})) \neq 0\}.$$

After this point, we have two goals. The first goal is to show that $\mathcal{C}_q(X) \subseteq \mathcal{G}(M_q)$. After showing this, the other goal is to show that we can bound the support of the Betti tables by using $\mathcal{C}_q(X)$ instead of $\mathcal{G}(M_q)$ for the restrictive case $n = 2$.

Definition 30. [14] Let $\mathbf{u} \in \mathbb{N}^n$, $q \in \mathbb{N}$, then we call the number of critical q-cells of V contained in $M^{\mathbf{u}} \setminus \bigcup_{i=1}^{n} M^{\mathbf{u}-e_i}$ as the q-th Morse number of V at $\mathbf{u}$, and we will denote it as $m_q(\mathbf{u})$.

Lemma 5.2.2. [14] Let $\mathbf{u} \in \mathbb{N}^n$, $q \in \mathbb{N}$. If $S \subseteq S' \subseteq Q = \{0,1\}^n$, then there are isomorphisms

$$\varphi_q^S : H_q(\bigcup_{s \in S} X^{\mathbf{u}-s}) \to H_q(\bigcup_{s \in S} M^{\mathbf{u}-s})$$

and

$$\varphi_q^{S'} : H_q(\bigcup_{s \in S'} X^{\mathbf{u}-s}) \to H_q(\bigcup_{s \in S'} M^{\mathbf{u}-s})$$

so that the following diagram

$$\begin{array}{ccc} H_q(\bigcup_{s \in S} X^{\mathbf{u}-s}) & \xrightarrow{\iota_1} & H_q(\bigcup_{s \in S'} X^{\mathbf{u}-s}) \\ \Big\downarrow \varphi_q^S & & \Big\downarrow \varphi_q^{S'} \\ H_q(\bigcup_{s \in S} M^{\mathbf{u}-s}) & \xrightarrow{\iota_2} & H_q(\bigcup_{s \in S'} M^{\mathbf{u}-s}) \end{array}$$

commutes where ι_1 and ι_2 are the maps induced by inclusions.

Proof. By Lemma 2.3.1, we already know that there exists a map $\Phi_{\mathbf{u}}$ which induces an isomorphism on all homology degrees such that the following diagram commutes.

$$\begin{array}{ccc} X^{\mathbf{u}} & \xrightarrow{\iota_1} & X^{\mathbf{u}} \\ \Big\downarrow \Phi_{\mathbf{u}} & & \Big\downarrow \Phi_{\mathbf{u}} \\ M^{\mathbf{u}} & \xrightarrow{\iota_2} & M^{\mathbf{u}} \end{array} \tag{5.7}$$

Since the above diagram is commutative, the following diagram is also commutative;

$$\begin{array}{ccc} \bigcup\limits_{s\in S} X^{\mathbf{u}-s} & \xrightarrow{\iota_1} & \bigcup\limits_{s\in S'} X^{\mathbf{u}-s} \\ \Big\downarrow \Phi_q^S & & \Big\downarrow \Phi_q^{S'} \\ \bigcup\limits_{s\in S} M^{\mathbf{u}-s} & \xrightarrow{\iota_2} & \bigcup\limits_{s\in S'} M^{\mathbf{u}-s} \end{array} \tag{5.8}$$

where Φ_q^S is the restriction of $\Phi_\mathbf{u}$ for all $q \in \mathbb{N}$, for all $S \subseteq Q$. Commutativity holds because horizontal maps are inclusion maps in both diagrams, and the vertical maps of the diagram 5.8 is the restriction of the map $\Phi_\mathbf{u}$ which is the vertical map of the diagram 5.7. Thus, the diagram 5.8 is also a commutative diagram. Now, if we apply q-th homology functor to the diagram 5.8, then it will still be a commutative diagram by the definition of functor. So, if we choose φ_q^S the induced map of Φ_q^S on homology for all $q \in \mathbb{N}$, for all $S \subseteq Q$, then it is enough to show that the induced map of the restriction of $\Phi_\mathbf{u}$ to $\bigcup\limits_{s\in S} X^{\mathbf{u}-s}$ on homology is an isomorphism for all $S \subseteq Q = \{0,1\}^n$. Let us prove this statement by using induction on the poset relation $\leq$ of $\mathbb{N}^n$. For the base step, assume that $S = \emptyset$, then the map will be $\Phi_\mathbf{u}$. By the Lemma 2.3.1, we already know that the induced map of $\Phi_\mathbf{u}$ on homology is an isomorphism. Now, assume that the induced map on homology is an isomorphism for all $S \subseteq Q$ such that $|S| \leq i$. Let us show that it is true for a subset of Q whose cardinality is $i+1$. Let A be the subset of Q such that $|A| = i+1$. We know that we can write A as $S \cup \{a\}$ for some $S \subseteq A$ such that $|S| = i$ and for some $a \in A$. Also, φ_q^S and $\varphi_q^{\{a\}}$ are isomorphisms by the inductive assumption. Let $T = \{t \in \mathbb{N}^n \mid t = \wedge(s,a) \text{ for some } s \in S\}$. Notice that $|T| = i$, $T \subseteq Q$,

$$\begin{aligned} (\bigcup_{s\in S} X^{\mathbf{u}-s}) \cap (X^{\mathbf{u}-a}) &= \bigcup_{s\in S}(X^{\mathbf{u}-s} \cap X^{\mathbf{u}-a}) \\ &= \bigcup_{s\in S}(X^{\mathbf{u}-\wedge(s,a)}) \\ &= \bigcup_{t\in T} X^{\mathbf{u}-t}, \end{aligned}$$

and

$$
\begin{aligned}
(\bigcup_{s\in S} M^{\mathbf{u}-s}) \cap (M^{\mathbf{u}-a}) &= \bigcup_{s\in S}(M^{\mathbf{u}-s}\cap M^{\mathbf{u}-a}) \\
&= \bigcup_{s\in S}(M^{\mathbf{u}-\wedge(s,a)}) \\
&= \bigcup_{t\in T} M^{\mathbf{u}-t},
\end{aligned}
$$

Thus,

$$\varphi_q^T : H_q(\bigcup_{s\in S}(X^{\mathbf{u}-s}\cap X^{\mathbf{u}-a})) \to \bigcup_{s\in S}(M^{\mathbf{u}-s}\cap M^{\mathbf{u}-a})$$

is an isomorphism by our induction assumption. At this moment, we have two different triads $(\bigcup_{s\in S'} X^{\mathbf{u}-s}, \bigcup_{s\in S} X^{\mathbf{u}-s}, X^{\mathbf{u}-a})$ and $(\bigcup_{s\in S'} M^{\mathbf{u}-s}, \bigcup_{s\in S} M^{\mathbf{u}-s}, M^{\mathbf{u}-a})$ such that $S \subseteq S' \subseteq Q$. Let us consider the Mayer-Vietoris exact homological sequence of these two triads. Since we have connecting maps between these Mayer-Vietoris exact homological sequences, we get the following diagram Mayer-Vietoris exact homological sequence of this triad

$$
\begin{array}{ccccc}
\dots H_q(\bigcup_{s\in S} X^{\mathbf{u}-s}) \oplus H_q(X^{\mathbf{u}-a}) & \to & H_q(\bigcup_{s\in S'} X^{\mathbf{u}-s}) & \to & H_{q-1}(\bigcup_{s\in S} X^{\mathbf{u}-s}\cap X^{\mathbf{u}-a})\dots \\
\downarrow{\scriptstyle \psi_q^S} & & \downarrow{\scriptstyle \varphi_q^{S'}} & & \downarrow{\scriptstyle \varphi_{q-1}^T} \\
\dots H_q(\bigcup_{s\in S} M^{\mathbf{u}-s}) \oplus H_q(M^{\mathbf{u}-a}) & \to & H_q(\bigcup_{s\in S'} M^{\mathbf{u}-s}) & \to & H_{q-1}(\bigcup_{s\in S} M^{\mathbf{u}-s}\cap M^{\mathbf{u}-a})\dots
\end{array}
$$

where $\psi_q^S = \varphi_q^S \oplus \varphi_q^{\{a\}}$. Since ψ_q^S and φ_{q-1}^T are isomorphism, $\varphi_q^{S'}$ is also an isomorphism by the Five Lemma. This finishes both induction and the proof of the lemma.

□

Lemma 5.2.3. [14] If $\mathbf{u} \in \mathbb{N}^n$, $q \in \mathbb{N}$, then

$$H_q(X^{\mathbf{u}}, \bigcup_{i=1}^{n} X^{\mathbf{u}-e_i}) \cong H_q(M^{\mathbf{u}}, \bigcup_{i=1}^{n} M^{\mathbf{u}-e_i}).$$

Proof. Let $S = \{e_1, e_2, \dots, e_n\}$, then we know that there exists an isomorphism

$$\varphi_q^S : H_q(\bigcup_{s\in S} X^{\mathbf{u}-s}) \to H_q(\bigcup_{s\in S} M^{\mathbf{u}-s})$$

for any $q \in \mathbb{N}$ by the proof of Lemma 5.2.2. For the sake of the proof, let $A_1 = X^{\mathbf{u}}$, $A_2 = \bigcup_{i=1}^{n} X^{\mathbf{u}-e_i}$, $B_1 = M^{\mathbf{u}}$ and $\bigcup_{i=1}^{n} M^{\mathbf{u}-e_i}$. Now, we have the following commutative diagram which is obtained by the relative homology of the long exact sequences of the pairs (A_1, A_2) and (B_1, B_2):

$$\begin{array}{ccccccccc}
H_q(A_2) & \longrightarrow & H_q(A_1) & \longrightarrow & H_q(A_1, A_2) & \longrightarrow & H_{q-1}(A_2) & \longrightarrow & H_{q-1}(A_1) \\
\downarrow\cong & & \downarrow\cong & & \downarrow & & \downarrow\cong & & \downarrow\cong \\
H_q(B_2) & \longrightarrow & H_q(B_1) & \longrightarrow & H_q(B_1, B_2) & \longrightarrow & H_{q-1}(B_2) & \longrightarrow & H_{q-1}(B_1)
\end{array}$$

By Five Lemma, $H_q(A_1, A_2) \cong H_q(B_1, B_2)$, that is,

$$H_q(X^{\mathbf{u}}, \bigcup_{i=1}^{n} X^{\mathbf{u}-e_i}) \cong H_q(M^{\mathbf{u}}, \bigcup_{i=1}^{n} M^{\mathbf{u}-e_i}).$$

□

Next proposition is important for us because we can easily conclude $\mathcal{C}_q(X) \subseteq \mathcal{G}(M_q)$ by using it.

Proposition 5.2.4. [14] Let $\mathbf{u} \in \mathbb{N}^n$ and $q \in \mathbb{N}$, then

$$m_q(\mathbf{u}) \geq dim(H_q(X^{\mathbf{u}}, \bigcup_{i=1}^{n} X^{\mathbf{u}-e_i})).$$

Proof. By the definition 30, we know that number of critical cells of the discrete gradient vector field V in $M^{\mathbf{u}} \setminus \bigcup_{i=1}^{n} M^{\mathbf{u}-e_i}$ is $m_q(\mathbf{u})$. Thus,

$$\begin{aligned} m_q(\mathbf{u}) &= dim(C_q(M^{\mathbf{u}})) - dim(C_q(\bigcup_{i=1}^{n} M^{\mathbf{u}-e_i})) \\ &= dim(C_q(M^{\mathbf{u}}, \bigcup_{i=1}^{n} M^{\mathbf{u}-e_i})). \end{aligned}$$

Moreover, we know that

$$\begin{aligned} dim(H_q(X^{\mathbf{u}}, \bigcup_{i_1}^{n} X^{\mathbf{u}-e_i})) &= dim(H_q(M^{\mathbf{u}}, \bigcup_{i_1}^{n} M^{\mathbf{u}-e_i})) \\ &= dim(ker(\partial_q^{rel})/im(\partial_{q+1}^{rel})) \\ &\leq dim(C_q(M^{\mathbf{u}}, \bigcup_{i_1}^{n} M^{\mathbf{u}-e_i})) \\ &= m_q(\mathbf{u}). \end{aligned}$$

where

$$\partial_q^{rel} : C_q(M^{\mathbf{u}}, \bigcup_{i_1}^{n} M^{\mathbf{u}-e_i}) \to C_{q-1}(M^{\mathbf{u}}, \bigcup_{i_1}^{n} M^{\mathbf{u}-e_i})$$

for all $q \in \mathbb{N}$, which is the relative boundary map of the pair $(M^{\mathbf{u}}, \bigcup_{i_1}^{n} M^{\mathbf{u}-e_i})$.

□

Corollary 5.2.5.

$$\mathcal{C}_q(X) \subseteq \mathcal{G}(M_q).$$

Proof. Let $\mathbf{u} \in \mathcal{C}_q(X)$, then this means that

$$dim(H_q(X^{\mathbf{u}}, \bigcup_{i=1}^{n} X^{\mathbf{u}-e_i})) \neq 0.$$

This implies that $m_q(\mathbf{u}) \geq 1$ by the proposition 5.2. Thus, there exists at least one q-dimensional critical cell whose entrance grade is $\mathbf{u}$ by the definition 30 of $m_q(\mathbf{u})$. By the definition 4.0.1 of $\mathcal{G}(M_q)$, $\mathbf{u} \in \mathcal{G}(M_q)$. Hence, $\mathcal{C}_q(X) \subseteq \mathcal{G}(M_q)$.

□

We just achieved our first goal with the proof of the last corollary, so after this point, we will fix $n = 2$ until the end of the thesis. Also, when we study with a fixed $\mathbf{u} \in \mathbb{N}^2$, we will set $\mathbf{x} = \mathbf{u} - e_1, \mathbf{y} = \mathbf{u} - e_2$ and $\mathbf{z} = \mathbf{u} - e_1 - e_2$.

Lemma 5.2.6. [13] If $\mathbf{u} \in \mathbb{N}^2$, $j \neq l$ such that $j, l \in \{1, 2\}$, then there existst a short exact sequence

$$0 \to C_*(X^{\mathbf{u}-e_l}, X^{\mathbf{z}}) \to C_*(X^{\mathbf{u}}, X^{\mathbf{u}-e_j}) \to C_*(X^{\mathbf{u}}, X^{\mathbf{x}} \cup X^{\mathbf{y}}) \to 0.$$

Remark 5.2.1. Before passing the proof, note that if $\mathbf{x} \notin \mathbb{N}^2$, then we assume that $X^{\mathbf{x}} = \emptyset$. Also, we assume the same for $X^{\mathbf{y}}$ and $X^{\mathbf{z}}$. This convention will be used in the rest of the chapter.

Proof. Let us prove the lemma for the case $l = 1$ and $j = 2$ without loss of generality. Now, we have the triple $X^{\mathbf{y}} \subseteq X^{\mathbf{x}} \cup X^{\mathbf{y}} \subseteq X^{\mathbf{u}}$, and we have the corresponding short exact sequence

$$0 \to C_*(X^{\mathbf{x}} \cup X^{\mathbf{y}}, X^{\mathbf{y}}) \to C_*(X^{\mathbf{u}}, X^{\mathbf{y}}) \to C_*(X^{\mathbf{u}}, X^{\mathbf{x}} \cup X^{\mathbf{y}}) \to 0.$$

At this point, we have to show that

$$C_*(X^{\mathbf{x}} \cup X^{\mathbf{y}}, X^{\mathbf{y}}) \cong C_*(X^{\mathbf{x}}, X^{\mathbf{z}})$$

as chain complexes, but it is enough to show that

$$C_q(X^{\mathbf{x}} \cup X^{\mathbf{y}}, X^{\mathbf{y}}) \cong C_q(X^{\mathbf{x}}, X^{\mathbf{z}})$$

for all $q \in \mathbb{N}$ because we know that differential maps of $C_*(X^{\mathbf{x}} \cup X^{\mathbf{y}}, X^{\mathbf{y}})$ and $C_*(X^{\mathbf{x}}, X^{\mathbf{z}})$ are induced by the differential of the same chain complex which is $C_*(X)$, so these isomorphisms will automatically commute with the differential maps of $C_*(X^{\mathbf{x}} \cup X^{\mathbf{y}}, X^{\mathbf{y}})$ and $C_*(X^{\mathbf{x}}, X^{\mathbf{z}})$. Now, let us show these isomorphisms. On the one hand,

$$\begin{aligned} C_q(X^{\mathbf{x}} \cup X^{\mathbf{y}}, X^{\mathbf{y}}) &= \frac{C_q(X^{\mathbf{x}} \cup X^{\mathbf{y}})}{C_q(X^{\mathbf{y}})} \text{ by definition} \\ &= \frac{C_q(X^{\mathbf{x}}) + C_q(X^{\mathbf{y}})}{C_q(X^{\mathbf{y}})} \end{aligned}$$

for all $q \in \mathbb{N}$. On the other hand,

$$\begin{aligned} C_q(X^{\mathbf{x}}, X^{\mathbf{z}}) &= \frac{C_q(X^{\mathbf{x}})}{C_q(X^{\mathbf{z}})} \text{ by definition} \\ &= \frac{C_q(X^{\mathbf{x}})}{C_q(X^{\mathbf{x}} \cap X^{\mathbf{y}})} \text{ by Remark 3.1.1} \\ &= \frac{C_q(X^{\mathbf{x}})}{C_q(X^{\mathbf{x}}) \cap C_q(X^{\mathbf{y}})}. \end{aligned}$$

for all $q \in \mathbb{N}$. Moreover, the second isomorphism theorem of vector spaces implies that

$$\frac{C_q(X^{\mathbf{x}}) + C_q(X^{\mathbf{y}})}{C_q(X^{\mathbf{y}})} \cong \frac{C_q(X^{\mathbf{x}})}{C_q(X^{\mathbf{x}}) \cap C_q(X^{\mathbf{y}})}.$$

Hence,

$$C_q(X^{\mathbf{x}} \cup X^{\mathbf{y}}, X^{\mathbf{y}}) \cong C_q(X^{\mathbf{x}}, X^{\mathbf{z}})$$

for all $q \in \mathbb{N}$. □

Corollary 5.2.7. [13] If $\mathbf{u} \in \mathbb{N}^2$, $q \in \mathbb{N}$, $j, l \in \{1, 2\}$ such that $j \neq l$ and $H_q(X^{\mathbf{u}}, X^{\mathbf{x}} \cup X^{\mathbf{y}}) = 0$, then

$$H_q(X^{\mathbf{u}}, X^{\mathbf{u}-e_j}) \neq 0 \Rightarrow H_q(X^{\mathbf{u}-e_l}, X^{\mathbf{z}}) \neq 0.$$

Proof. By our assumption and Lemma 5.2.6, we know that there exists a short exact sequence

$$0 \to C_*(X^{\mathbf{u}-e_l}, X^{\mathbf{z}}) \to C_*(X^{\mathbf{u}}, X^{\mathbf{u}-e_j}) \to C_*(X^{\mathbf{u}}, X^{\mathbf{x}} \cup X^{\mathbf{y}}) \to 0.$$

Thus, we have a associated homology long exact sequence

$$... \to H_q(X^{\mathbf{u}-e_l}, X^{\mathbf{z}}) \to H_q(X^{\mathbf{u}}, X^{\mathbf{u}-e_j}) \to H_q(X^{\mathbf{z}}, X^{\mathbf{x}} \cup X^{\mathbf{y}}) \to ...$$

By our assumption, we already know that $H_q(X^{\mathbf{u}}, X^{\mathbf{x}} \cup X^{\mathbf{y}}) = 0$, so the map from $H_q(X^{\mathbf{u}-e_l}, X^{\mathbf{z}})$ to $H_q(X^{\mathbf{u}}, X^{\mathbf{u}-e_j})$ is surjective which automatically implies that if $H_q(X^{\mathbf{u}}, X^{\mathbf{u}-e_j})$ is non-zero, then $H_q(X^{\mathbf{u}-e_l}, X^{\mathbf{z}})$ is also non-zero.

□

Before passing our last results, we are going to prove a lemma and a corollary of this lemma, and this will be useful in the proof of the proposition...

Lemma 5.2.8. [13] If $\mathbf{u} \in supp\, \xi_2^{q-1}$, then for all $i \in \{1, 2\}$ there exists $\lambda \in \mathbb{N} \setminus \{0\}$ such that

$$H_q(X^{\mathbf{u}-\lambda e_i}, X^{\mathbf{u}-(\lambda+1)e_i} \cup X^{\mathbf{u}-\lambda e_i - e_j}) \neq 0,$$

where $j \in \{1, 2\}$ such that $i \neq j$, for all $q \in \mathbb{N}$.

Proof. Let $q \in \mathbb{N}$, and i be an arbitrary element of $\{1, 2\}$. Assume the contrary that $H_q(X^{\mathbf{u}-\lambda e_i}, X^{\mathbf{u}-(\lambda+1)e_i} \cup X^{\mathbf{u}-\lambda e_i - e_j}) = 0$ for all $\lambda \in \mathbb{N} \setminus \{0\}$. Thus,

$$H_q(X^{\mathbf{u}-\lambda e_i}, X^{\mathbf{u}-\lambda e_i - e_j}) \neq 0 \Rightarrow H_q(X^{\mathbf{u}-(\lambda+1)e_i}, X^{\mathbf{u}-(\lambda+1)e_i - e_j}) \neq 0$$

for all $\lambda \in \mathbb{N} \setminus \{0\}$ by Corollary 5.2.7. Now, let us try to show that

$$H_q(X^{\mathbf{u}-\lambda e_i}, X^{\mathbf{u}-\lambda e_i - e_j}) \neq 0$$

for all $\lambda \in \mathbb{N} \setminus \{0\}$ by using induction. As a base step of the induction, let $\lambda = 1$. We know that

$$\iota_{q-1}^{\mathbf{u}-e_1-e_2, \mathbf{u}-e_1} : H_q(X^{\mathbf{u}-e_1-e_2}) \to H_{q-1}(X^{\mathbf{u}-e_1})$$

has non-zero kernel since $\mathbf{u} \in supp\, \xi_2^{q-1}$, so $H_q(X^{\mathbf{u}-e_1}, X^{\mathbf{u}-e_1-e_2}) \neq 0$. Now, assume that $H_q(X^{\mathbf{u}-\lambda e_i}, X^{\mathbf{u}-\lambda e_i - e_j}) \neq 0$ for a fixed λ, then we know that this implies that

$$H_q(X^{\mathbf{u}} - (\lambda+1)e_i, X^{\mathbf{u}-(\lambda+1)e_i - e_j}) \neq 0.$$

Thus, $H_q(X^{\mathbf{u}-\lambda e_i}, X^{\mathbf{u}-\lambda e_i - e_j}) \neq 0$ for all $\lambda \in \mathbb{N} \setminus \{0\}$ by induction. On the other hand, we know that $\mathbf{u} - \lambda e_i$ and $\mathbf{u} - \lambda e_i - e_j$ are not an element of $\mathbb{N}^2$ for a sufficiently large $\lambda \in \mathbb{N}$, so $X^{\mathbf{u}-\lambda e_i} = X^{\mathbf{u}-\lambda e_i - e_j} = \emptyset$ for a sufficiently large $\lambda \in \mathbb{N}$, but this implies that

$$H_q(X^{\mathbf{u}-\lambda e_i}, X^{\mathbf{u}-\lambda e_i - e_j}) = 0$$

for a sufficiently large $\lambda \in \mathbb{N}$. This is a contradiction, hence there exists a $\lambda \in \mathbb{N} \setminus \{0\}$ such that

$$H_q(X^{\mathbf{u}-\lambda e_i}, X^{\mathbf{u}-(\lambda+1)e_i} \cup X^{\mathbf{u}-\lambda e_i - e_j}) \neq 0.$$

□

Corollary 5.2.9. [13] $supp\, \xi_2^{q-1} \subseteq \overline{\mathcal{C}_q(X)}$ for all $q \in \mathbb{N}$.

Proof. Let $\mathbf{u}$ be an arbitrary element of $supp\, \xi_2^{q-1}$. By Lemma 5.2.8, we know that there exists $\lambda_1, \lambda_2 \in \mathbb{N} \setminus \{0\}$ such that

$$H_q(X^{\mathbf{u}-\lambda_1 e_1}, X^{\mathbf{u}-(\lambda_1+1)e_1} \cup X^{\mathbf{u}-\lambda_1 e_1 - e_2}) \neq 0$$

and

$$H_q(X^{\mathbf{u}-\lambda_2 e_2}, X^{\mathbf{u}-(\lambda_2+1)e_2} \cup X^{\mathbf{u}-\lambda_2 e_2 - e_1}) \neq 0.$$

Thus, $(\mathbf{u}-\lambda_1 e_1), (\mathbf{u}-\lambda_2 e_2) \in \mathcal{C}_q(X)$. Also, we know that $(\mathbf{u}-\lambda_1 e_1) \vee (\mathbf{u}-\lambda_2 e_2) = \mathbf{u}$, so $\mathbf{u} \in \overline{\mathcal{C}_q(X)}$.

□

Proposition 5.2.10. [13] $supp\, \xi_0^q \cup supp\, \xi_1^{q-1} \cup supp\, \xi_2^{q-1} \subseteq \overline{\mathcal{C}_q(X)}$ for all $q \in \mathbb{N}$.

Proof. Let $q \in \mathbb{N}$ and $\mathbf{u} \notin \overline{\mathcal{C}_q(X)}$. Then, $\mathbf{u} \notin supp\, \xi_2^{q-1}$ by the Corollary 5.2.9, so $\xi_2^{q-1}(\mathbf{u}) = 0$. Also, $\mathbf{u} \notin \overline{\mathcal{C}_q(X)}$ implies that $H_q(X^{\mathbf{u}}, X^{\mathbf{u}-e_1} \cup X^{\mathbf{u}-e_2}) = 0$ by the definition of $\mathcal{C}_q(X)$. Thus, Corollary 5.1.4 implies that $\xi_0^q(\mathbf{u}) + \xi_1^{q-1}(\mathbf{u}) - \xi_2^{q-1}(\mathbf{u}) = 0$, and we already know that $\xi_2^{q-1}(\mathbf{u}) = 0$, so $\xi_0^q(\mathbf{u}) + \xi_1^{q-1}(\mathbf{u}) = 0$. This implies that $\xi_0^q(\mathbf{u}) = \xi_1^{q-1}(\mathbf{u}) = 0$ because we know that $\xi_0^q(\mathbf{u})$ and $\xi_1^{q-1}(\mathbf{u})$ are non-negative integers. Thus, $\mathbf{u} \notin supp\, \xi_0^q$ and $\mathbf{u} \notin supp\, \xi_1^{q-1}$. Hence,

$$\mathbf{u} \notin supp\, \xi_0^q \cup supp\, \xi_1^{q-1} \cup supp\, \xi_2^{q-1}.$$

□

Corollary 5.2.11. [13] $supp\,\xi_0^q \cup supp\,\xi_1^q \cup supp\,\xi_2^q \subseteq \overline{\mathcal{C}_q(X)} \cup \overline{\mathcal{C}_{q+1}(X)}$ for all $q \in \mathbb{N}$.

Proof. By proposition 5.5, we know that $supp\,\xi_0^q \subseteq \overline{\mathcal{C}_q(X)}$, and $supp\,\xi_1^q \cup supp\,\xi_2^q \subseteq \overline{C_{q-1}(X)}$ for all $q \in \mathbb{N}$. Thus,

$$supp\,\xi_0^q \cup supp\,\xi_1^q \cup supp\,\xi_2^q \subseteq \overline{\mathcal{C}_q(X)} \cup \overline{\mathcal{C}_{q+1}(X)}$$

for all $q \in \mathbb{N}$. □

Corollary 5.2.12. [13] $\bigcup_q \mathcal{C}_q(X) \subseteq \bigcup_{q,i} supp\,\xi_i^q \subseteq \bigcup_q \overline{\mathcal{C}_q(X)}$.

Proof. Let $\mathbf{u} \in \bigcup_q \mathcal{C}_q(X)$, then $\mathbf{u} \in \mathcal{C}_q(X)$ for some $q \in \mathbb{N}$, so $dim(H_q(X^{\mathbf{u}}, X^{\mathbf{u}-e_1} \cup X^{\mathbf{u}-e_2})) \neq 0$ by the definition of $\mathcal{C}_q(X)$. Also, we know that

$$dim(H_q(X^{\mathbf{u}}, x^{\mathbf{u}-e_1} \cup X^{\mathbf{u}-e_2})) \leq \xi_0^q(\mathbf{u}) + \xi_1^{q-1}(\mathbf{u}) + \xi_2^{q-2}(\mathbf{u})$$

by corollary 5.1.4. Thus, $\mathbf{u} \in supp\,\xi_0^q \cup supp\,\xi_1^{q-1} \cup supp\,\xi_2^{q-2}$. Hence, $\mathbf{u} \in \bigcup_{q,i} supp\,\xi_i^q$. On the other hand, if $\mathbf{u} \in \bigcup_{q,i} supp\,\xi_i^q$, then $\mathbf{u} \in supp\,\xi_i^q$ for some $i \in \{0, 1, 2\}$, $q \in \mathbb{N}$ and

$$\begin{aligned} supp\,\xi_i^q &\subseteq supp\,\xi_0^q \cup supp\,\xi_1^q \cup supp\,\xi_2^q \\ &\subseteq \overline{\mathcal{C}_q(X)} \cup \overline{\mathcal{C}_{q+1}(X)} \\ &\subseteq \bigcup_q \overline{\mathcal{C}_q(X)}. \end{aligned}$$

Hence, $\mathbf{u} \in \bigcup_q \overline{C_q(X)}$. □

CHAPTER 6

CONCLUSION

In this thesis, our aim was to give a proof of the Theorem 4.0.8. In that theorem, a bound for the support of the Betti tables of multiparameter persistence modules which is obtained by 1-critical filtration of a finite cell complex by using the set of the entrance grades of a discrete Morse complex corresponding to a discrete gradient vector field that is consistent with the filtration. For this reason, a detailed explanation of discrete Morse theory was given in the background section. Moreover, the iterative construction of Koszul complexes by utilization of the mapping cones had a key role in the proof of the Theorem 4.0.8.

Beyond that, a stronger proof was given in the case $n = 2$ for the same type of multiparameter persistence modules on chapter 5. This time the bound was given by the homological critical grades which is always a subset of the entrance grades of discrete Morse complex by Corollary 5.2.5. Also, the Corollary 5.1.4 had a key role in the proof.

For future work, one can try to generalize the bound which is given by homological critical grades for the case $n = 2$ to $n = 3$ case. We tried it by changing the grades of the statements which we prove in the Lemma 5.2.6. For example, we have changed the grades of the sequence in the Lemma 5.2.6 as $\mathbf{u}-e_{j_1}-e_{j_2}$ for $\mathbf{u}-e_l$ and $\mathbf{u}-e_{j_3}$ for $\mathbf{u}-e_j$ where j_1, j_2, j_3 are different elements of $\{1, 2, 3\}$. After that we have changed to adjust the other grades with respect to these two grades. However, we could not get the result which as in the Lemma 5.2.8. Other trials about changing grades also did not give the desired result. Probably, we should use a new approach other than the ones used in this article [13].

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