EFFECT OF TIME DELAYS ON THE CONVERGENCE SPEED OF CONSENSUS DYNAMICS

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We certify that we have read this thesis and that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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ABSTRACT

EFFECT OF TIME DELAYS ON THE CONVERGENCE SPEED OF CONSENSUS DYNAMICS

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We discuss consensus problems under time delays. The presence of time delays results in an infinite-dimensional system rather than a system of ordinary differential equations. It has been shown that information transmission delays do not influence whether the system converges to a consensus value; however, further effects of delays are unknown. We show that time delays in most graphs decreases the convergence speed; while somewhat surprisingly, they can improve convergence in certain special graphs. We discuss the structure of graphs for which such improvement is possible.

Keywords: Consensus problems, time delays, convergence speed, undirected graphs, directed graphs, graph operations, normalized Laplacian, stability.

ÖZET

UZLAŞMA PROBLEMLERİNDE GECİKME TERİMİNİN YAKINSAMA HIZINA ETKİSİ

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Matematik, Yüksek Lisans
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Ocak 2020

Bu çalışmada zaman gecikme teriminin uzlaşma dinamiği üzerindeki etkileri incelenmektedir. Gecikme teriminin varlığı, problemi, bir adi diferansiyel denklem sistemi yerine sonsuz boyutlu bir uzayda ele almayı gerektirir. Gecikme teriminin kararlılığı etkilemediği literatürde gösterilmiş olmakla beraber, bunun ötesindeki detaylı etkileri bilinmemektedir. Bu çalışmada, gecikme teriminin çoğu ağ için yakınsama hızını yavaşlattığı, fakat bazı özel ağlar içinse tam tersine hızlandırdığı gösterilmektedir. Gecikme altında hızlanan bu özel ağların yapısal özellikleri detaylı olarak incelenmektedir.

Anahtar sözcükler: Uzlaşma problemi, senkronizasyon, gecikme, yakınsama hızı, yönsüz ağlar, yönlü ağlar, ağ işlemleri, Laplace matrisi, kararlılık.

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Chapter 1

Introduction

Consensus or coordination problems are fundamental models having a wide range of applications in distributed computing [1], management science and statistics [2], distributed control [3], flocking/swarming theory [4], distributed sensor networks [5] and many more.

In a network, consensus refers to agreement among a number of agents. The consensus algorithm (or protocol) is the interaction rule that specifies the information exchange between an agent and its neighbors on a network. In the literature, there are many works relating to consensus problems modelled in a related field of interest [6–30].

Consensus problems in the field of communication under time delays can be modelled into delayed information processing and delayed information transmission. It is known that the consensus problem under information processing delay reaches consensus depending on the time delay under undirected network (see Theorem 2.4.1 in [9]). On the other hand, the consensus problem under information transmission delay reaches consensus regardless of time delay, provided the underlying network has a spanning tree (see Theorem 2.4.2 in [7]). Since in the latter case reaching consensus does not depend on the presence or the value of the delay, a natural question arises:

QUESTION 1: So, for the consensus problem under transmission delays, what precisely is the effect of delays?

Regarding this question, one might think that the main effect of delays would be to slow down the convergence speed, as the information flowing around in the network incurs delays. This is indeed the case with most graphs, as we will see. However, our main result points to an interesting outcome:

MAIN RESULT: For certain graphs, convergence to consensus can be *faster* under time delays.

Our investigation does not stop here, however. Since the main result holds for some graphs and not for others, we are led to the second question:

QUESTION 2: What role does the network structure play in the effects of the delays?

We will use the term *FD-graph* (for Faster under Delay) to denote graphs that achieve consensus faster at some positive value of the delay as compared to the undelayed dynamics.

The main difficulty in addressing these two questions is that the presence of time delays results in an infinite-dimensional system, making the analysis difficult. We will tackle both questions in the following chapters using a variety of methods from the theory of delay differential equations, dynamical systems, and graph theory.

The thesis is organized in five chapters. In the second chapter, we give an introduction to the consensus problem and time delays. We present an analytical solution to the consensus problem modelled with time delays. Basics from graph theory are explained. We define graph Laplacians which is a key factor as far as reaching consensus is concerned.

In the third chapter, we introduce a background on delay differential equations. We state basic theorems for the existence and uniqueness of the solutions to the delay differential equations. A method for solving delay differential equations is also explained in this chapter. We introduce Lambert W function and show how to use it in solving the transcendental equation numerically.

In the fourth chapter, the main results of the thesis are presented. We formally define convergence speed of the linear consensus problem and the information transmission delay. We measure the convergence speed of consensus dynamics by the spectral gap, namely, by the real part of the eigenvalue closest to the imaginary axis (the slowest mode). This leads us to the definition of FD-graphs, for which a spectrum condition is derived. Stability conditions for the information transmission delays is studied and an exact stability region is obtained. We introduced distributed delays under information transmission consensus problem.

Finally, in the last chapter of this thesis, we present a main result on the structure of the FD-graphs. We give definitions of various graph operations such as Cartesian product, join, H-Join, adding and removing edges. With these graph operations, we are able to build new FD-graphs from existing FD-graphs.

Chapter 2

The Consensus Problem and Time Delays

2.1 The Consensus Algorithm

The consensus problem on a network can be formulated in continuous time $t \in \mathbb{R}$ as the system of differential equations

$$\dot{x}_i(t) = \sum_{j=1}^{n} a_{ij}(x_j(t) - x_i(t)), \quad i = 1, \dots, n. \tag{2.1}$$

There is also a normalized version of the consensus problem is formulated as

$$\dot{x}_i(t) = \frac{1}{d_i} \sum_{j=1}^{n} a_{ij}(x_j(t) - x_i(t)), \quad i = 1, \dots, n. \tag{2.2}$$

Here $x_i(t) \in \mathbb{R}$ denotes the state (opinion) of the agent i at time t, $a_{ij} \geq 0$ is the strength of the influence of agent j on i and $d_i := \sum_{j=1}^{n} a_{ij}$ is the in-degree. The basic idea of consensus dynamics is to update the current opinion of each agent by comparing the opinion of its neighbors and its own. The vector form of the linear consensus problem in (2.1) is

$$
\begin{aligned}
\dot{x}_i(t) &= \sum_{j=1}^{n} a_{ij}(x_j(t) - x_i(t)) \\
&= \sum_{j=1}^{n} a_{ij}x_j(t) - x_i(t)\sum_{j=1}^{n} a_{ij} \\
&= \sum_{j=1}^{n} a_{ij}x_j(t) - x_i(t)d_i \\
&= -\sum_{j=1}^{n} (\delta_{ij}d_i - a_{ij})x_j(t) \\
&= -\sum_{j=1}^{n} (l_{ij})x_j(t), \quad \text{where} \quad l_{ij} = \delta_{ij}d_i - a_{ij}, \\
&= -Lx(t)
\end{aligned}
$$

$$\dot{x}(t) = -Lx(t), \tag{2.3}$$

where $L = [l_{ij}]$ is the Laplacian matrix. Similarly, the vector form for the normalized consensus problem (2.2) is

$$\dot{x}(t) = -\mathcal{L}x(t), \tag{2.4}$$

where $\mathcal{L}$ is the normalized Laplacian matrix. We will define Laplacian and normalized Laplacian in the next section.

Definition 2.1.1. The consensus problem reaches consensus if for any set of initial conditions there exists some $c \in \mathbb{R}$ such that $\lim_{t\to\infty} x_i(t) = c$ for all i. In this case c is called the consensus value.

Communication over networks often involve delays due to the finite speed of information propagation or processing.

The consensus problem which involves time delays can be modelled in two ways, corresponding to information processing delays

$$\dot{x}_i(t) = \frac{1}{d_i}\sum_{j=1}^{n} a_{ij}(x_j(t-\tau) - x_i(t-\tau)). \tag{2.5}$$

and information transmission delays

$$\dot{x}_i(t) = \frac{1}{d_i}\sum_{j=1}^{n} a_{ij}(x_j(t-\tau) - x_i(t)). \tag{2.6}$$

The agent i of model (2.5) requires a certain time $\tau \geq 0$ for the processing of its information on the state difference $x_j - x_i$ to refresh its state, whereas in model (2.6), the agent i compares its current state with its neighbor j as the information is being delayed by τ. The model (2.5) in an non-normalized form have been studied in [9] and model (2.6) with both fixed and distributed delays have been studied in [7]. Our main objective in this thesis is to study both models (2.2) and (2.6). We will look at how delay enhances (speeds up) convergence of consensus problem by comparing both undelayed and delayed consensus problems in models (2.2) and (2.6), respectively.

Let $x = (x_1, \ldots, x_n)^\top \in \mathbb{R}^n$. The models (2.5) and (2.6) can then be written in vector from as

$$\dot{x}(t) = -\mathcal{L}x(t-\tau) \tag{2.7}$$

and

$$\begin{aligned}
\dot{x}_i(t) &= \frac{1}{d_i}\sum_{j=1}^{n} a_{ij}(x_j(t-\tau) - x_i(t)) \\
&= \frac{1}{d_i}\sum_{j=1}^{n} a_{ij}x_j(t-\tau) - x_i(t)\frac{1}{d_i}\sum_{j=1}^{n} a_{ij},
\end{aligned}$$

$$\dot{x}(t) = -x(t) + D^{-1}Ax(t-\tau), \tag{2.8}$$

respectively. A is the adjacency matrix of a graph G and $D = \operatorname{diag}\{d_1, \ldots, d_n\}$ is the diagonal matrix of in-degrees.

2.2 Basics from graph theory

A graph is a diagram consisting of points called vertices (nodes) that are connected by lines called edges or arcs. Let $G = (V, E)$ be a graph, where V is the non-empty set of vertices $V = \{1, 2, \ldots, n\}$, and $E \subset V \times V$ is the set of edges.

All graphs considered in this thesis are assumed to be simple graphs. Simple graphs are graphs that have no loops and no multiple edges. A graph with only one node is called a trivial graph. A graph in which it is possible to reach any vertex by traversing the edges from one vertex to another is said to be connected. The set of edges used (not necessarily distinct) is called a path between the given vertices. The degree of a vertex is the number of edges connected to that vertex.

2.2.1 Adjacency Matrix

An adjacency matrix is a square matrix used to represent a finite graph. Although graphs are usually shown diagrammatically, graphs can be represented in a matrix form which can be analyzed by using the well known operations of matrices.

Definition 2.2.1. Let G be a graph of order n, then its adjacency matrix A is an $n \times n$ square matrix, where each row and column corresponds to a vertex of G. The element a_{ij} specifies the connection between i and j.

$$a_{ij} = \begin{cases} 1 & \text{if there is an edge from vertex } j \text{ to vertex } i \\ 0 & \text{otherwise} \end{cases}$$

Remark 2.2.2. In our notation, the adjacency matrix is the transpose of the adjacency matrix used in graph theory.

Example 2.2.3. *Consider the complete graph K_5 in Figure 2.2, the adjacency matrix is*

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}.$$

2.2.2 Undirected Graphs

Undirected graph is a graph in which the direction of the edge is not defined. If there is an edge between vertex i and vertex j, then there is a path from vertex i to j and also from j to i.

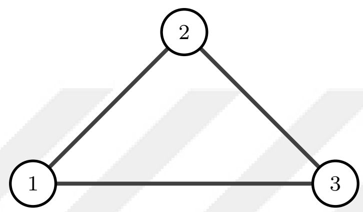

Figure 2.1: Example of an undirected graph

2.2.3 Complete Graphs

A complete graph is a simple graph in which each vertex is connected to all other vertices. A complete graph is denoted by K_n where n is the number of vertices and all the vertices have $n-1$ degree. A complete graph has $\frac{n(n-1)}{2}$ edges. An example of a complete graph is shown in Figure 2.2.

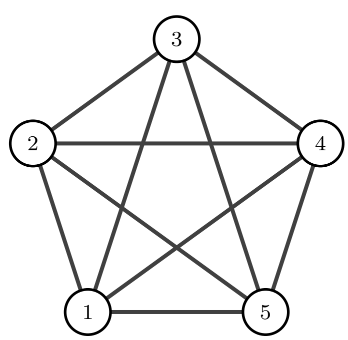

Figure 2.2: A complete graph K_5

2.2.4 Directed Graphs

A directed graph (or digraph) is a graph in which the direction of the edge is defined.

2.2.5 Definitions

Let X be a digraph on n vertices with vertex set $V(X) = \{1, 2, \ldots, n\}$ and edge set $E(X) \subset X \times X$, where $(x, y) \in E(X)$ if there exists a directed edge from y to x. The out-degree and in-degree of the vertex x are defined by

$$d_{out}(x) = \#\{y : (y, x) \in E(X)\}$$

$$d_{in}(x) = \#\{y : (x, y) \in E(X)\}$$

If $d_{out}(x) = d_{in}(x) = k$ for each vertex $x \in X$, we call X a regular graph. The adjacency matrix $A = A(X)$ of X has ij-entry equal to 1 if $(i, j) \in E(X)$ and 0 otherwise. Hence, $d_{in}(i) = \sum_{j=1}^{n} a_{ij}$.

Example 2.2.4. *The adjacency matrix of the directed graph in Figure 2.3 is*

$$A = \begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{bmatrix}.$$

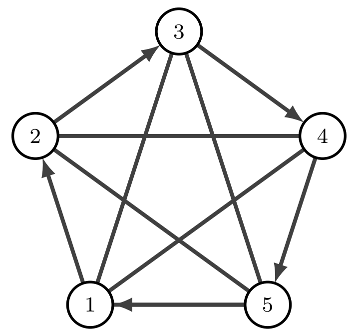

Figure 2.3: Example of a directed graph

2.3 The Laplacian matrix

Let $G = (V, E)$ be a graph of order n, where V is the set of vertices $V = \{1, \ldots, n\}, E \subseteq V \times V$ is the set of edges. Let $A = [a_{ij}]$ be the adjacency matrix with elements a_{ij}. Let $d_i = \sum_{j \in V} a_{ij}$ be the total in-degree of the node i and $D = \mathrm{diag}\{d_1, \ldots, d_n\}$ be the diagonal matrix of the vertex in-degrees. The Laplacian matrix is defined as

$$L = D - A,$$

where the elements of $L = [l_{ij}]$ are defined by

$$l_{ij} = \begin{cases} d_i & \text{if } i = j, \\ -1 & \text{if } i \neq j \text{ and } i \text{ is adjacent to } j, \\ 0 & \text{otherwise.} \end{cases}$$

2.3.1 Symmetric normalized Laplacian

The symmetric normalized Laplacian is defined as

$$L^{sym} = D^{-\frac{1}{2}} L D^{-\frac{1}{2}} = I - D^{-\frac{1}{2}} A D^{-\frac{1}{2}}.$$

The elements of L^{sym} are given by

$$l_{ij}^{sym} = \begin{cases} d_i & \text{if } i = j \text{ and } d_i \neq 0, \\ -\frac{1}{\sqrt{d_i d_j}} & \text{if } i \neq j \text{ and } i \text{ is adjacent to } j, \\ 0 & \text{otherwise.} \end{cases}$$

2.3.2 Normalized Laplacian

The normalized Laplacian matrix is defined as

$$\mathcal{L} = D^{-1}L = I - D^{-1}A.$$

It can be seen that

$$\mathcal{L} = I - D^{-\frac{1}{2}}(I - L^{sym})D^{\frac{1}{2}},$$

so $\mathcal{L}$ and L^{sym} have the same spectrum. The elements of $\mathcal{L}$ are given by

$$\mathcal{L}_{ij} = \begin{cases} 1 & \text{if } i = j \text{ and } d_i \neq 0, \\ -\frac{1}{d_i} & \text{if } i \neq j \text{ and } i \text{ is adjacent to } j, \\ 0 & \text{otherwise.} \end{cases}$$

Example 2.3.1. *The normalized Laplacian matrix $\mathcal{L}$ of the adjacency matrix A in Example 2.2.3 and Example 2.2.4 are*

$$\mathcal{L} = \begin{bmatrix} 1 & -1/4 & -1/4 & -1/4 & -1/4 \\ -1/4 & 1 & -1/4 & -1/4 & -1/4 \\ -1/4 & -1/4 & 1 & -1/4 & -1/4 \\ -1/4 & -1/4 & -1/4 & 1 & -1/4 \\ -1/4 & -1/4 & -1/4 & -1/4 & 1 \end{bmatrix}$$

and

$$\mathcal{L} = \begin{bmatrix} 1 & 0 & -1/3 & -1/3 & -1/3 \\ -1/3 & 1 & 0 & -1/3 & -1/3 \\ -1/3 & -1/3 & 1 & 0 & -1/3 \\ -1/3 & -1/3 & -1/3 & 1 & 0 \\ 0 & -1/3 & -1/3 & -1/3 & 1 \end{bmatrix},$$

respectively.

Let $\{\lambda_1, \ldots, \lambda_n\}$ be the set of eigenvalues of the normalized Laplacian $\mathcal{L}$. Zero is always an eigenvalue of $\mathcal{L}$ since it has zero row sums. By Gershgorin's circle theorem [31], we have the following property:

$$\mid 1 - \lambda_i \mid \leq 1, \quad \forall i. \tag{2.9}$$

It is thus convenient to work with the normalized Laplacian as the spectrum is bounded irrespective of the graph size. For undirected graphs we denote the eigenvalues of the normalized Laplacian $\mathcal{L}$ by $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. For directed graphs, although the eigenvalues may be complex numbers, we still set $\lambda_1 = 0$, as 0 is always an eigenvalue of the normalized Laplacian $\mathcal{L}$ matrix. Let $\{v_1, \ldots, v_n\}$ be the eigenvectors of the corresponding eigenvalues λ_i; we assume that there is a complete set of eigenvectors, $\mathcal{L}v_i = \lambda v_i$, for $i = 1, \ldots, n$. Note that, the eigenvector v_1 corresponding to $\lambda_1 = 0$ is $v_1 = (1, 1, \ldots, 1)^\top$. Let $\{u_1, \ldots, u_n\}$ be the dual basis corresponding to v_i, that is, $u_i\mathcal{L} = \lambda_i u_i$. The u_i are the left eigenvectors of $\mathcal{L}$. The undelayed consensus problem (2.3) has the solution

$$x(t) = e^{-\mathcal{L}t}x(0), \tag{2.10}$$

where $x(0) = x_0$ is the set of initial opinions of the agents. In the eigenbasis of $\mathcal{L}$ we can express x as

$$x(t) = \sum_{i=1}^{n} \alpha_i(t)v_i, \tag{2.11}$$

where $\alpha_i(t) = \langle u_i, x(t)\rangle$. By substituting (2.11) into (2.10) we get

$$\alpha_i(t) = \alpha_i(0)e^{-\lambda_i t}. \tag{2.12}$$

For the delayed consensus problem (2.8), if we substitute (2.11) into (2.8) we obtain the

$$\begin{aligned}
\frac{d}{dt}\sum_{i=1}^{n} \alpha_i(t)v_i &= -\sum_{i=1}^{n} \alpha_i(t)v_i + D^{-1}A\sum_{i=1}^{n} \alpha_i(t-\tau)v_i \\
\sum_{i=1}^{n} \frac{d}{dt}\alpha_i(t)v_i &= -\sum_{i=1}^{n} \alpha_i(t)v_i + D^{-1}A\sum_{i=1}^{n} \alpha_i(t-\tau)v_i \\
\sum_{i=1}^{n} \dot{\alpha}_i(t)v_i &= -\sum_{i=1}^{n} \alpha_i(t)v_i + D^{-1}A\alpha_i(t-\tau)v_i \\
\sum_{i=1}^{n} [\dot{\alpha}_i(t)v_i &+ \alpha_i(t)v_i - D^{-1}A\alpha_i(t-\tau)v_i] = 0.
\end{aligned} \tag{2.13}$$

We know that

$$\begin{aligned}
\mathcal{L}v_i &= \lambda_i v_i \\
(I - D^{-1}A)v_i &= \lambda_i v_i \\
v_i - D^{-1}Av_i &= \lambda_i v_i \\
D^{-1}Av_i &= (1 - \lambda_i)v_i
\end{aligned} \tag{2.14}$$

ans substituting equation (2.14) into equation (2.13) we obtain

$$\dot{\alpha}_i(t) = -\alpha_i(t) + (1 - \lambda_i)\alpha_i(t - \tau). \tag{2.15}$$

Assuming exponential solutions $\alpha_i(t) = e^{st}$ and substituting into (2.15) yields

$$se^{st} = -e^{st} + (1 - \lambda_i)e^{s(t-\tau)},$$

which shows that s is a root of the characteristic

$$s = -1 + (1 - \lambda_i)e^{-s\tau} \tag{2.16}$$

corresponding to (2.15). This characteristic equation has infinitely many solutions $s \in \mathbb{C}$, which is expected because a delay differential equation has infinite-dimensional state space.

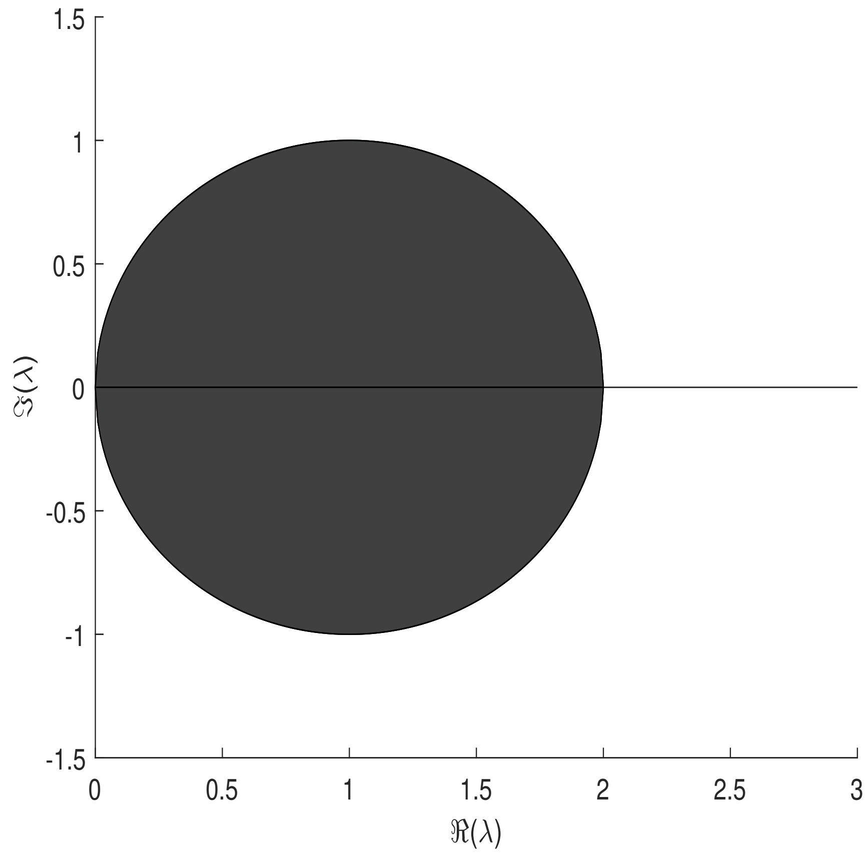

Figure 2.4: All the eigenvalues λ of the normalized Laplacian $\mathcal{L}$ belong to the shaded region $|1-\lambda| \leq 1$.

2.4 Consensus problem under time delay

Consensus as we have already defined in dynamical systems as an agreement among number of agents in related field of interest. In the field of communication, information can be processed and can be transmitted among various agents in suitable network.

Information Processing Delays:

$$\dot{x}_i(t) = \frac{1}{d_i}\sum_{j=1}^{n} a_{ij}(x_j(t-\tau) - x_i(t-\tau))$$

The non-normalized version of the information processing delay has been studied in [9]. They showed the condition under which the system reaches consesus asymptotically in an undirected network.

Theorem 2.4.1 (Olfati-Saber and Murray, 2004)**.** *The system under information processing delay τ on a connected undirected graph reaches consensus from arbitrary initial states if and only if $0 \leq \tau < \frac{\pi}{2\lambda_n}$, where λ_n denotes the largest eigenvalue of the Laplacian matrix.*

Proof. We refer to [9] for the proof. □

Information transmission delays can also take the following form:

$$\dot{x}_i(t) = \frac{1}{d_i}\sum_{j=1}^{N} a_{ij}\left(\int_0^{\tau} f(r)x_j(t-r)dr - x_i(t)\right) \quad \text{distributed delay} \tag{2.17}$$

Information transmission delays have been studied in [7], where exact conditions are derived under which a directed network can reach consensus. It was shown that the network under information transmission delay reaches consensus if and only if zero is a simple (unrepeated) eigenvalue of the Laplacian, which is a condition irrespective of the time delay.

Theorem 2.4.2 (Atay, 2013)**.** *The system (2.17) under information transmission delay reaches consensus from arbitrary initial states if and only if zero is a simple eigenvalue of the normalized Laplacian $\mathcal{L}$. Furthermore, the consensus value is given by*

$$c = \frac{1}{1+\bar{\tau}}\langle u^1, x(0) + \int_0^{\tau}\int_{-\theta}^{0} f(\theta)x(\xi)\,d\xi\,d\theta\rangle,$$

where u^1 is the left eigenvector of $\mathcal{L}$ corresponding to the zero eigenvalue and $\bar{\tau} = \int_0^{\tau} r f(r)\,dr$ is the mean of the delay distribution f. For the special case (2.6) of a single discrete delay τ, the consensus value is

$$c = \frac{1}{1+\tau}\langle u^1, x(0) + \int_{-\tau}^{0} x(\xi)\,d\xi\rangle.$$

Proof. We refer to [7] for the proof. $\square$

Chapter 3

Delay Differential Equations (DDEs)

Delay differential equations (DDEs) are a type of differential equation in which the derivative of the unknown function at a certain time is given in terms of the values of the function at previous times. Let $\tau \geq 0$ and let $C([-\tau, 0], \mathbb{R}^n)$ be the Banach space of continuous functions mapping the interval $[-\tau, 0]$ into $\mathbb{R}^n$ with the topology of uniform convergence. The general form of delay differential equation of retarded type can be written as

$$\dot{x}(t) = g(t, x_t) \tag{3.1}$$

where $x(t) \in \mathbb{R}^n$, $x_t \in C([-\tau, 0], \mathbb{R}^n)$ is the function defined by $x_t(\theta) = x(t + \theta)$ for $\theta \in [-\tau, 0]$,

$$g : D \to \mathbb{R}^n, \quad \text{where} \quad D \subset \mathbb{R} \times C([-\tau, 0], \mathbb{R}^n).$$

Consider the equation

$$\begin{aligned} \dot{x}(t) &= g(t, x_t) \quad t > t_0 \\ x(t) &= \phi(t) \quad \text{for} \quad t \in [t_0 - \tau, t_0]. \end{aligned} \tag{3.2}$$

where $\phi(t)$ is the initial function defined on the interval $[t_0 - \tau, t_0]$. Equation (3.2) is the initial value problem of equation (3.1) also known as the Cauchy problem.

Definition 3.0.1. A function x is said to be a solution of equation (3.1) on $[t_0-\tau, t_0+A)$ if there are $t_0 \in \mathbb{R}$ and $A > 0$ such that $x \in C([t_0-\tau, t_0+A), \mathbb{R}^n)$, $(t, x_t) \in D$ and $x(t)$ satisfies equation (3.1) for $t \in [t_0, t_0+A)$. For given initial time $t_0 \in \mathbb{R}$ and initial data $\phi \in C([-\tau, 0], \mathbb{R}^n)$, we say that $x(t_0, \phi, g)$ is a solution of equation (3.2) with initial value ϕ at t_0 if there is an $A > 0$ such that $x(t_0, \phi)$ is a solution of (3.2) on $[t_0-\tau, t_0+A)$ and $x_{t_0}(t_0, \phi) = \phi$.

Remark 3.0.2. The delay differential equation in (3.1) includes, as special cases, ordinary differential equations when $\tau = 0$,

$$\dot{x}(t) = G(t, x(t))$$

as well as integro-differential equations of the form

$$\dot{x}(t) = \int_{-\tau}^{0} g(t, \theta, x(t+\theta))\, d\theta.$$

Remark 3.0.3. [26] If $t_0 \in \mathbb{R}$, $\phi \in C([-\tau, 0], \mathbb{R}^n)$ are given, and $g(t, \phi)$ is continuous, then finding a solution of equation (3.1) through (ϵ, ϕ) is equivalent to solving the integral equation

$$\begin{aligned} x_{t_0} &= \phi \\ x(t) &= \phi(t_0) + \int_{t_0}^{t} g(\xi, x_\xi)\, d\xi, \quad t \geq t_0. \end{aligned}$$

We note that solution of the delay differential equation (3.1) becomes smoother in time, if the right hand side is smooth.

Definition 3.0.4. We say that g satisfies a Lipschitz condition with a Lipschitz constant K on the set $D \subset \mathbb{R} \times C([-\tau, 0], \mathbb{R}^n)$ if

$$\|g(t, \psi_1) - g(t, \psi_2)\| \leq K\|\psi_1 - \psi_2\|$$

for all (t, ψ_1) and (t, ψ_2) in D. We say that g is a Lipschitzian on D.

If g is Lipschitzian on $\mathbb{R} \times C([-\tau, 0], \mathbb{R}^n)$, it can be shown that the delay differential equation (3.2) has a unique solution.

Theorem 3.0.5. *[26] Suppose Ω is an open set in $\mathbb{R} \times C([-\tau, 0], \mathbb{R}^n)$, $g : \Omega \to \mathbb{R}^n$ is continuous, and $g(t, \phi)$ is Lipschitizian in ϕ in each compact set in Ω. If $(t_0, \phi) \in \Omega$, then there is a unique solution of equation (3.2).*

3.1 Solving DDEs

A conceptually simple method for solving equations with discrete delays is known as the method of steps. Let us consider the delay differential equation

$$\dot{x}(t) = g(t, x(t-\tau)) \quad \text{for} \quad t \geq 0, \tag{3.3}$$

and the initial condition

$$x(t) = \phi(t) \quad \text{for} \quad t \in [-\tau, 0], \tag{3.4}$$

ϕ is the intial function defined on the interval $[-\tau, 0]$. If we assume the ϕ to be continuous on the interval $[-\tau, 0]$, then it is easy to establish the existence and uniqueness of a solution of equation (3.3) on the interval $[0, \tau]$ subject to the initial condition (3.4) by the method called the method of steps. To be able to find a unique solution for the interval $[0, \tau]$, $g(t, \phi(t-\tau))$ must be integrable. On the interval $[0, \tau]$, equation (3.3) becomes a first order ordinary differential equation with the initial condition. Consider the initial value problem

$$\begin{aligned} &\dot{x}(t) = g(t, x(t-\tau)) \quad \text{for} \quad t \geq 0 \\ &x(0) = \phi(0), \end{aligned} \tag{3.5}$$

the solution of this initial value problem is

$$\int_0^t \dot{x}(t')dt' = \int_0^t g(t', \phi(t'-\tau))dt'$$

$$x(t) = \phi(0) + \int_0^t g(t', \phi(t'-\tau))dt', \quad \text{for} \quad t \in [0, \tau]. \tag{3.6}$$

To find a solution to equation (3.3) for the next interval $[\tau, 2\tau]$, we use the solution of (3.6) as the initial condition. Continuing this way, the solution to equation (3.3) can be found on successive intervals by using the solution from the previous interval as the initial condition.

Example 3.1.1. *Consider the delay-differential equation*

$$\dot{x}\,(t) = -x\,(t-1) \quad t \geq 0$$

$$x(t) = 1, \quad t \in [-1, 0] .$$

For the interval $t \in [0, 1]$,

$$\begin{aligned} x(t) &= x(0) + \int_0^t \frac{d}{dt} x(t') dt' \\ &= 1 - \int_0^t x(t'-1) dt' \\ &= 1 - \int_1^t 1 dt' \\ &= 1 - t. \end{aligned}$$

For the interval $t \in [1, 2]$,

$$\begin{aligned} x(t) &= x(1) + \int_1^t \frac{d}{dt} x(t') dt' \\ &= 0 - \int_1^t x(t'-1) dt' \\ &= -\int_0^{t-1} x(s) ds \\ &= -\int_0^{t-1} (1-s)\, ds \\ &= \frac{t^2}{2} - 2t + \frac{3}{2}. \end{aligned}$$

For the interval $t \in [2, 3]$, *we have*

$$\begin{aligned} x(t) &= -\frac{1}{6}(t-1)^3 + (t-1)^2 - \frac{3}{2}t + \frac{5}{3} \\ x(2) &= -\frac{1}{2}. \end{aligned}$$

The solution for $x(t)$ continues analogously for the successive intervals.

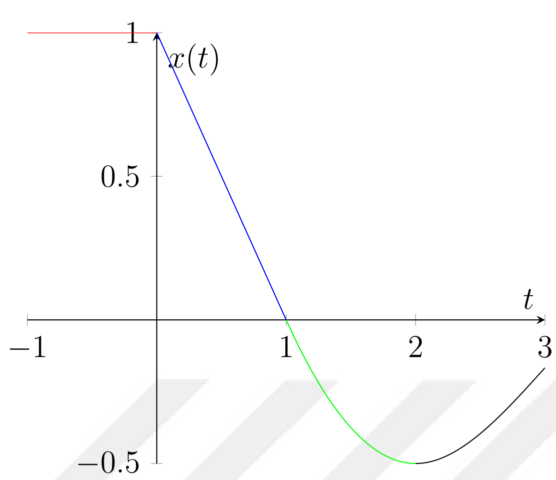

Figure 3.1: The plot of the solutions to the delay differential equation in Example 3.1.1

3.2 Lambert W Function

The Lambert W function is defined as the inverse of the mapping

$$z \mapsto ze^z, \quad z \in \mathbb{C}.$$

Hence, it satisfies $z = W(ze^z)$ [13]. Let $z_0 = ze^z$, we have $z_0 = W(z_0)e^{W(z_0)}$ for $z_0 \in \mathbb{C}$. $W(z_0)$ is a multivalued function except at zero. The Lambert W function with complex argument z is a complex valued function with infinite branches, denoted $W_k, k = 0, \pm1, \pm2, \ldots$. Each branch W_k is a single-valued function, and W_0 is the principal branch.

Lemma 3.2.1 (Shinozaki and Mori, 2006)**.** *For arbitrary $z \in \mathbb{C}$,*

$$\max\{\mathrm{Re}(W_k(z)) \mid k = 0, \pm1, \pm2, \ldots\} = \mathrm{Re}(W_0(z)).$$

Proof. We refer to [12] for the proof. $\square$

By Lemma 3.2.1, it is enough to check the principal branch for the stability analysis and also the analysis of the convergence speed which we will be looking at in the following chapters.

3.2.1 Using the Lambert W function

By using the Lambert W function we calculate the roots of the characteristic equation

$$s = -1 + (1 - \lambda_i)\, e^{-s\tau}$$

corresponding to the delay differential equation

$$\dot{\alpha}_i(t) = -\alpha_i(t) + (1 - \lambda_i)\alpha_i(t - \tau)$$

Letting $\tilde{s} = s\tau$ and $b = (1 - \lambda_i)$, we have

$$\begin{aligned}
\frac{\tilde{s}}{\tau} &= -1 + be^{-\tilde{s}} \\
\tilde{s} &= -\tau + b\tau e^{-\tilde{s}} \\
\tilde{s} + \tau &= b\tau e^{-\tilde{s}} \\
\tilde{s} + \tau &= b\tau e^{-\tilde{s}} e^{\tau} e^{-\tau} \\
\tilde{s} + \tau &= b\tau e^{-(\tilde{s}+\tau)} e^{\tau} \\
(\tilde{s} + \tau)\, e^{(\tilde{s}+\tau)} &= b\tau e^{\tau} \\
\tilde{s} + \tau &= W_k\left(b\tau e^{\tau}\right) \\
\tilde{s} &= -\tau + W_k\left(b\tau e^{\tau}\right) \\
s\tau &= -\tau + W_k\left(b\tau e^{\tau}\right) \\
s &= -1 + \frac{1}{\tau} W_k\left(b\tau e^{\tau}\right)
\end{aligned}$$

Example 3.2.2. *Consider the transcendental equation*

$$s = -1 - \frac{1}{2} e^{-\frac{1}{2}s}.$$

Using the Lambert W function we can write its solutions as

$$s = -1 + 2W_k\left(-\frac{1}{4} e^{\frac{1}{2}}\right).$$

The numerical solution to this example is shown in Figure 3.2.

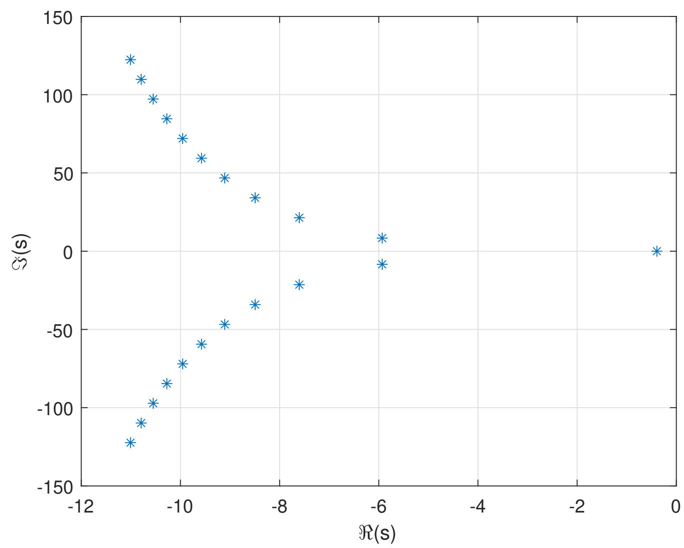

Figure 3.2: The plot of $s = -1 + 2W_k\left(-\frac{1}{4}e^{\frac{1}{2}}\right)$ for $k = 0, \pm 1, \pm 2, \ldots, \pm 10$.

Chapter 4

Effects of Small Delays

4.1 Convergence Speed of Consensus Problems

The normalized consensus problem on a network continuous in $t \in \mathbb{R}$ as a system of differential equations given in (2.2)

$$\dot{x}_i(t) = \frac{1}{d_i} \sum_{j=1}^{n} a_{ij}(x_j(t) - x_i(t)), \quad i = 1, \dots, n.$$

The vector form as given in (2.4) is

$$\dot{x}(t) = -\mathcal{L}x(t),$$

with solution $x(t) = e^{-\mathcal{L}t}x(0)$, where $x(0)$ is the initial condition. In the eigenbasis of $\mathcal{L}$, we express x as $x(t) = \sum_{i=1}^{n} \alpha_i(t)v_i$, where $\alpha_i(t) = \langle u_i, x(t) \rangle$. So, the solution is

$$x(t) = e^{-\mathcal{L}t}x(0) = \sum_{i=1}^{n} \langle u_i, x(0) \rangle e^{-\lambda_i t} v_i, \tag{4.1}$$

where the λ_i are the eigenvalues of the normalized Laplacian $\mathcal{L}$. Recall that $\lambda_1 = 0$ and $v_1 = (1, 1, \dots, 1)^\top$. So, the solution $x(t)$ in (4.1) converges to $\langle u_1, x(0) \rangle v_1$, that is, the system (2.2) reaches consensus from arbitrary initial conditions, if and only if

$$\operatorname{Re} \lambda_i > 0 \quad \text{for } i \geq 2. \tag{4.2}$$

Under condition (4.2), $x(t) \to \langle u_1, x(0)\rangle v_1$, with the consensus value given by $c = \langle u_1, x(0)\rangle$. For undirected graphs, where the eigenvalues are real, the slowest mode converges at the rate $e^{-\lambda_2 t}$. Hence, λ_2 serves as a measure of the consensus speed in the undelayed problem. In graph theory, the quantity λ_2 is called the spectral gap.

The consensus problem under the information transmission delay given in (2.6),

$$\dot{x}_i(t) = \frac{1}{d_i} \sum_{j=1}^{n} a_{ij}(x_j(t-\tau) - x_i(t)), \quad i = 1, \dots, n,$$

can be written in vector form as given in (2.8):

$$\dot{x}(t) = -x(t) + D^{-1}Ax(t-\tau).$$

Similar to above, in the eigenbasis of $\mathcal{L}$ we express x as $x(t) = \sum_{i=1}^{n} \alpha_i(t) v_i$, where $\alpha_i(t) = \langle u_i, x(t)\rangle$. Assuming an exponential solution $\alpha_i(t) = e^{st}$ we obtain the characteristic equation (2.16):

$$s = -1 + (1 - \lambda_i)e^{-s\tau}.$$

We will measure the convergence speed of the system (2.6) by the characteristic value s closest to the imaginary axis, that is, $\min\{-\operatorname{Re}(s)\}$ where the minimum is taken over all solutions s of the characteristic equation (except the zero root, which is always present). This is well defined since $\operatorname{Re}(s) < 0$ for all characteristic roots (except for a single zero root).

The system (2.6) is known to reach consensus if and only if zero is a simple eigenvalue of the normalized Laplacian $\mathcal{L}$ [7]. We will always assume this condition throughout the thesis.

4.2 The effect of delays on the convergence speed

From the literature, we see that time delay has no effect on the information transmission delay reaching consensus. Surprisingly, time delay in certain networks can speed up the convergence to consensus, which is the main topic of this thesis. The effect of delays on the convergence speed depends largely on the structure of the graph. Similarly, we know that delays can slow down the convergence speed so as it can speed up the convergence speed provided the graph structure satisfy certain conditions, which we are going to analyze.

Lets us consider two simple graphs, the directed cycle C_4 and the complete graph K_4, as depicted in Figure 4.1. The system (2.6) defined over these graphs reaches consensus regardless of the time delay and the initial conditions.

(a) Directed cycle graph C_4

(b) Complete graph K_4

Figure 4.1: The networks (graphs) used in the simulations of Figure 4.2 and Figure 4.3.

The system (2.6) over the graph in Figure 4.1a reaches consensus for $\tau = 0$ and $\tau = 0.5$, as shown in Figure 4.2. Likewise, (2.6) reaches consensus over the graph in Figure 4.1b for $\tau = 0$ and $\tau = 0.5$, as shown in Figure 4.3.

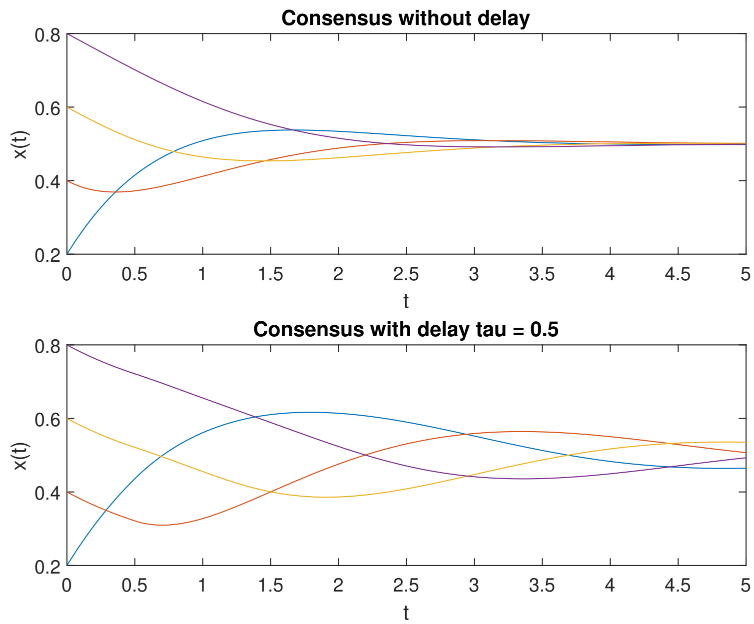

Figure 4.2: The top plot is the simulation of the system (2.2), while the bottom plot is the simulation of the system (2.6) using the graph in Figure 4.1a.

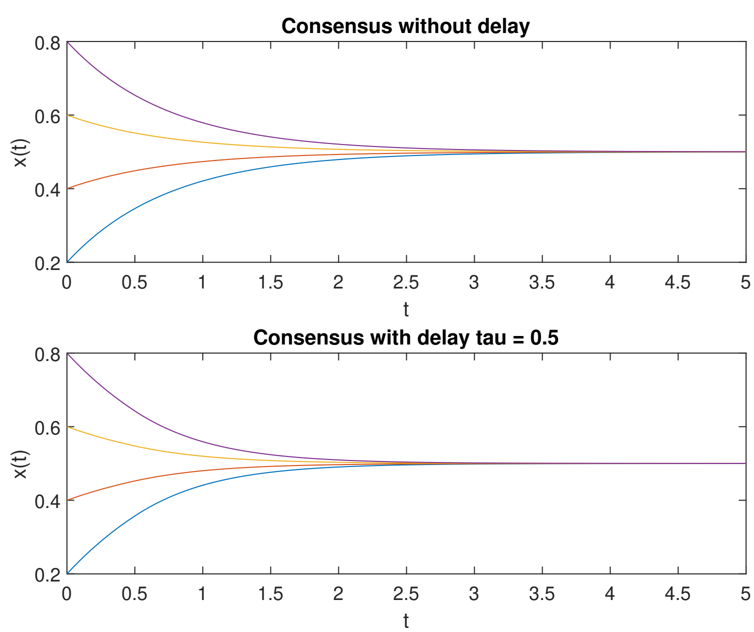

Figure 4.3: The top plot is the simulation of the system (2.2), while the bottom plot is the simulation of the system (2.6) using the graph in Figure 4.1b.

From the plots we see that time delay can speed up or slow down the convergence speed depending on the structure of the graph. We compare the convergence speed of the delayed system (2.6) and the undelayed consensus problem in equation (2.2) under the two simple graphs in Figure 4.1. For each graph, we compare the standard deviation of the states of various agents with the delay consensus problem and without the delay consensus problem.

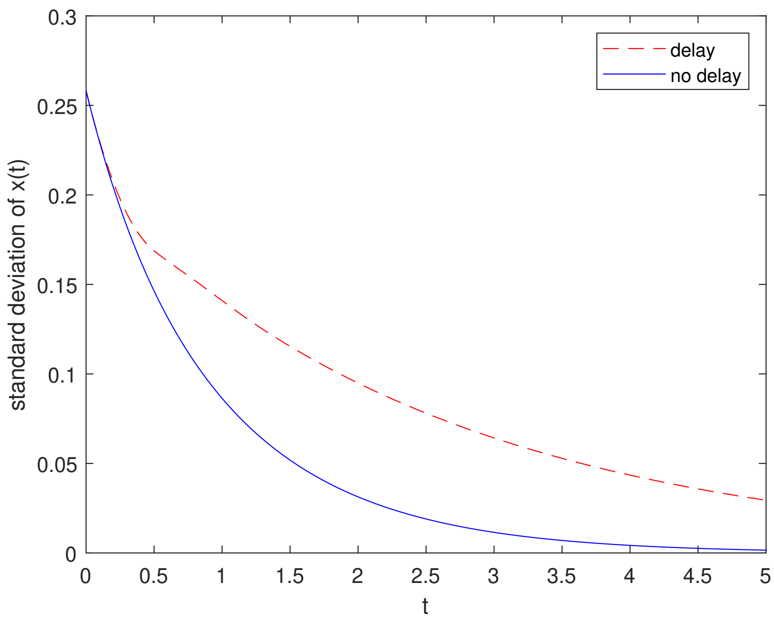

Figure 4.4: The plot shows how delay worsens the convergence speed

The network in Figure 4.1a is used in the consensus dynamics (2.2) without delays, as well as in (2.6) with time delay $\tau = 0.5$, in both cases with the same initial states ($x_1 = 0.2, x_2 = 0.4, x_3 = 0.6, x_4 = 0.8$). The standard deviation of the states of the agents is plotted against time in the Figure 4.4. It is observed that under this network, the time delay worsens the convergence speed.

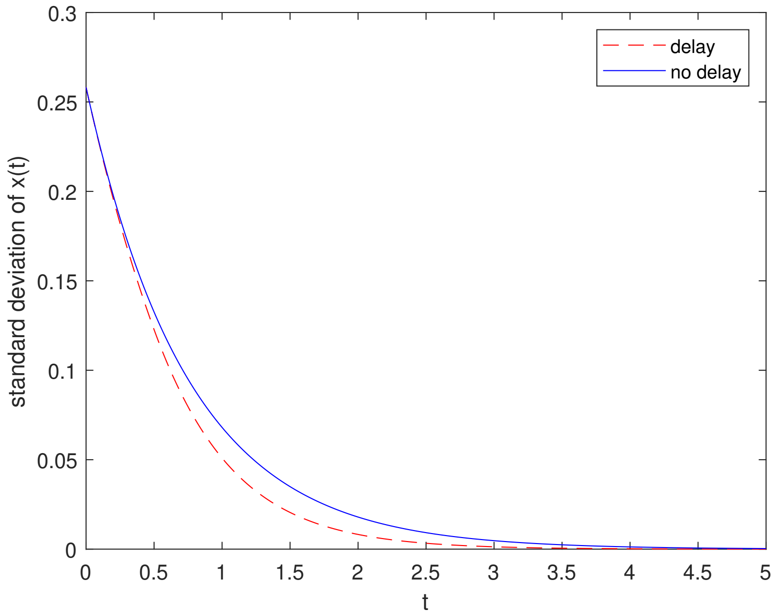

Figure 4.5: The plot shows how delay improves the convergence speed.

The network in Figure 4.1b is used in the consensus dynamics (2.2) without delays, as well as in (2.6) with time delay $\tau = 0.5$ with the same initial conditions $(x_1 = 0.2, x_2 = 0.4, x_3 = 0.6, x_4 = 0.8)$. The standard deviation of the states of the agents is plotted against time in the Figure 4.5. It is observed that under this network, time delay improves the convergence speed.

4.3 Analytical explanation for small delays

We have observed that for certain graphs the consensus speed can be improved by the presence of delays. We shall now study the spectrum of such graphs.

Consider the system (2.8) under information transmission delays

$$\dot{x}_i(t) = \frac{1}{d_i}\sum_{j=1}^{n} a_{ij}(x_j(t-\tau) - x_i(t)), \quad i = 1, 2, \ldots, n,$$

or, in vector form (2.6),

$$\dot{x}(t) = -x(t) + D^{-1}Ax(t-\tau).$$

Carrying out the decomposition as $x(t) = \sum_{i=1}^{n} \alpha_i(t)v_i$ as in equation (2.15), we have

$$\dot{\alpha}_i(t) = -\alpha_i(t) + (1-\lambda_i)\alpha_i(t-\tau).$$

The characteristic equation is

$$s = -1 + (1-\lambda_i)e^{-s\tau}, \tag{4.3}$$

where λ_i is the eigenvalues of the normalized Laplacian $\mathcal{L}$. We will study the characteristic roots s and the spectrum of the networks that make the roots to be in the left half plane. Implicit differentiation of (4.3) gives

$$\begin{aligned}
\frac{\partial s}{\partial \tau} &= -s(1-\lambda_i)e^{-s\tau} - \tau(1-\lambda_i)e^{-s\tau}\frac{\partial s}{\partial \tau} \\
(1+\tau(1-\lambda_i)e^{-s\tau})\frac{\partial s}{\partial \tau} &= -s(1-\lambda_i)e^{-s\tau} \\
\frac{\partial s}{\partial \tau} &= \frac{-s(1-\lambda_i)e^{-s\tau}}{1+\tau(1-\lambda_i)e^{-s\tau}}
\end{aligned} \tag{4.4}$$

at $\tau = 0$, $s = -\lambda_i$ and substituting into equation (4.4) we have

$$\left.\frac{\partial s}{\partial \tau}\right|_{\tau=0} = \lambda_i(1-\lambda_i). \tag{4.5}$$

Hence, near $\tau = 0$, the real part of the characteristic roots can move to the left or right depending on the spectrum $\{\lambda_i\}$ of the underlying graph, as given by (4.5). Recall that the λ_i lie in the shaded region of Figure 2.4.

Let $\lambda_i = \gamma_i + i\delta_i$, $\gamma_i, \delta_i \in \mathbb{R}$ where γ_i and δ_i are the real and imaginary parts of the eigenvalues λ_i. Then we have

$$\begin{aligned}
\left.\frac{\partial s}{\partial \tau}\right|_{\tau=0} &= (\gamma_i + i\delta_i)(1-\gamma_i - i\delta_i) \\
&= \gamma_i - \gamma_i^2 - i\gamma_i\delta_i + i\delta_i - \gamma_i\delta_i + \delta_i^2 \\
&= (\gamma_i - \gamma_i^2 + \delta_i^2) + i(\delta_i - 2\gamma_i\delta_i) \\
&= \gamma_i(1-\gamma_i) + \delta_i^2 + i(\delta_i - 2\gamma_i\delta_i).
\end{aligned}$$

The real part is given as

$$\operatorname{Re}\left(\frac{\partial s}{\partial \tau}\right)\Bigg|_{\tau=0} = \gamma_i(1-\gamma_i) + \delta_i^2. \tag{4.6}$$

When $\operatorname{Re}(\frac{\partial s}{\partial \tau})\Big|_{\tau=0} < 0$, small time delays can improve the convergence speed of the information transmission delay consensus problem. This condition depends on the spectrum of the underlying network. Hence, when

$$\gamma_i(\gamma_i - 1) < \delta_i^2 \quad \forall i \geq 2, \tag{4.7}$$

small delays improves the convergence speed.

Definition 4.3.1. A graph G whose eigenvalues satisfy condition (4.7) will be called an *FD-graph* (for *F*aster under *D*elay).

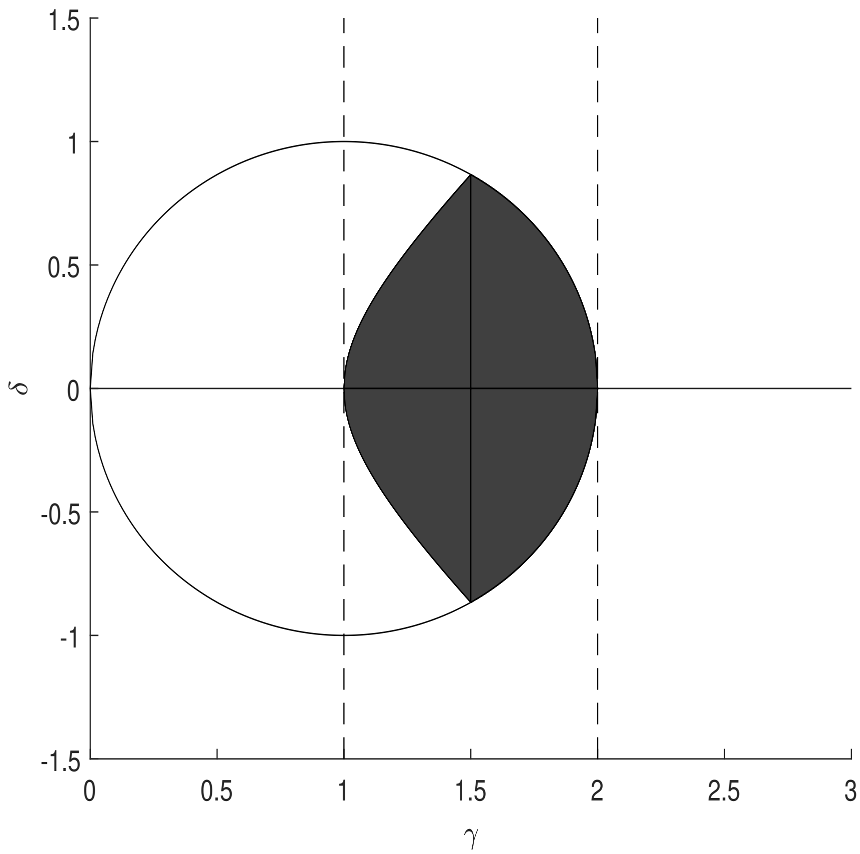

Figure 4.6: The shaded region shows complex numbers $\lambda = \gamma + i\delta$ that satisfy (4.7). The eigenvalues of FD-graphs belong to this region.

From the analysis of the characteristic equation of the information transmission delays under small delays, the following results are obtained.

Theorem 4.3.2. *If the eigenvalues of the normalized Laplacian $\mathcal{L}$ satisfy the condition* $\mathrm{Re}(\lambda_i)(\mathrm{Re}(\lambda_i) - 1) < [\mathrm{Im}(\lambda_i)]^2$ *for $i \geq 2$, then small delays improve the convergence speed of the system* (2.8).

Recall that the eigenvalues of the normalized Laplacian for undirected graphs are real, so we can order them as $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq 2$.

Lemma 4.3.3 (Chung and Graham, 1997)**.** *For an undirected graph G on n vertices, we have*

(i): $\sum \lambda_i \leq n$*, with equality holding if and only if G has no isolated vertices;*

(ii): $\lambda_2 \leq \frac{n}{n-1}$*, with equality holding if and only if G is complete graph on n vertices. Also for a graph G without isolated vertices, we have* $\lambda_n \geq \frac{n}{n-1}$*;*

(iii): for a graph which is not a complete graph, we have $\lambda_2 \leq 1$.

Proof. We refer to [16] for the proof. □

Theorem 4.3.4. *For an undirected graph G, small delays improve the convergence speed* $\left(\mathrm{Re}(\frac{\partial s}{\partial \tau})\Big|_{\tau=0} < 0\right)$ *if and only if G is a complete graph.*

Proof. ($\Rightarrow$) For an undirected graph G, if small delays improves convergence speed then,

$$\mathrm{Re}\left(\frac{\partial s}{\partial \tau}\right)\Bigg|_{\tau=0} < 0 \implies \lambda_2(1 - \lambda_2) < 0$$

So

$$\lambda_2(1 - \lambda_2) < 0$$

$$\lambda_2 > 1.$$

Therefore the graph is complete by Lemma 4.3.3 part (*ii*) and (*iii*).

($\Leftarrow$) If G is a complete graph, then from the Lemma 4.3.3 part (*ii*), the eigenvalue of the normalized Laplacian $\lambda_2 = \frac{n}{n-1}$; so we have

$$\begin{aligned}
\mathrm{Re}\left(\frac{\partial s}{\partial \tau}\right)\Bigg|_{\tau=0} &= \lambda_2(1-\lambda_2) \\
&= \frac{n}{n-1}\left(1-\frac{n}{n-1}\right) \\
&= -\frac{n}{(n-1)^2} \\
&< 0.
\end{aligned}$$

The proof is completed. □

4.4 Stability Analysis

Consider the characteristic equation of the information transmission delay

$$s = -1 + (1-\lambda_i)e^{-s\tau}.$$

When there is no delay, the characteristic roots are the negatives of the Laplacian eigenvalues:

$$s = -\lambda_i \quad \text{when } \tau = 0. \tag{4.8}$$

Since $\mathrm{Re}(\lambda_i) > 0 \ \forall i \geq 2$, it is clear that the system in (2.3) is stable for any eigenvalue of the normalized Laplacian matrix. We will study the position of the characteristic root s for $\tau > 0$.

Let $\sigma_0 = \min_{i\geq 2}\{\mathrm{Re}\,\lambda_i\}$; σ_0 is the spectral gap of the network, which is a measure of the convergence speed of the undelayed system. Our main objective is to check for the condition $\mathrm{Re}(s) < -\sigma_0$ for the characteristic roots s under delays: If $\mathrm{Re}(s) < -\sigma_0$, then the system will converge faster under delays. Let us consider the equivalent condition $\mathrm{Re}(s + \sigma_0) < 0$, and consider $z = s + \sigma_0$.

Substituting $s = z - \sigma_0$ into the characteristic equation, we get

$$\begin{aligned} s &= -1 + (1-\lambda_i)e^{-s\tau} \\ z - \sigma_0 &= -1 + (1-\lambda_i)e^{-(s)\tau} \\ z &= \sigma_0 - 1 + (1-\lambda_i)e^{\sigma_0\tau}e^{-z\tau}. \end{aligned} \tag{4.9}$$

The last line is the characteristic equation of the delay equation

$$\dot{x}(t) = (\sigma_0 - 1)x(t) + (1-\lambda_i)e^{\sigma_0\tau}x(t-\tau), \tag{4.10}$$

4.4.1 Exact stability region for real coefficients

Consider the delay differential equation (4.10) and its characteristic equation

$$z = a + be^{-z\tau}. \tag{4.11}$$

Here, $a = (\sigma - 1) \in \mathbb{R}$, and for a real Laplacian eigenvalues, $\sigma_0 = \lambda_i$, $b = (1-\lambda_i)e^{\lambda_i\tau} \in \mathbb{R}$. We make a re-scaling of time to reduce the number of parameters. In (4.10), let $r = t/\tau$, $\tau > 0$. Then $t = r\tau$ and $dt = \tau dr$, so that

$$\frac{d}{\tau dr}x(r\tau) = ax(r\tau) + bx(r\tau - \tau)$$

$$\frac{d}{dr}x(r\tau) = a\tau x(r\tau) + b\tau x(r\tau - \tau),$$

Let $u(r) = x(r\tau)$, so that $x(r\tau - \tau) = x((r-1)\tau) = u(r-1)$ and

$$\frac{d}{dr}u(r) = a\tau u(r) + b\tau u(r-1).$$

Denoting again $t = r$, the rescaled equation has the form

$$\dot{u}(t) = a\tau u(t) + b\tau u(t-1), \tag{4.12}$$

which has a unit delay and τ appearing as a parameter of the equation. With $v = z\tau$, the characteristic equation for (4.12) is

$$v = a\tau + b\tau e^{-v} := \alpha + \beta e^{-v}. \tag{4.13}$$

Since $v = z\tau$, the real parts of v and z have the same sign; hence, roots v of (4.13) have negative real parts if and only if roots z of (4.11) have negative real parts. In other words, the zero solution of (4.10) is asymptotically stable if and only if all the roots v of the characteristic equation (4.13) have negative real parts. A necessary and sufficient condition that all the roots of (4.13) be in the left half plane has been given by Hayes [18] in the following theorem.

Theorem 4.4.1 (Hayes, 1950)**.** *The roots v of (4.13) are all in the left half plane if and only if $\alpha < 1$ and $\alpha < -\beta < \sqrt{\nu^2 + \alpha^2}$, where ν is the root of the $\nu \cot \nu = \alpha$ satisfying $0 < \nu < \pi$.*

Here, we relate Theorem 4.4.1 to our characteristic equation obtained in (4.13):

$$\alpha = a\tau = \tau(\lambda_i - 1) < 1,$$

so

$$\tau < \frac{1}{\lambda_i - 1} \tag{4.14}$$

and

$$\tau(\lambda_i - 1) < \tau(\lambda_i - 1)e^{\lambda_i \tau} < \sqrt{V^2 + \tau^2(\lambda_i - 1)^2}. \tag{4.15}$$

For a given range of λ we can analyze the exact stability region corresponding to τ using the two conditions from (4.14) and (4.15) and the exact stability region is shown in Figure 4.7.

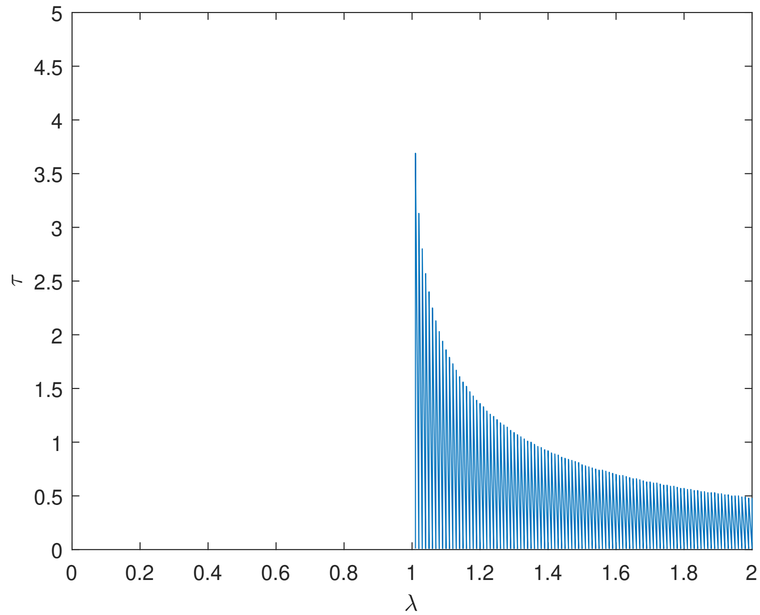

Figure 4.7: The shaded region shows the parameter values for which a delayed system with real spectrum reaches consensus faster than the undelayed system..

Theorem 4.4.2. *Consider* (4.10) *with* $\lambda_i \in \mathbb{R}$*, the zero solution is*

(i): unstable for all $\tau > 0$ *if* $\lambda_i < 1$*;*

(ii): asymptotically stable if $\lambda_i > 1$ *for* $0 < \tau < T$*, and is unstable for* $\tau > T$*.*

Proof. From the delay differential equation (4.10),

$$\dot{x}(t) = (\lambda_i - 1)x(t) + (1 - \lambda_i)e^{\lambda_i \tau}x(t - \tau).$$

(i): If $\lambda_i < 1$, $(1 - \lambda_i)e^{\lambda_i \tau} > 0$ since $e^{\lambda_i \tau} > 1$ for $\tau > 0$ and $\lambda_i > 0$. For the zero solution of equation (4.10) to be stable, $(\lambda_i - 1) + (1 - \lambda_i)e^{\lambda_i \tau}$ must be strictly negative by Hayes.

$$\begin{aligned}(\lambda_i - 1) + (1 - \lambda_i)e^{\lambda_i \tau} &= (1 - \lambda_i)e^{\lambda_i \tau} - (1 - \lambda_i)\\ &= (1 - \lambda_i)(e^{\lambda_i \tau} - 1)\\ &> 0 \quad \text{for} \quad \lambda_i < 1.\end{aligned}$$

(ii): If $\lambda_i > 1$, $(1-\lambda_i)e^{\lambda_i \tau} < 0$. For the zero solution of equation (4.10) to be stable, $(\lambda_i - 1) + (1-\lambda_i)e^{\lambda_i \tau}$ must be strictly negative by Hayes.

$$\begin{aligned}(\lambda_i - 1) + (1-\lambda_i)e^{\lambda_i \tau} &= (1-\lambda_i)e^{\lambda_i \tau} - (1-\lambda_i) \\ &= (1-\lambda_i)(e^{\lambda_i \tau} - 1) \\ &< 0 \quad \text{for} \quad \lambda_i > 1.\end{aligned}$$

□

Corollary 4.4.3. *For a graph with real eigenvalues:*

(i): if there exists an eigenvalue less than one, then the system (2.8) converges slower for all positive values of delay;

(ii): if all eigenvalues are greater than one, then the system (2.8) converges faster for a range of delays.

4.4.2 Exact stability region for complex coefficients

Consider the delay equation in (4.10)

$$\dot{x}(t) = (\sigma_0 - 1)x(t) + (1-\lambda_i)e^{\sigma_0 \tau}x(t-\tau).$$

For complex eigenvalues λ_i, $\sigma_0 - 1$ is a real number because $\sigma_0 = \min_{i \geq 2}\{\operatorname{Re}\lambda_i\}$ and $(1-\lambda_i)e^{\sigma_0 \tau}$ is a complex number. Letting $p = \sigma_0 - 1$ and $q = (1-\lambda_i)e^{\sigma_0 \tau}$, we obtain a delay equation with complex coefficient given as

$$\dot{x}(t) = px(t) + qx(t-\tau). \tag{4.16}$$

Let $x(t) = \phi(t) + i\Phi(t)$ be a solution of equation (4.16)) and $\lambda_i = \gamma_i + i\delta_i$.

$$\begin{aligned}q &= (1-\gamma_i - i\delta_i)e^{\sigma_0 \tau} \\ &= (1-\gamma_i)e^{\sigma_0 \tau} + i(-\delta_i)e^{\sigma_0 \tau} \\ &= q_1 + iq_2\end{aligned}$$

We have

$$\begin{aligned}\dot{\phi}(t)+i\dot{\Phi}(t) &= p(\phi(t)+i\Phi(t))+(q_1+iq_2)(\phi(t-\tau)+i\Phi(t-\tau))\\ &= p\phi(t)+ip\Phi(t)+q_1\phi(t-\tau)+iq_1\Phi(t-\tau)+iq_2\phi(t-\tau)-q_2\Phi(t-\tau)\\ &= [p\phi(t)+q_1\phi(t-\tau)-q_2\Phi(t-\tau)]+i[p\Phi(t)+q_1\Phi(t-\tau)+q_2\phi(t-\tau)],\end{aligned}$$

where $q_1=(1-\gamma_i)e^{\sigma_0\tau}$ and $q_2=-\delta_i e^{\sigma_0\tau}$. Separating the real and imaginary parts, we obtain

$$\begin{aligned}\dot{\phi}(t) &= p\phi(t)+q_1\phi(t-\tau)-q_2\Phi(t-\tau)\\ \dot{\Phi}(t) &= p\Phi(t)+q_1\Phi(t-\tau)+q_2\phi(t-\tau).\end{aligned} \tag{4.17}$$

The characteristic equation of (4.17) is

$$\begin{vmatrix}\eta-p-q_1e^{-\eta\tau} & -q_2e^{-\eta\tau}\\ -q_2e^{-\eta\tau} & \eta-p-q_1e^{-\eta\tau}\end{vmatrix}=0$$

$$(\eta-p-q_1e^{-\eta\tau})^2+q_2^2e^{-2\eta\tau}=0. \tag{4.18}$$

Equation (4.18) is equivalent to the system

$$(\eta-p-q_1e^{-\eta\tau}-iq_2e^{-\eta\tau})(\eta-p-q_1e^{-\eta\tau}+iq_2e^{-\eta\tau})=0$$

$$\eta-p-q_1e^{-\eta\tau}-iq_2e^{-\eta\tau}=0 \tag{4.19}$$

and

$$\eta-p-q_1e^{-\eta\tau}+iq_2e^{-\eta\tau}=0 \tag{4.20}$$

It is easy to show that η is a root of equation (4.19) if and only if its conjugate is a root of (4.20). So, studying the roots of (4.19) is enough to investigate the asymptotic stability of (4.16) since (4.16) is equivalent to (4.17). The trivial solution of (4.16) is asymptotically stable if and only if all the roots of (4.19) have negative real parts, and unstable if and only if (4.19) has a root with positive real part. To be able to study the distribution of the roots of (4.19), we need the results of Ruan and Wei [32].

Theorem 4.4.4 (Ruan and Wei, 2003)**.** *For the transcendental equation*

$$\begin{aligned}
&\eta^n + p_1^{(0)}\eta^{n-1} + \cdots + p_{n-1}^{(0)}\eta + p_n^{(0)} \\
&\quad + [p_1^{(1)}\eta^{n-1} + \cdots + p_{n-1}^{(1)}\eta + p_n^{(1)}]e^{-\eta\tau_1} \\
&\quad + \cdots + [p_1^{(m)}\eta^{n-1} + \cdots + p_{n-1}^{(m)}\eta + p_n^{(m)}]e^{-\eta\tau_m} = 0
\end{aligned} \tag{4.21}$$

as $(p_1^{(0)}, \ldots, p_1^{(0)}, \ldots, p_1^{(1)}, \ldots, p_1^{(m)}; \tau_1, \ldots, \tau_m)$ *varies, the sum of the orders of the zeros of* (4.21) *in the open right half-plane can change only if a zero solution appears on or crosses the imaginary axis. Here* $\tau_i \geq 0 (i = 1, 2, \ldots, m)$ *and* $p_i^j (i = 0, 1, \ldots, m; j = 1, 2, \ldots, n)$ *are constants.*

Proof. We refer to [32] for the proof. □

Let $\eta = i\omega$ be the root of (4.19). We have

$$i\omega - p - q_1 e^{-i\omega\tau} - iq_2 e^{-i\omega\tau} = 0$$

$$i\omega - p - q_1\cos(\omega\tau) + iq_1\sin(\omega\tau) - iq_2\cos(\omega\tau) - q_2\sin(\omega\tau) = 0.$$

Separating the real and imaginary parts, we get

$$\begin{aligned}
-p &= q_1\cos(\omega\tau) + q_2\sin(\omega\tau) \\
w &= -q_1\sin(\omega\tau) + q_2\cos(\omega\tau)
\end{aligned} \tag{4.22}$$

and

$$\omega^2 = q_1^2 + q_2^2 - p^2. \tag{4.23}$$

Equation (4.23) cannot hold if

$$q_1^2 + q_2^2 < p^2. \tag{4.24}$$

Lemma 4.4.5 (Wei and Zhang, 2004)**.** *If*

$$p + q_1 < 0$$

and (4.24) *are satisfied, then all the roots of* (4.19) *have negative real parts.*

Proof. We refer to [19] for the proof. □

Lemma 4.4.6 (Wei and Zhang, 2004)**.** *If*

$$p + q_1 > 0$$

and equation (4.24) are satisfied, then equation (4.19) has at least one root with positive real part for all $\tau \geq 0$.

Proof. We refer to [19] for the proof. □

4.5 Distributed delays

A more general delay model is given by distributed delays, which accounts for data distributed over an interval rather than specified at a single past time point. The general setting of consensus problem under distributed information transmission delays can be written as

$$\dot{x}_i(t) = \frac{1}{d_i} \sum_{j=1}^{n} a_{ij} \left(\int_0^\tau f(r) x_j(t-r)\, dr - x_i(t) \right) \tag{4.25}$$

$$\dot{x}(t) = -x(t) + D^{-1} A \int_0^\tau f(r) x(t-r)\, dr. \tag{4.26}$$

The function $f(r)$ is a distributed delay kernel over the interval $[0, \tau]$. We assume $f(r) \geq 0$ for $r \in [0, \tau]$. Here, with $x(t) = \sum_{i=1}^{n} \alpha_i(t) v_i$, while $\alpha_i(t) = \langle u_i, x(t) \rangle$. We have

$$\dot{\alpha}_i(t) = -\alpha_i(t) + (1 - \lambda_i) \int_0^\tau f(r) \alpha_i(t-r) dr \tag{4.27}$$

and its characteristic equation is

$$X_i(\varphi) := \varphi + 1 - (1 - \lambda_i) F(\varphi) = 0,$$

with

$$F(\varphi) = \int_0^\tau e^{-\varphi r} f(r)\, dr.$$

We will study the nature of the roots of the characteristic equation by transforming the equation to an ordinary differential equation.

4.5.1 Transformation to ODE

Consider the family of Gamma distributions

$$f_k(\tau) = \frac{\mu^{k+1}}{k!}\tau^k e^{-\mu\tau} \tag{4.28}$$

where $\tau \geq 0, k \in \mathbb{N}$ and $\mu > 0$, as the delay kernel in (4.26). The mean and the variance of the Gamma distribution are $\frac{k+1}{\mu}$ and $\frac{k+1}{\mu^2}$, respectively. Substituting (4.28) into (4.27), we obtain

$$\dot{\alpha}_i(t) = -\alpha_i(t) + (1-\lambda_i)\frac{\mu^{k+1}}{k!}\int_0^\infty \tau^k e^{-\mu\tau}\alpha_i(t-\tau)d\tau. \tag{4.29}$$

We can transform the delay equation (4.29) into an ordinary differential equation.
Case 1: For $k = 0$,

$$\begin{aligned}\dot{\alpha}_i(t) &= -\alpha_i(t) + (1-\lambda_i)\mu\int_0^\infty e^{-\mu\tau}\alpha_i(t-\tau)d\tau\\ \dot{\alpha}_i(t) + \alpha_i(t) &= (1-\lambda_i)\mu\int_0^\infty e^{-\mu\tau}\alpha_i(t-\tau)d\tau.\end{aligned}$$

Let

$$\begin{aligned}y_{0i}(t) &= (1-\lambda_i)\mu\int_0^\infty e^{-\mu\tau}\alpha_i(t-\tau)d\tau\\ &= (1-\lambda_i)\mu\int_{-\infty}^t e^{-\mu(t-\rho)}\alpha_i(\rho)d\rho.\end{aligned}$$

Taking derivative of $y_{0i}(t)$ with respect to t,

$$\begin{aligned}\dot{y}_{0i}(t) &= (1-\lambda_i)\mu\alpha_i(t) - (1-\lambda_i)\mu^2\int_{-\infty}^t e^{-\mu(t-\rho)}\alpha_i(\rho)d\rho\\ &= (1-\lambda_i)\mu\alpha_i(t) - \mu y_{0i}(t)\end{aligned}$$

yields to two linear ordinary differential equations

$$\begin{aligned}\dot{\alpha}_i(t) &= -\alpha_i(t) + y_{0i}(t)\\ \dot{y}_{0i}(t) &= (1-\lambda_i)\mu\alpha_i(t) - \mu y_{0i}(t).\end{aligned}$$

The characteristic equation is

$$\begin{vmatrix} -1-s & 1\\ (1-\lambda_i)\mu & -\mu-s\end{vmatrix} = 0,$$

which simplifies to

$$s^2 + (1+\mu)s + \lambda_i \mu = 0.$$

Case 2: For $k = 1$,

$$\begin{aligned} \dot{\alpha}_i(t) &= -\alpha_i(t) + (1-\lambda_i)\mu^2 \int_0^\infty \tau e^{-\mu\tau} \alpha_i(t-\tau) d\tau \\ \dot{\alpha}_i(t) + \alpha_i(t) &= (1-\lambda_i)\mu^2 \int_0^\infty \tau e^{-\mu\tau} \alpha_i(t-\tau) d\tau. \end{aligned}$$

Let

$$\begin{aligned} y_{1i}(t) &= (1-\lambda_i)\mu^2 \int_0^\infty \tau e^{-\mu\tau} \alpha_i(t-\tau) d\tau \\ &= (1-\lambda_i)\mu^2 \int_{-\infty}^t (t-\rho) e^{-\mu(t-\rho)} \alpha_i(\rho) d\rho. \end{aligned}$$

Taking derivative of $y_{1i}(t)$ with respect to t,

$$\begin{aligned} \dot{y}_{1i}(t) &= (1-\lambda_i)\mu^2 \int_{-\infty}^t e^{-\mu(t-\rho)} \alpha_i(\rho) d\rho - (1-\lambda_i)\mu^3 \int_{-\infty}^t (t-\rho) e^{-\mu(t-\rho)} \alpha_i(\rho) d\rho \\ &= \mu y_{0i}(t) - \mu y_{1i}(t) \end{aligned}$$

yields to three linear ordinary differential equations

$$\begin{aligned} \dot{\alpha}_i(t) &= -\alpha_i(t) + y_{1i}(t) \\ \dot{y}_{0i}(t) &= (1-\lambda_i)\mu\alpha_i(t) - \mu y_{0i}(t) \\ \dot{y}_{1i}(t) &= \mu y_{0i}(t) - \mu y_{1i}(t). \end{aligned}$$

The characteristic equation is

$$\begin{vmatrix} -1-s & 0 & 1 \\ (1-\lambda_i)\mu & -\mu-s & 0 \\ 0 & \mu & -\mu-s \end{vmatrix} = s^3 + (1+2\mu)s^2 + (\mu^2+2\mu)s + \lambda_i\mu^2 = 0.$$

In the general case, by following the same process above, the system yields $k+1$ additional variables for which the problem can be written as a system of ordinary differential equations. The characteristic equation is

$$\begin{vmatrix} -1-s & 0 & 0 & \cdots & & 0 & 1 \\ (1-\lambda_i)\mu & -\mu-s & 0 & \cdots & & 0 & 0 \\ 0 & \mu & -\mu-s & \ddots & & \vdots & \vdots \\ 0 & 0 & \mu & -\mu-s & & & \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & & -\mu-s & 0 \\ 0 & 0 & \cdots & 0 & & \mu & -\mu-s \end{vmatrix} = 0,$$

which is polynomial equation of order $k+2$:

$$(-(s+\mu))^k(s^2+(\mu+1)s+\mu)+(1-\lambda_i)(-\mu)^{k+1}=0. \tag{4.30}$$

4.5.2 The characteristic equation

The general characteristic equation for the consensus problem under information transmission delays with Gamma distribution in equation (4.25) is

$$(-(s+\mu))^k(s^2+(\mu+1)s+\mu)+(1-\lambda_i)(-\mu)^{k+1}=0.$$

For $k=0$,

$$f_0(\tau)=\mu e^{-\mu\tau}$$

which is the exponential distribution. The mean and the variance of the exponential distribution are

$$\bar{\tau}=\frac{1}{\mu} \tag{4.31}$$

and

$$\mathrm{Var}(\tau)=\frac{1}{\mu^2}, \tag{4.32}$$

respectively. The characteristic equation is

$$s^2+(1+\mu)s+\lambda_i\mu=0. \tag{4.33}$$

The roots s of (4.33) can be calculated explicitly by the quadratic formula. Here we consider undirected graph where λ_i is real, we have

$$s=\frac{-(1+\mu)\pm\sqrt{(1+\mu)^2-4\lambda_i\mu}}{2}. \tag{4.34}$$

If the discriminant in (4.34) is zero we will get negative real roots since $\mu > 0$. This happens when

$$
\begin{aligned}
(1+\mu)^2 - 4\lambda_i\mu &= 0, \\
\lambda_i = \frac{(1+\mu)^2}{4\mu} &= \frac{(1+\bar{\tau})^2}{4\bar{\tau}}.
\end{aligned}
$$

If the discriminant in (4.34) is greater than zero we will obtain distinct negative real roots, which happens when

$$
\begin{aligned}
(1+\mu)^2 - 4\lambda_i\mu &> 0, \\
\lambda_i < \frac{(1+\mu)^2}{4\mu} &= \frac{(1+\bar{\tau})^2}{4\bar{\tau}}.
\end{aligned}
$$

If the discriminant in (4.34) is less than zero we will obtain imaginary roots with negative real parts, which happens when

$$
\begin{aligned}
(1+\mu)^2 - 4\lambda_i\mu &< 0 \\
\lambda_i > \frac{(1+\mu)^2}{4\mu} &= \frac{(1+\bar{\tau})^2}{4\bar{\tau}}.
\end{aligned}
$$

Chapter 5

Building networks that converge faster with delays

In this chapter we will investigate the structure of FD-networks, namely networks that reach consensus faster under delays.

Lemma 5.0.1. *The characteristic equation of a normalized Laplacian matrix of a complete graph on n vertices is*

$$(1-\lambda)\begin{vmatrix} 1-\lambda & \frac{-1}{n-1} & \cdots & \frac{-1}{n-1} \\ \frac{-1}{n-1} & 1-\lambda & & \vdots \\ \vdots & & \ddots & \\ \frac{-1}{n-1} & \cdots & & 1-\lambda \end{vmatrix}_{(n-1)\times(n-1)} + \begin{vmatrix} 1-\lambda & \frac{-1}{n-1} & \cdots & \frac{-1}{n-1} \\ \frac{-1}{n-1} & 1-\lambda & & \vdots \\ \vdots & & \ddots & \\ \frac{-1}{n-1} & \cdots & & 1-\lambda \end{vmatrix}_{(n-1)\times(n-1)}$$

$$= \lambda\left(\lambda - \frac{n}{n-1}\right)^{n-1} = 0.$$

This is a well-known result, but we give an elementary proof that also illustrates the kind of calculations we will be using later.

Proof. For a complete graph of size n, the adjacency matrix and the normalized

Laplacian are

$$A=\begin{bmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 1 \\ \vdots & & \ddots & & \vdots \\ & & & & 1 \\ 1 & \cdots & & 1 & 0 \end{bmatrix}_{n\times n} \quad \text{and } \mathcal{L}=\begin{bmatrix} 1 & \frac{-1}{n-1} & \frac{-1}{n-1} & \cdots & \frac{-1}{n-1} \\ \frac{-1}{n-1} & 1 & \frac{-1}{n-1} & \cdots & \frac{-1}{n-1} \\ \vdots & & & \ddots & \\ & & & & \frac{-1}{n-1} \\ \frac{-1}{n-1} & \cdots & & \frac{-1}{n-1} & 1 \end{bmatrix}_{n\times n}.$$

The characteristic equation for the eigenvalues is

$$\begin{vmatrix} 1-\lambda & \frac{-1}{n-1} & \frac{-1}{n-1} & \cdots & \frac{-1}{n-1} & \frac{-1}{n-1} \\ \frac{-1}{n-1} & 1-\lambda & \frac{-1}{n-1} & \cdots & \frac{-1}{n-1} & \frac{-1}{n-1} \\ \frac{-1}{n-1} & \frac{-1}{n-1} & 1-\lambda & & \vdots & \vdots \\ \vdots & \vdots & & \ddots & \frac{-1}{n-1} & \vdots \\ \vdots & \vdots & & & 1-\lambda & \frac{-1}{n-1} \\ \frac{-1}{n-1} & \frac{-1}{n-1} & \cdots & & \frac{-1}{n-1} & 1-\lambda \end{vmatrix}_{n\times n}=0.$$

By cofactor expansion along the first row, we have

$$(1-\lambda)\begin{vmatrix} 1-\lambda & \frac{-1}{n-1} & \cdots & \frac{-1}{n-1} & \frac{-1}{n-1} \\ \frac{-1}{n-1} & 1-\lambda & & \vdots & \vdots \\ \vdots & & \ddots & \frac{-1}{n-1} & \vdots \\ \vdots & & & 1-\lambda & \frac{-1}{n-1} \\ \frac{-1}{n-1} & \cdots & & \frac{-1}{n-1} & 1-\lambda \end{vmatrix}+\frac{1}{n-1}\begin{vmatrix} \frac{-1}{n-1} & \frac{-1}{n-1} & \cdots & \frac{-1}{n-1} & \frac{-1}{n-1} \\ \frac{-1}{n-1} & 1-\lambda & & \vdots & \vdots \\ \vdots & & \ddots & \frac{-1}{n-1} & \vdots \\ \vdots & & & 1-\lambda & \frac{-1}{n-1} \\ \frac{-1}{n-1} & \cdots & & \frac{-1}{n-1} & 1-\lambda \end{vmatrix}+$$

$$\frac{-1}{n-1}\begin{vmatrix} \frac{-1}{n-1} & 1-\lambda & \cdots & \frac{-1}{n-1} & \frac{-1}{n-1} \\ \frac{-1}{n-1} & \frac{-1}{n-1} & & \vdots & \vdots \\ \vdots & \vdots & \ddots & \frac{-1}{n-1} & \vdots \\ \vdots & \vdots & & 1-\lambda & \frac{-1}{n-1} \\ \frac{-1}{n-1} & \frac{-1}{n-1} & \cdots & \frac{-1}{n-1} & 1-\lambda \end{vmatrix}+\cdots+(-1)^{n+1}\frac{1}{n-1}\begin{vmatrix} \frac{-1}{n-1} & 1-\lambda & \frac{-1}{n-1} & \cdots & \frac{-1}{n-1} \\ \frac{-1}{n-1} & \frac{-1}{n-1} & 1-\lambda & & \vdots \\ \vdots & \vdots & & \ddots & \vdots \\ \vdots & \vdots & & & \frac{-1}{n-1} \\ \frac{-1}{n-1} & \frac{-1}{n-1} & & \cdots & 1-\lambda \end{vmatrix}$$

$$+(-1)^{n}\frac{1}{n-1}\begin{vmatrix} \frac{-1}{n-1} & 1-\lambda & \frac{-1}{n-1} & \cdots & \frac{-1}{n-1} \\ \frac{-1}{n-1} & \frac{-1}{n-1} & 1-\lambda & & \vdots \\ \vdots & \vdots & & \ddots & \frac{-1}{n-1} \\ \vdots & \vdots & & & 1-\lambda \\ \frac{-1}{n-1} & \frac{-1}{n-1} & \cdots & & \frac{-1}{n-1} \end{vmatrix}=0.$$

Interchanging various rows in the third and subsequent cofactors of the above cofactor expansions, so that it will have the same rows and columns entries as that of the second cofator above. We obtain the following expansion

$$(1-\lambda)\begin{vmatrix} 1-\lambda & \frac{-1}{n-1} & \cdots & \frac{-1}{n-1} & \frac{-1}{n-1} \\ \frac{-1}{n-1} & 1-\lambda & & \vdots & \vdots \\ \vdots & & \ddots & \frac{-1}{n-1} & \vdots \\ \vdots & & & 1-\lambda & \frac{-1}{n-1} \\ \frac{-1}{n-1} & \cdots & & \frac{-1}{n-1} & 1-\lambda \end{vmatrix} + \frac{1}{n-1}\begin{vmatrix} \frac{-1}{n-1} & \frac{-1}{n-1} & \cdots & \frac{-1}{n-1} & \frac{-1}{n-1} \\ \frac{-1}{n-1} & 1-\lambda & & \vdots & \vdots \\ \vdots & & \ddots & \frac{-1}{n-1} & \vdots \\ \vdots & & & 1-\lambda & \frac{-1}{n-1} \\ \frac{-1}{n-1} & \cdots & & \frac{-1}{n-1} & 1-\lambda \end{vmatrix}$$

$$+\frac{1}{n-1}\begin{vmatrix} \frac{-1}{n-1} & \frac{-1}{n-1} & \cdots & \frac{-1}{n-1} & \frac{-1}{n-1} \\ \frac{-1}{n-1} & 1-\lambda & & \vdots & \vdots \\ \vdots & & \ddots & \frac{-1}{n-1} & \vdots \\ \vdots & & & 1-\lambda & \frac{-1}{n-1} \\ \frac{-1}{n-1} & \cdots & & \frac{-1}{n-1} & 1-\lambda \end{vmatrix} + \cdots + \frac{1}{n-1}\begin{vmatrix} \frac{-1}{n-1} & \frac{-1}{n-1} & \cdots & \frac{-1}{n-1} & \frac{-1}{n-1} \\ \frac{-1}{n-1} & 1-\lambda & & \vdots & \vdots \\ \vdots & & \ddots & \frac{-1}{n-1} & \vdots \\ \vdots & & & 1-\lambda & \frac{-1}{n-1} \\ \frac{-1}{n-1} & \cdots & & \frac{-1}{n-1} & 1-\lambda \end{vmatrix}$$

$$+\frac{1}{n-1}\begin{vmatrix} \frac{-1}{n-1} & \frac{-1}{n-1} & \cdots & \frac{-1}{n-1} & \frac{-1}{n-1} \\ \frac{-1}{n-1} & 1-\lambda & & \vdots & \vdots \\ \vdots & & \ddots & \frac{-1}{n-1} & \vdots \\ \vdots & & & 1-\lambda & \frac{-1}{n-1} \\ \frac{-1}{n-1} & \cdots & & \frac{-1}{n-1} & 1-\lambda \end{vmatrix} = 0.$$

There are $(n-1)$ multiples of

$$\frac{1}{n-1}\begin{vmatrix} \frac{-1}{n-1} & \frac{-1}{n-1} & \cdots & \frac{-1}{n-1} & \frac{-1}{n-1} \\ \frac{-1}{n-1} & 1-\lambda & & \vdots & \vdots \\ \vdots & & \ddots & \frac{-1}{n-1} & \vdots \\ \vdots & & & 1-\lambda & \frac{-1}{n-1} \\ \frac{-1}{n-1} & \cdots & & \frac{-1}{n-1} & 1-\lambda \end{vmatrix}.$$

Hence

$$(1-\lambda)\begin{vmatrix} 1-\lambda & \frac{-1}{n-1} & \cdots & \frac{-1}{n-1} & \frac{-1}{n-1} \\ \frac{-1}{n-1} & 1-\lambda & & \vdots & \vdots \\ \vdots & & \ddots & \frac{-1}{n-1} & \vdots \\ \vdots & & & 1-\lambda & \frac{-1}{n-1} \\ \frac{-1}{n-1} & \cdots & & \frac{-1}{n-1} & 1-\lambda \end{vmatrix}_{(n-1)\times(n-1)} + \begin{vmatrix} \frac{-1}{n-1} & \frac{-1}{n-1} & \cdots & \frac{-1}{n-1} & \frac{-1}{n-1} \\ \frac{-1}{n-1} & 1-\lambda & & \vdots & \vdots \\ \vdots & & \ddots & \frac{-1}{n-1} & \vdots \\ \vdots & & & 1-\lambda & \frac{-1}{n-1} \\ \frac{-1}{n-1} & \cdots & & \frac{-1}{n-1} & 1-\lambda \end{vmatrix}_{(n-1)\times(n-1)}$$

$$= \lambda\left(\lambda - \frac{n}{n-1}\right)^{n-1} = 0.$$

□

5.1 Graph Operations

We consider various ways of creating new graphs from existing ones. The new graph formed could be larger or smaller than the existing one. Our aim is to predict the spectrum of the newly formed graph from the known spectrum of the original graphs. Larger networks can be formed from smaller ones with known spectrum. We will see how we can build FD-graphs from existing FD-graphs using various graph operations. Graph operations we shall be discussing are the Cartesian product, join, H-join and adding and removing edges. From the literature, Atay and Bıyıkoğlu [11] have studied the Cartesian product and join of undirected graphs, they obtained the results for non-normalized Laplacian matrix in Propositions 5.1.2 and 5.1.4, respectively. We will study these graph operations for both undirected and directed graphs.

5.1.1 Cartesian Product

Definition 5.1.1. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two nonempty graphs. The Cartesian product $G_1 \square G_2$ is a graph with vertex set $V_2 \times V_2$, and $(x_1, x_2)(y_1, y_2)$ is an edge in $E(G_1 \Box G_2)$ if and only if either $x_2 = y_2$ and $x_1y_1 \in E(G_1)$ or if $x_1 = y_1$ and $x_2y_2 \in E(G_2)$. An example is given in Figure 5.1.

Proposition 5.1.2 (Atay and Bıyıkoğlu, 2005)**.** *Let G_1 and G_2 be undirected graphs. The eigenvalues of the Laplacian Λ for the cartesian product $G_1 \Box G_2$ satisfy*

$$\Lambda_{\min}(G_1 \Box G_2) = \min\{\Lambda_{\min}(G_1), \Lambda_{\min}(G_2)\},$$

$$\Lambda_{\max}(G_1 \Box G_2) = \Lambda_{\max}(G_1) + \Lambda_{\max}(G_2),$$

$$\frac{\Lambda_{\min}(G_1 \Box G_2)}{\Lambda_{\max}(G_1 \Box G_2)} < \min\left\{\frac{\Lambda_{\min}(G_1)}{\Lambda_{\max}(G_1)}, \frac{\Lambda_{\min}(G_2)}{\Lambda_{\max}(G_2)}\right\}.$$

Proof. We refer to [11] for the proof. $\square$

Let us consider the Cartesian product depicted in Figure 5.1, the two graphs (K_2) and the directed cycle of size 3 used in the Cartesian products are both FD-graphs but the resulting graph is not.

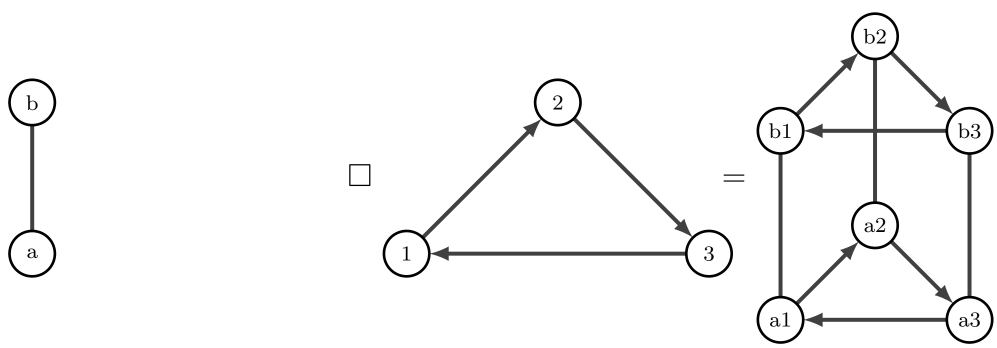

Figure 5.1: The Cartesian product of K_2 and directed cycle of size 3. The spectrum of the resulting graph is $(0, 0.7500 + 0.4330i, 0.7500 - 0.4330i, 1, 1.7500 - 0.4330i, 1.7500 - 0.4330i)$.

We conclude that the Cartesian product is not helpful in building FD-graphs.

5.1.2 Join

Definition 5.1.3. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs on disjoint sets of n and m vertices, respectively. Their disjoint union $G_1 + G_2$ is the graph $G_1 + G_2 = (V_1 \cup V_2, E_1 \cup E_2)$, and their join $G_1 * G_2$ is the graph on $r = n + m$ vertices obtained from $G_1 + G_2$ by inserting undirected edges from each vertex of G_1 to each vertex of G_2. An example is given in Figure 5.2.

Proposition 5.1.4 (Atay and Bıyıkoğlu, 2005)**.** *Let G and H be undirected graphs on n and m vertices, respectively. Then the eigenvalues of the Laplacian Λ for the join $G * H$ satisfy*

$$\begin{aligned}\Lambda_{\min}(G * H) &= \min\{\Lambda_{\min}(G) + m, \Lambda_{\min}(H) + n\}\\ &\geq \Lambda_{\min}(G) + \Lambda_{\min}(H).\end{aligned}$$

and

$$\Lambda_{\max}(G * H) = m + n.$$

If G and H have the same number of vertices, then

$$\frac{\Lambda_{\min}(G * H)}{\Lambda_{\max}(G * H)} > \frac{1}{2}.$$

Proof. We refer to [11] for the proof. □

Conjecture 5.1.5. *Let G_1 and G_2 be FD-graphs on n_1 and n_2 vertices, respectively. Then the eigenvalues of the normalized Laplacian for the join $G_1 * G_2$ satisfy the FD-graph condition.*

Generation of the spectrum of the normalized Laplacian for the join of directed graphs is very complex, thus making the proof of the Conjecture 5.1.5 difficult. However, some special directed graphs with simple spectrum will be analyzed in the subsequent subsections. For the case of undirected graphs we obtain the result in Preposition 5.1.6.

Proposition 5.1.6. *Let G_1 and G_2 be undirected graphs on n_1 and n_2 vertices, respectively. If G_1 and G_2 are FD-graphs, then the join $G_1 * G_2$ is an FD-graph.*

Proof. It is a straightforward proof by Theorem 4.3.4, if G_1 and G_2 are FD-graphs, then they are complete graphs. By the definition of join operation, $G_1 * G_2$ is a complete graph on $n = n_1 + n_2$ vertices, which completes the proof. $\square$

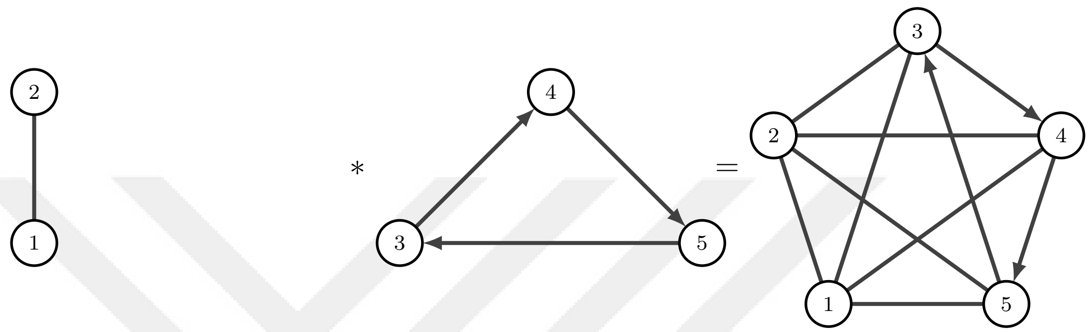

Figure 5.2: The join of K_2 and directed cycle of size 3 The resulting graph is an FD-graph with the spectrum $(0, 1.1667 + 0.2887i, 1.1667 - 0.2887i, 1.25, 1.4167)$.

5.1.3 Spectra of H-Join of Regular Graphs

In this section we look at the generalized form of join operation known as H-join operation on regular graphs [33]. The spectra of the normalized Laplacian under this operation will be discussed.

Definition 5.1.7. Let H be a graph with $V(H) = \{1, 2, \ldots, k\}$ and G_i be disjoint graphs of order n_i, $i = 1, 2, \ldots, k$. Then the H-join is the graph $V_H\{G_1, G_2, \ldots, G_k\}$ is formed by taking the graphs $G_1, G_2, \ldots, G_k$ and joining every vertex of G_i to every vertex of G_j whenever i is adjacent to j in H.

For a graph of H of order k, in the same manner, we consider the H-join graph $V_H\{G_1, G_2, \ldots, G_k\}$, where G_i is a r_i-regular graph of order $n_i (i = 1, 2, \ldots, k)$. We define $N_i = \sum_{j \in N_H(i)} n_j$ for each $i \in V(H)$ and a new matrix as follows

$$C_{\mathcal{L}}(H) = (c_{ij})_{k\times k} \quad \text{with} \quad c_{ij} = \begin{cases} \dfrac{N_i}{r_i + N_i}, & i = j \quad \text{and} \quad d_H(i) \neq 0, \\ -\sqrt{\dfrac{n_i n_j}{(r_i + N_i)(r_j + N_j)}}, & ij \in E(H), \\ 0 & \text{otherwise.} \end{cases}$$

Let $M = M(G)$ be a graph matrix associated with a graph G. For a given square matrix M, the M-polynomial is defined as $P_M(G,x) = \det(x\mathbb{I} - M)$ where $\mathbb{I}$ is the identity matrix. The H-join operation is trivial when H is trivial. In this case $C_{\mathcal{L}}(H) = 0$. The matrix $C_{\mathcal{L}}(H)$ depends only on the graph H and the orders and degrees, not the structures, of regular graphs.

Theorem 5.1.8 (Wu, Lou and He, 2014)**.** *Let H be a graph with no isolated vertices and $V(H) = \{1, 2, \ldots, k\}$, and let G_i be r_i-regular graphs of order n_i, $i = 1, 2, \ldots, k$. If $G = V_H\{G_1, G_2, \ldots, G_k\}$, then*

$$\mathrm{Spec}_{\mathcal{L}}(G) = \left(\bigcup_{i=1}^{k}\left(\frac{N_i}{r_i+N_i} + \frac{r_i}{r_i+N_i}\left(\mathrm{Spec}_{\mathcal{L}}(G_i)\setminus\{0\}\right)\right)\right) \cup \mathrm{Spec}(C_{\mathcal{L}}(H)),$$

i.e.,

$$P_{\mathcal{L}}(G,x) = P_{\mathcal{L}}(C_{\mathcal{L}}(H),x)\prod_{i=1}^{k}\frac{P_{\mathcal{L}}\left(G_i, g_i\left(x - \frac{N_i}{r_i+N_i}\right)\right)}{g_i^{n_i}\left(x - \frac{N_i}{r_i+N_i}\right)},$$

where $g_i = \dfrac{r_i+N_i}{r_i}$ *if* $r_i \neq 0$ *and* $g_i = 1$ *otherwise* $(i = 1, 2, \ldots, k)$.

Proof. We refer to [33] for the proof. □

Corollary 5.1.9 (Wu, Lou and He, 2014)**.** *Let G_i be a r_i-regular graph of order n_i $(i = 1, 2)$. Then the $\mathcal{L}$-polynomial of $G_1 * G_2$ is*

$$P_{\mathcal{L}}(G_1 * G_2, x) = \frac{P_{\mathcal{L}}\left(G_1, g_1\left(x - \frac{n_2}{r_1+n_2}\right)\right)P_{\mathcal{L}}\left(G_2, g_2\left(x - \frac{n_1}{r_2+n_1}\right)\right)}{g_1^{n_1}g_2^{n_2}\left(x - \frac{n_2}{r_1+n_2}\right)\left(x - \frac{n_1}{r_2+n_1}\right)}f(x),$$

where $f(x) = x\left(x - \dfrac{n_2}{r_1+n_2} - \dfrac{n_1}{r_2+n_1}\right)$, $g_i = 1$ *if* $r_i = 0$ $(i = 1, 2)$ *and* $g_1 = \dfrac{r_1+n_2}{r_1}$, $g_2 = \dfrac{r_2+n_1}{r_2}$ *otherwise.*

Proof. We refer to [33] for the proof. □

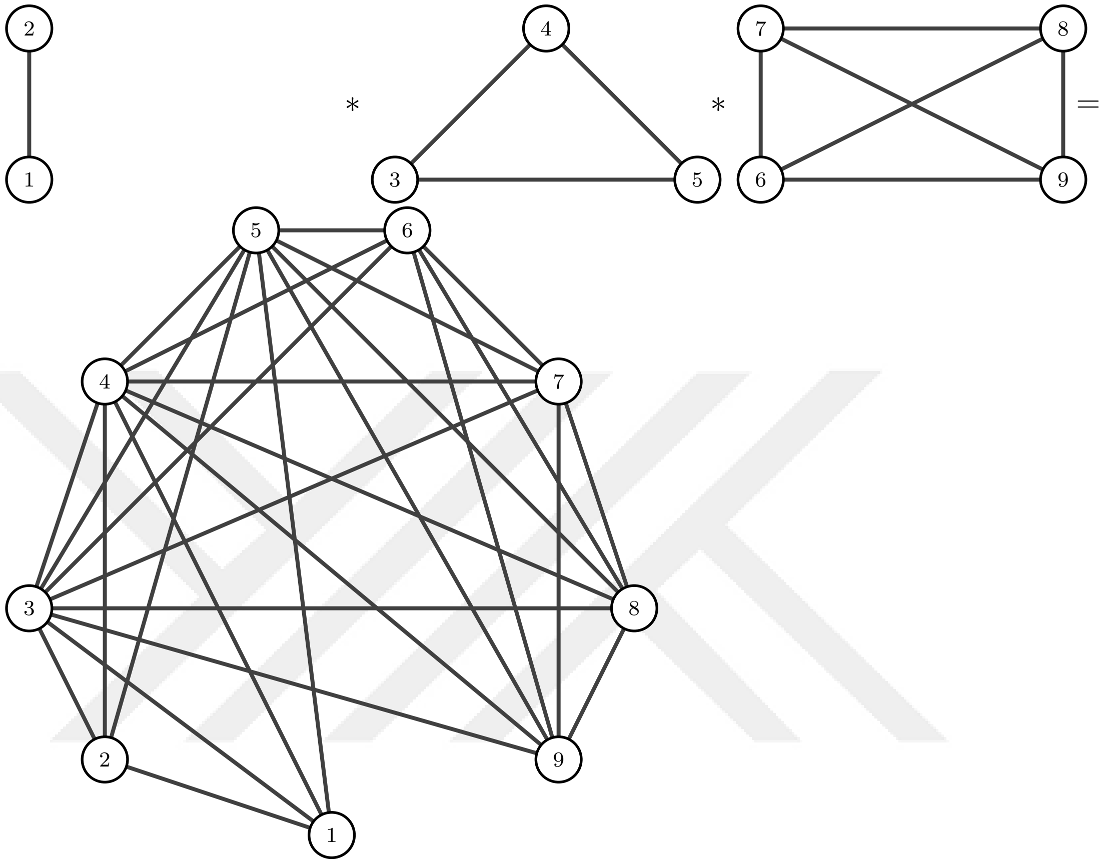

Figure 5.3: H-Join of $\{K_2, K_3, K_4\}$.

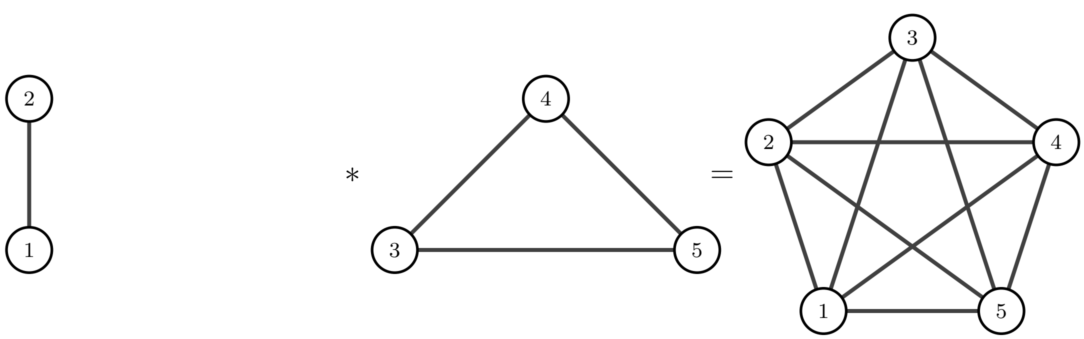

Figure 5.4: H-Join of $\{K_2, K_3\}$.

Example 5.1.10. *Consider the H-join depicted in Figure 5.3. We calculate the spectrum of the graph G. We note that:*

$$\left(\frac{N_1}{r_1+N_1}, \frac{N_2}{r_2+N_2}, \frac{N_3}{r_3+N_3}\right) = \left(\frac{3}{4}, \frac{3}{4}, \frac{1}{2}\right)$$

$$\left(\frac{r_1}{r_1+N_1},\frac{r_2}{r_2+N_2},\frac{r_3}{r_3+N_3}\right)=\left(\frac{1}{4},\frac{1}{4},\frac{1}{2}\right)$$

$$\frac{n_1n_2}{(r_1+N_1)(r_2+N_2)}=\frac{3}{16}$$

$$\frac{n_2n_3}{(r_2+N_2)(r_3+N_3)}=\frac{1}{4}$$

$$C_{\mathcal{L}}(H)=\begin{bmatrix}\frac{3}{4} & -\frac{\sqrt{3}}{4} & 0\\ -\frac{\sqrt{3}}{4} & \frac{3}{4} & -\frac{1}{2}\\ 0 & -\frac{1}{2} & \frac{1}{2}\end{bmatrix}$$

$$\text{Spec}_{\mathcal{L}}(G_1)=\{0,2\}$$

$$\text{Spec}_{\mathcal{L}}(G_2)=\left\{0,\frac{3}{2},\frac{3}{2}\right\}$$

$$\text{Spec}_{\mathcal{L}}(G_3)=\left\{0,\frac{4}{3},\frac{4}{3},\frac{4}{3}\right\}$$

$$\text{Spec}(C_{\mathcal{L}}(H))=\{0,0.6464,1.3536\}$$

$$\text{Spec}_{\mathcal{L}}(G)=\left(\frac{3}{4}+\frac{1}{4}\{2\}\right)\cup\left(\frac{3}{4}+\frac{1}{4}\left\{\frac{3}{2},\frac{3}{2}\right\}\right)\cup\left(\frac{1}{2}+\frac{1}{2}\left\{\frac{4}{3},\frac{4}{3},\frac{4}{3}\right\}\right)\cup\{0,0.6464,1.3536\}$$

$$=\{1.250\}\cup\{1.1250,1.1250\}\cup\{1.1667,1.1667,1.1667\}\cup\{0,0.6464,1.3536\}$$

$$=\{0,0.6464,1.2500,1.1250,1.1250,1.1667,1.1667,1.1667,1.3536\}$$

The H-join in Example 5.1.10 shows that with the H join, we may not necessarily obtain a FD-graph from joining three or more FD-graphs. As seen, we have joined three FD-graphs K_2, K_3 and K_4 in which all of them having a spectral gap greater than one ($\lambda_2 > 1$) but the H-join of the three graphs produce a graph with spectral gap less than one ($\lambda_2 = 0.6464$) so not an FD-graph. We can only obtain an FD-graph from the H-join operation if and only if we join two FD-graphs. We see that in 5.1.11.

Example 5.1.11. *Consider the H-join depicted in 5.4. We calculate the spectrum of the graph G. We note that*

$$\left(\frac{N_1}{r_1+N_1},\frac{N_2}{r_2+N_2}\right)=\left(\frac{1}{2},\frac{3}{4}\right)$$

$$\left(\frac{r_1}{r_1+N_1}, \frac{r_2}{r_2+N_2}\right) = \left(\frac{1}{2}, \frac{1}{4}\right)$$

$$\frac{n_1 n_2}{(r_1+N_1)(r_2+N_2)} = \frac{3}{8}$$

$$C_L(H) = \begin{bmatrix} \frac{3}{4} & -\frac{\sqrt{6}}{4} \\ -\frac{\sqrt{6}}{4} & \frac{1}{2} \end{bmatrix}$$

$$\mathrm{Spec}_L(G_1) = \{0, 2\}$$

$$\mathrm{Spec}_L(G_2) = \left\{0, \frac{3}{2}, \frac{3}{2}\right\}$$

$$\mathrm{Spec}(C_L(H)) = \{0, 1.2500\}$$

$$\begin{aligned} \mathrm{Spec}_L(G) &= \left(\frac{3}{4} + \frac{1}{4}\{2\}\right) \cup \left(\frac{1}{2} + \frac{1}{2}\left\{\frac{3}{2}, \frac{3}{2}\right\}\right) \cup \{0, 1.2500\} \\ &= \{1.2500\} \cup \{1.2500, 1.2500\} \cup \{0, 1.2500\} \\ &= \{0, 1.2500, 1.2500, 1.2500, 1.2500\}. \end{aligned}$$

5.1.4 Join of directed graphs

Definition 5.1.12. Let K_n be the complete graph on n vertices. Let H be a new graph formed by removing a directed cycle C_n from K_n in the order $i \longrightarrow (i+1) \longrightarrow \cdots n \longrightarrow i$ starting from some vertex i. We denote the resulting graph as $H = K_n - C_n$. We also call such graphs K-C graphs, for "complete minus cycle".

The graph in Figure 2.3 shows $K_5 - C_5$.

Conjecture 5.1.13. *Let G_1 and G_2 be K-C graphs on n_1 and n_2 vertices, respectively, where n_1 and n_2 are both odd. If the eigenvalues of the normalized Laplacian of G_1 and G_2 satisfy the FD-graph condition $\left(\mathrm{Re}\left(\frac{\partial s}{\partial \tau}\right)\Big|_{\tau=0} < 0\right)$, then the join $G_1 * G_2$ of size $n = n_1 + n_2$ also satisfies the FD-graph condition.*

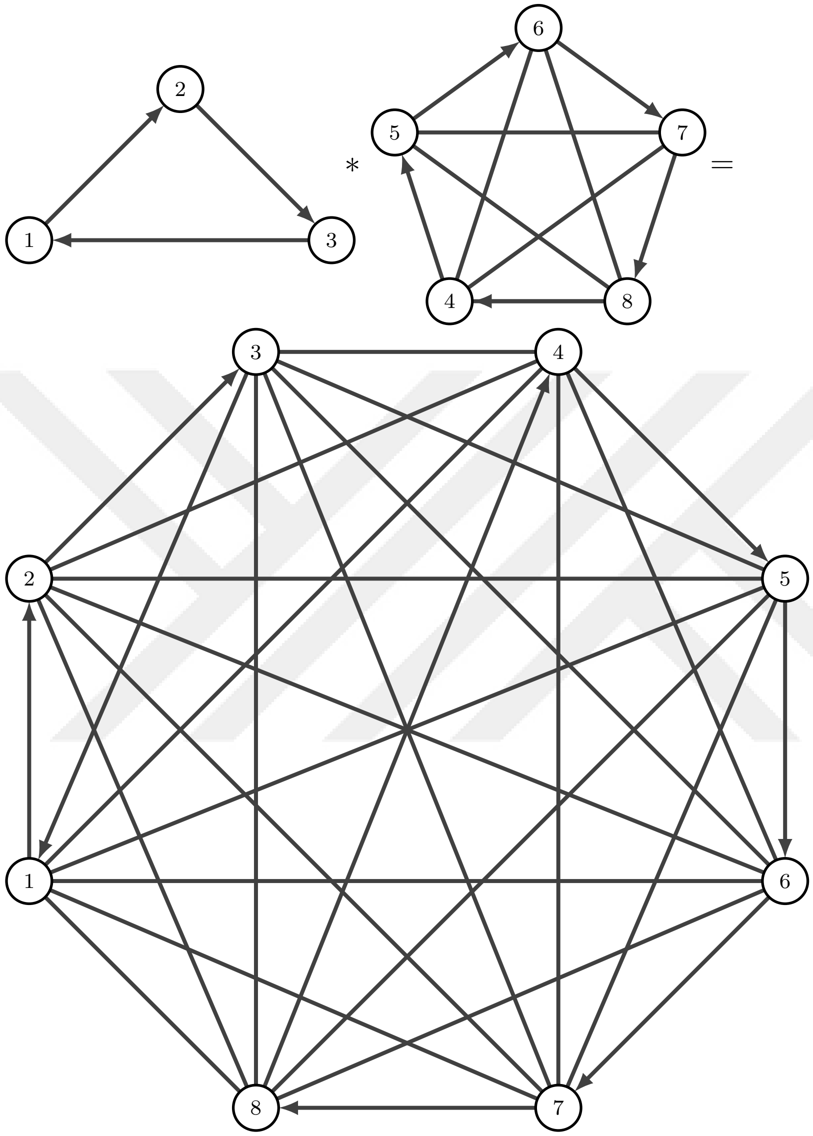

Figure 5.5: $G = G_1 * G_2$ the join of G_1 and G_2

Example 5.1.14. *Consider the join graph $G_1 * G_2$ depicted in Figure 5.5 The eigenvalues of the normalized Laplacian corresponding to G_1 and G_2 are*

$$\{0, 1.5 + 0.8660i, 1.5 - 0.8660i\}$$

and

$$\{0, 1.4363 + 0.3170i, 1.4363 - 0.3170i, 1.0637 + 0.1959i, 1.0637 - 0.1959i\},$$

respectively. The eigenvalues of the normalized Laplacian corresponding to the graphs G_1 and G_2 satisfy the FD-graph condition; so will the join $G_1 *$

G_2 also satisfy the FD-graph condition? The eigenvalues of the normalized Laplacian corresponding to the join $G_1 * G_2$ are $0, 1.0833 + 0.1443i, 1.0833 - 0.1443i, 1.3333, 1.2182 + 0.1585i, 1.2182 - 0.1585i, 1.0318 + 0.0980i, 1.0318 - 0.0980i$, *and by checking the FD-graph condition, all these eigenvalues satisfy the FD-graph condition so $G = G_1 * G_2$ is an FD-graph.*

5.1.5 Adding and removing edges

Definition 5.1.15. If $G = (V, E)$ is a graph, then $G - e$ denotes the graph obtained from G by removing the edge $e \in E(G)$. If $e \notin (G)$, then $G + e$ is the graph obtained from G by adding an edge e.

Adding or removing an edge from a graph can alter the spectrum of the new graph formed. Interestingly, for certain graphs the spectrum remains the same after removing some specific edge(s) from the graph.

Theorem 5.1.16. *Let $G = K_n$ be a complete graph of n vertices and let $H = G - e_{ij}$ be the graph formed by removing a single directed edge from vertex j to vertex i. Then the spectrum of the normalized Laplacian of graph H is the same as the complete graph G.*

Proof. The complete graph $G = K_n$ has the adjacency matrix and a normalized Laplacian

$$A_G = \begin{bmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 1 \\ \vdots & & \ddots & & \vdots \\ 1 & \cdots & & & 0 \end{bmatrix} \text{ and } \mathcal{L}_G = \begin{bmatrix} 1 & \frac{-1}{n-1} & \cdots & \frac{-1}{n-1} \\ \frac{-1}{n-1} & 1 & & \vdots \\ \vdots & & \ddots & \\ \frac{-1}{n-1} & \cdots & & 1 \end{bmatrix}_{n\times n},$$

respectively. The characteristic equation of the normalized Laplacian is

$$(1-\lambda)\begin{vmatrix} 1-\lambda & \frac{-1}{n-1} & \cdots & \frac{-1}{n-1} \\ \frac{-1}{n-1} & 1-\lambda & & \vdots \\ \vdots & & \ddots & \\ \frac{-1}{n-1} & \cdots & & 1-\lambda \end{vmatrix}_{(n-1)\times(n-1)} + \begin{vmatrix} 1-\lambda & \frac{-1}{n-1} & \cdots & \frac{-1}{n-1} \\ \frac{-1}{n-1} & 1-\lambda & & \vdots \\ \vdots & & \ddots & \\ \frac{-1}{n-1} & \cdots & & 1-\lambda \end{vmatrix}_{(n-1)\times(n-1)}$$

$$= \lambda\left(\lambda - \frac{n}{n-1}\right)^{n-1} = 0,$$

which is the same as in Lemma 5.0.1. Let us consider removing the edge e_{1n} (that is, an edge from vertex n to vertex 1) from the graph G, then A_H will be the adjacency matrix of the graph H and $\mathcal{L}_H$ be the normalized Laplacian of the graph H.

$$A_H = \begin{bmatrix} 0 & 1 & 1 & \cdots & 1 & 0 \\ 1 & 0 & 1 & \cdots & 1 & 1 \\ \vdots & & \ddots & & & \vdots \\ \vdots & & & & & 1 \\ 1 & \cdots & & & 1 & 0 \end{bmatrix} \text{ and } \mathcal{L}_H = \begin{bmatrix} 1 & \frac{-1}{n-2} & \frac{-1}{n-2} & \cdots & \frac{-1}{n-2} & 0 \\ \frac{-1}{n-1} & 1 & \frac{-1}{n-1} & \cdots & \frac{-1}{n-1} & \frac{-1}{n-1} \\ \vdots & & \ddots & & & \vdots \\ \vdots & & & & & \frac{-1}{n-1} \\ \frac{-1}{n-1} & \cdots & & & \frac{-1}{n-1} & 1 \end{bmatrix}$$

The characteristic equation of the normalized Laplacian is

$$\begin{vmatrix} 1-\lambda & \frac{-1}{n-2} & \frac{-1}{n-2} & \cdots & \frac{-1}{n-2} & 0 \\ \frac{-1}{n-1} & 1-\lambda & \frac{-1}{n-1} & \cdots & \frac{-1}{n-1} & \frac{-1}{n-1} \\ \frac{-1}{n-1} & \frac{-1}{n-1} & 1-\lambda & & \vdots & \vdots \\ \vdots & \vdots & & \ddots & \frac{-1}{n-1} & \vdots \\ \vdots & \vdots & & & 1-\lambda & \frac{-1}{n-1} \\ \frac{-1}{n-1} & \frac{-1}{n-1} & \cdots & & \frac{-1}{n-1} & 1-\lambda \end{vmatrix}_{n\times n} = 0.$$

By cofactor expansion along the first row, we have

$$
(1-\lambda)\begin{vmatrix}
1-\lambda & \tfrac{-1}{n-1} & \cdots & \tfrac{-1}{n-1} & \tfrac{-1}{n-1}\\
\tfrac{-1}{n-1} & 1-\lambda & & \vdots & \vdots\\
\vdots & & \ddots & \tfrac{-1}{n-1} & \vdots\\
\vdots & & & 1-\lambda & \tfrac{-1}{n-1}\\
\tfrac{-1}{n-1} & \cdots & & \tfrac{-1}{n-1} & 1-\lambda
\end{vmatrix}
+\frac{1}{n-2}\begin{vmatrix}
\tfrac{-1}{n-1} & \tfrac{-1}{n-1} & \cdots & \tfrac{-1}{n-1} & \tfrac{-1}{n-1}\\
\tfrac{-1}{n-1} & 1-\lambda & & \vdots & \vdots\\
\vdots & & \ddots & \tfrac{-1}{n-1} & \vdots\\
\vdots & & & 1-\lambda & \tfrac{-1}{n-1}\\
\tfrac{-1}{n-1} & \cdots & & \tfrac{-1}{n-1} & 1-\lambda
\end{vmatrix}
$$

$$
+\frac{-1}{n-2}\begin{vmatrix}
\tfrac{-1}{n-1} & 1-\lambda & \cdots & \tfrac{-1}{n-1} & \tfrac{-1}{n-1}\\
\tfrac{-1}{n-1} & \tfrac{-1}{n-1} & & \vdots & \vdots\\
\vdots & \vdots & \ddots & \tfrac{-1}{n-1} & \vdots\\
\vdots & \vdots & & 1-\lambda & \tfrac{-1}{n-1}\\
\tfrac{-1}{n-1} & \tfrac{-1}{n-1} & \cdots & \tfrac{-1}{n-1} & 1-\lambda
\end{vmatrix}
+\cdots+
$$

$$
(-1)^{n+1}\frac{1}{n-2}\begin{vmatrix}
\tfrac{-1}{n-1} & 1-\lambda & \tfrac{-1}{n-1} & \cdots & \tfrac{-1}{n-1}\\
\tfrac{-1}{n-1} & \tfrac{-1}{n-1} & 1-\lambda & & \vdots\\
\vdots & \vdots & & \ddots & \vdots\\
\vdots & \vdots & & & \tfrac{-1}{n-1}\\
\tfrac{-1}{n-1} & \tfrac{-1}{n-1} & \cdots & & 1-\lambda
\end{vmatrix}=0.
$$

Interchanging various rows in the third and subsequent cofactors of the above cofactor expansions, so that we can have the same rows and columns entries as that of the second cofator. We obtain

$$
(1-\lambda)\begin{vmatrix}
1-\lambda & \tfrac{-1}{n-1} & \cdots & \tfrac{-1}{n-1} & \tfrac{-1}{n-1}\\
\tfrac{-1}{n-1} & 1-\lambda & & \vdots & \vdots\\
\vdots & & \ddots & \tfrac{-1}{n-1} & \vdots\\
\vdots & & & 1-\lambda & \tfrac{-1}{n-1}\\
\tfrac{-1}{n-1} & \cdots & & \tfrac{-1}{n-1} & 1-\lambda
\end{vmatrix}
+\frac{1}{n-2}\begin{vmatrix}
\tfrac{-1}{n-1} & \tfrac{-1}{n-1} & \cdots & \tfrac{-1}{n-1} & \tfrac{-1}{n-1}\\
\tfrac{-1}{n-1} & 1-\lambda & & \vdots & \vdots\\
\vdots & & \ddots & \tfrac{-1}{n-1} & \vdots\\
\vdots & & & 1-\lambda & \tfrac{-1}{n-1}\\
\tfrac{-1}{n-1} & \cdots & & \tfrac{-1}{n-1} & 1-\lambda
\end{vmatrix}+
$$

$$
\frac{1}{n-2}\begin{vmatrix}
\tfrac{-1}{n-2} & \tfrac{-1}{n-1} & \cdots & \tfrac{-1}{n-1} & \tfrac{-1}{n-1}\\
\tfrac{-1}{n-1} & 1-\lambda & & \vdots & \vdots\\
\vdots & & \ddots & \tfrac{-1}{n-1} & \vdots\\
\vdots & & & 1-\lambda & \tfrac{-1}{n-1}\\
\tfrac{-1}{n-1} & \cdots & & \tfrac{-1}{n-1} & 1-\lambda
\end{vmatrix}
+\cdots+\frac{1}{n-2}\begin{vmatrix}
\tfrac{-1}{n-1} & \tfrac{-1}{n-1} & \cdots & \tfrac{-1}{n-1} & \tfrac{-1}{n-1}\\
\tfrac{-1}{n-1} & 1-\lambda & & \vdots & \vdots\\
\vdots & & \ddots & \tfrac{-1}{n-1} & \vdots\\
\vdots & & & 1-\lambda & \tfrac{-1}{n-1}\\
\tfrac{-1}{n-1} & \cdots & & \tfrac{-1}{n-1} & 1-\lambda
\end{vmatrix}=0.
$$

There are $(n-2)$ multiples of

$$\frac{1}{n-2}\begin{vmatrix} \frac{-1}{n-1} & \frac{-1}{n-1} & \cdots & \frac{-1}{n-1} & \frac{-1}{n-1} \\ \frac{-1}{n-1} & 1-\lambda & & \vdots & \vdots \\ \vdots & & \ddots & \frac{-1}{n-1} & \vdots \\ \vdots & & & 1-\lambda & \frac{-1}{n-1} \\ \frac{-1}{n-1} & \cdots & & \frac{-1}{n-1} & 1-\lambda \end{vmatrix}$$

Hence

$$(1-\lambda)\begin{vmatrix} 1-\lambda & \frac{-1}{n-1} & \cdots & \frac{-1}{n-1} & \frac{-1}{n-1} \\ \frac{-1}{n-1} & 1-\lambda & & \vdots & \vdots \\ \vdots & & \ddots & \frac{-1}{n-1} & \vdots \\ \vdots & & & 1-\lambda & \frac{-1}{n-1} \\ \frac{-1}{n-1} & \cdots & & \frac{-1}{n-1} & 1-\lambda \end{vmatrix} + \begin{vmatrix} \frac{-1}{n-1} & \frac{-1}{n-1} & \cdots & \frac{-1}{n-1} & \frac{-1}{n-1} \\ \frac{-1}{n-1} & 1-\lambda & & \vdots & \vdots \\ \vdots & & \ddots & \frac{-1}{n-1} & \vdots \\ \vdots & & & 1-\lambda & \frac{-1}{n-1} \\ \frac{-1}{n-1} & \cdots & & \frac{-1}{n-1} & 1-\lambda \end{vmatrix}$$

$$= \lambda\left(\lambda - \frac{n}{n-1}\right)^{n-1} = 0$$

□

(a) $G = K_5$ (b) $H = G - e_{51}$

Figure 5.6: Both graphs G and H have the same spectrum. The eigenvalues of the normalized Laplacian are $\lambda_1 = 0$ and $\lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = 5/4$.

Conjecture 5.1.17. *Let G_1 be a graph on n_1 vertices formed from the complete graph K_{n_1} by removing a single directed edge. Let G_2 be a graph on n_2 vertices formed from the complete graph K_{n_2} by removing a single directed edge. Let $G = G_1 * G_2$ be the join of G_1 and G_2. Then the eigenvalues of the normalized*

Laplacian of G are $\lambda_1 = 0, \lambda_2 = \frac{n-1}{n-2}, \lambda_3 = \frac{(n-1)(n-2)+(n-3)}{(n-1)(n-2)}$ and $\lambda_4 = \lambda_5 = \cdots = \lambda_n = \frac{n}{n-1}$.

Example 5.1.18. *Consider the join of $K_2 - e_{12}$ and $K_3 - e_{35}$ depicted in Figure 5.7.*

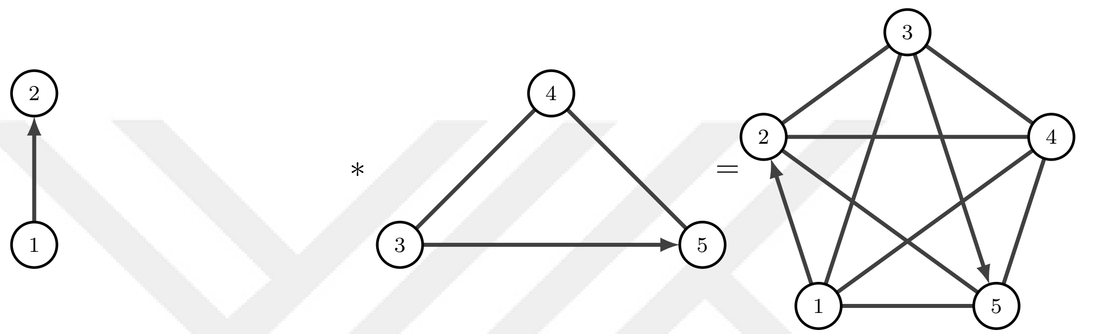

Figure 5.7: The eigenvalues of the normalized Laplacian of graph formed from the join operation are $\lambda_1 = 0, \lambda_2 = 4/3, \lambda_3 = 7/6$ and $\lambda_4 = \lambda_5 = 5/4$.

5.1.6 FD-graphs

We illustrate for different sizes of graphs up to size 6 in the following figures. All the graphs in the figures are FD-graphs, that is they reach consensus faster with the introduction of time delay.

Figure 5.8: For a graph of size 2.

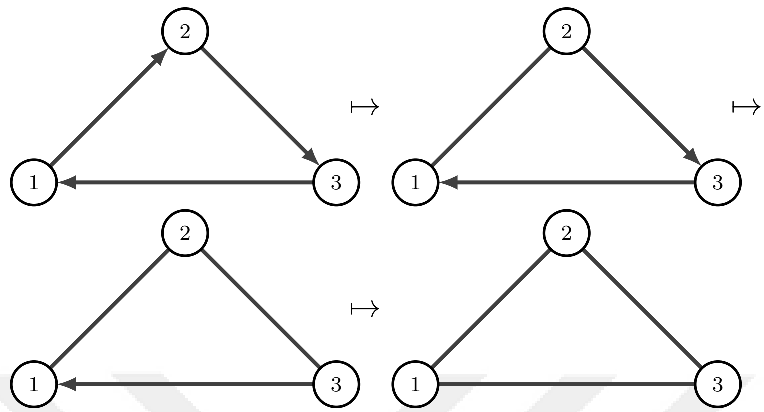

Figure 5.9: For graphs of size 3.

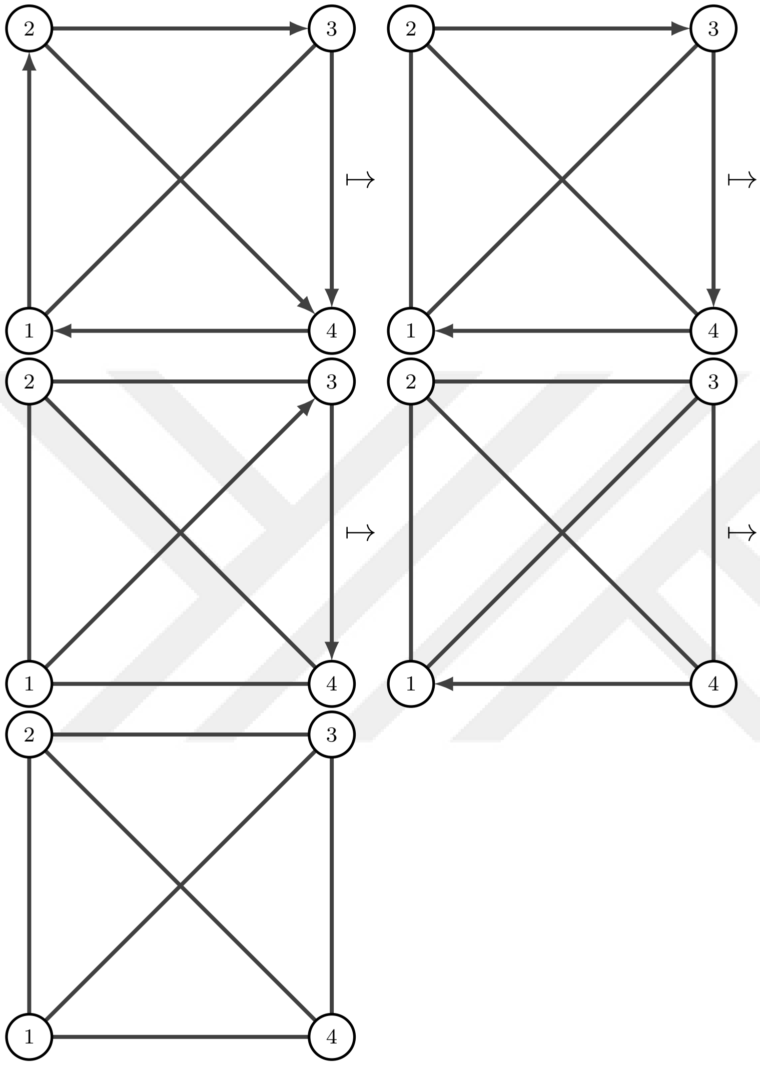

Figure 5.10: For graphs of size 4.

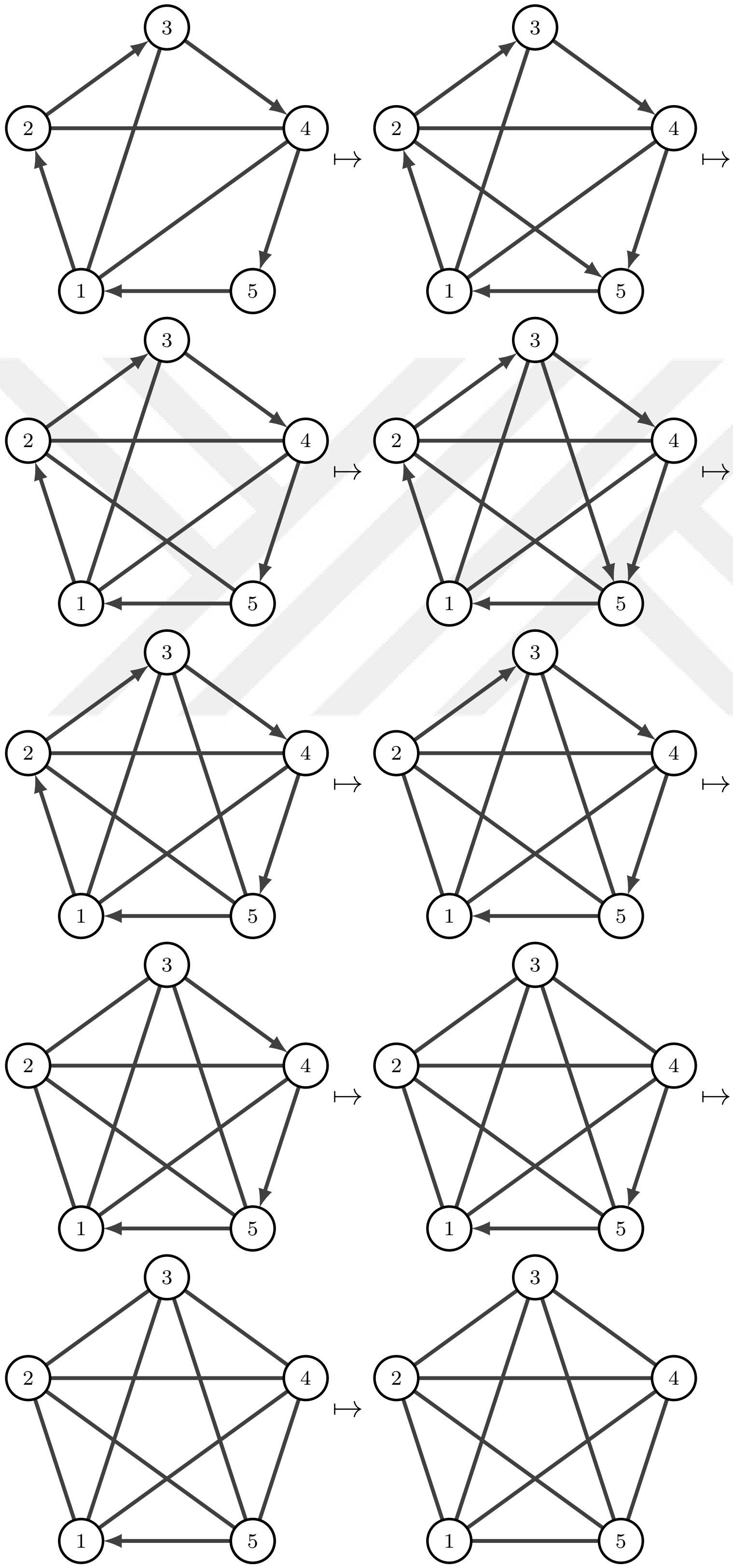

Figure 5.11: For graphs of size 5.

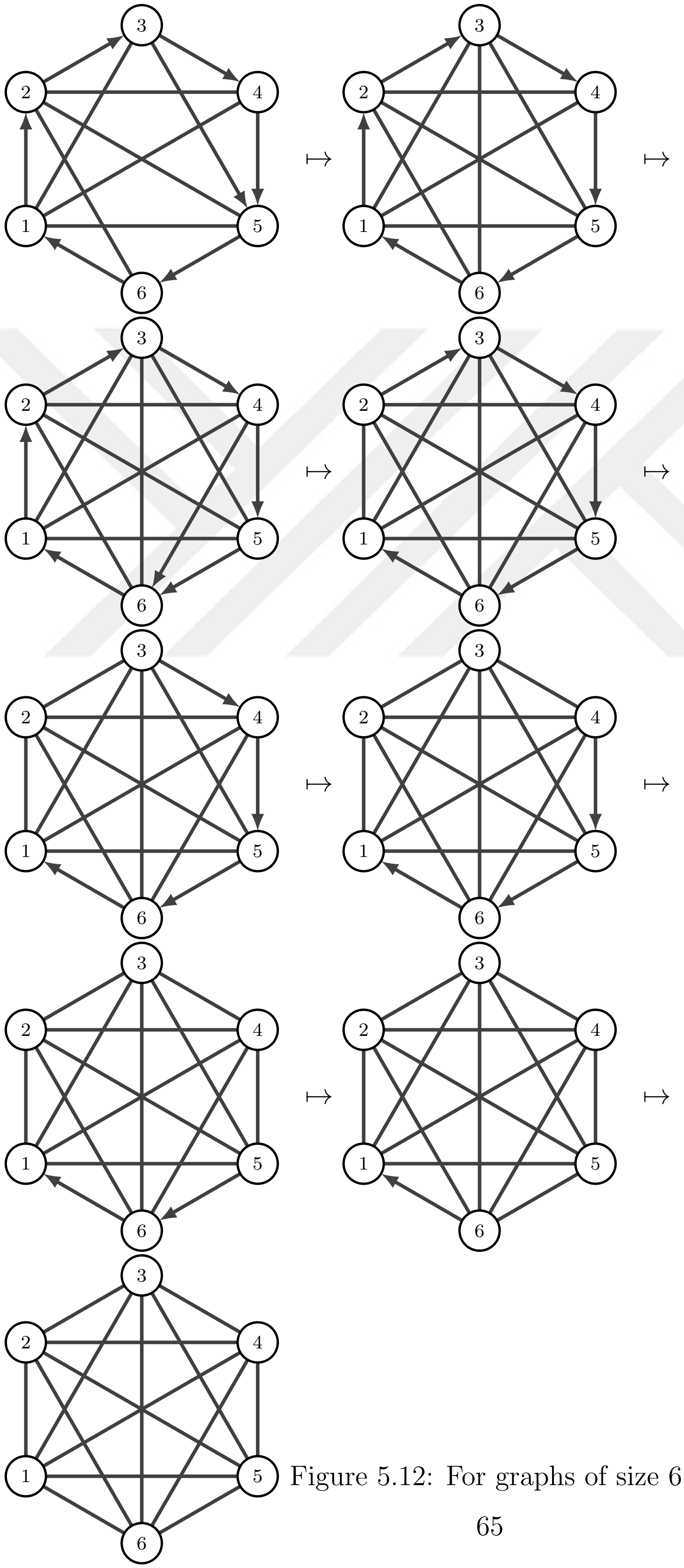

Figure 5.12: For graphs of size 6.

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