HOMOGENEOUS BASES AND THEIR APPLICATIONS

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ABSTRACT

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An algorithm for computation of H-bases for polynomial ideals is developed. The known methods of computations of H-bases use the division algorithm and Gröbner Basis. The suggested algorithm uses only some vector space computations on matrices. Furthermore, it is also shown that how H-bases can be applied to some Numerical Analysis problems.

Keywords: H-basis, Gröbner basis, Syzygy Module, Hilbert function, Quasi row reduced form.

ÖZET

HOMOJEN TABANLAR VE UYGULAMALARI

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Polinom ideallerinin H-tabanlarını hesaplamak için bir algoritma geliştirildi. Homojen tabanları hesablamak için bilinen metodlar bölme algoritmalarını ve Grobner tabanını kullanmaktadır. Önerilen algoritma sadece matrisler üzerinde bazı vektör uzayı hesaplamalarını kullanmaktadır. Ayrıca H-tabanlarının bazı Sayısal Analiz problemlerine nasıl uygulanabileceği de gösterilmiştir.

Anahtar Kelimeler: H-tabanı, Gröbner tabanı, Syzygy Modül, Hilbert fonksiyonu, Yarı satır indirgenmiş form.

To my family and my supervisor

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CHAPTER 1

INTRODUCTION

Macaulay [5] introduced the notion of H-bases in 1916. His original motivation was the transformation of systems of polynomial equations into simpler ones. The power of this concept was not really understood presumably because of the lack of facilities for symbolic computations. In 1965, Buchberger [1] invented Gröbner bases for computing multiplication tables for factor rings. When Computer Algebra Systems came up, Buchberger's Algorithm for computing Gröbner basis is implemented to this programs. H-bases have not been used in computational problems as extensively as Gröbner basis. However, Moller and Thomas ([6, 7, 8]) show that H-bases yield a perfect replacement for the Gröbner bases in some numerical analysis problems. All approaches related to Gröbner Bases are fundamentally tied to on term orders which leads to asymmetry among the variables to be considered. On the other hand, the concept of H-bases is based solely on homogeneous terms of a polynomial. Hence H-bases lead to a significant stabilization of the computations when they used instead of Gröbner bases.

It is well known that a Gröbner basis with respect to a degree compatible ordering is an H-basis as well. However, this Gröbner basis may contain some unnecessary elements. Yılmaz and Kılıçarslan [10] gives a method of eliminating of these unnecessary elements during the Gröbner basis computation. H-basis have not been preferred over Gröbner basis because there is no general algorithm for computing it. In [6] an algorithm is proposed but it relies on the a priori knowledge of a system of generator of the syzygy module which cannot be expected to be known in advance in most situations. The only known method for computation of a basis of syzygy module depends on Gröbner basis computation.

In this thesis, we give a method for computation of module of syzygies using only linear algebraic techniques. Our method computes module of syzygies only for homogeneous ideals. However, this is enough to give an algorithm for obtaining H-basis not involving any Gröbner basis computation. We also consider two Numerical Analysis problem, polynomial interpolation and solving system of polynomial equations, and show that how H-bases can be applied to them.

CHAPTER 2

HOMOGENEOUS BASES FOR POLYNOMIAL IDEALS

2.1 Basic Concepts

In this thesis, we consider polynomials in n variables $x_1, \ldots, x_n$ with coefficients from a field $\mathbb{K}$. For short, we write

$$\mathcal{P} := \mathbb{K}[x_1, \ldots, x_n].$$

Definition 2.1. *A monomial in $x_1, \ldots, x_n$ is a product of the form*

$$x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}$$

where all of the exponents $\alpha_1, \ldots, \alpha_n$ are nonnegative integers. The total degree of this monomial is the sum $\alpha_1 + \cdots + \alpha_n$. For brevity, let $x = (x_1, \ldots, x_n)$ and $\alpha = (\alpha_1, \ldots, \alpha_n)$. Then the monomial can be written as x^α. We also let $|\alpha| = \alpha_\mathbf{1} + \cdots + \alpha_\mathbf{n}$ denote the total degree of the monomial x^α.

Definition 2.2. *A polynomial $f \in \mathcal{P}$ is a finite linear combination of monomials with coefficients in a field $\mathbb{K}$. A polynomial f will be written in the form*

$$f = \sum_\alpha a_\alpha x^\alpha$$

where the sum over a finite number of n-tuples $\alpha = (\alpha_1, \ldots, \alpha_n)$ *and* $a_\alpha \in k - \{0\}$*. The total degree of* f*, denoted deg*(f)*, is the maximum* $|\alpha| = \alpha_1 + \cdots + \alpha_n$ *in the expression of* f*.*

Definition 2.3. *Let* f *be a nonzero polynomial in* $\mathcal{P}$ *and let* $>$ *an admissible order on monomials.*

(i) The multidegree of f *is* $multideg(f) = max\{\alpha \mid a_\alpha \neq 0\}$

(ii) The leading coefficient of f is $LC(f) = a_{multideg(f)}$

(iii) The leading monomial of f *is* $LM(f) = x^{multideg(f)}$

(iv) The leading term of f *is* $LT(f) = LC(f)LM(f)$*.*

Definition 2.4. *Let* F *be a set of polynomials, the set*

$$I = \left\{ \sum_{f \in F} h_f f \mid h_f \in \mathcal{P}, \text{ only finitely many } h_f \neq 0 \right\}$$

is the ideal generated by F*. We express this dependence shortly by* $\langle F \rangle = I$*. Note that the Hilbert Basis Theorem implies that* F *can always be replaced by a finite subset of* F*, hence , we can always assume* F *to be finite. Finite generating sets of an ideal are usually called (ideal) bases.*

Definition 2.5. *Let* $I \subset \mathcal{P}$ *be an ideal.*

(i) We denoted by $LT(I)$ *the set of leading forms of elements of* I*. Thus,*

$$LT(I) = \{LT(f) : f \in I\}.$$

(ii) We denoted by $\langle LT(I) \rangle$ *the ideal generated by* $LT(I)$*.*

Definition 2.6. *Choose an admissible order on monomials. Let* $I = \langle f_1, f_2, \ldots, f_s \rangle \subset \mathcal{P}$ *be an ideal.* $H = \{f_1, f_2, \ldots, f_s\}$ *is called a Grobner Basis for* I*, if* $\langle LT(I) \rangle = \langle LT(f_1), LT(f_2), \ldots, LT(f_s) \rangle$*.*

Definition 2.7. *A polynomial f is called homogeneous of total degree d, if every term appearing in f has total degree d.*

Definition 2.8. *An ideal $I \subseteq \mathcal{P}$ is called homogeneous ideal if I has a generating set consist of only homogeneous polynomials.*

Definition 2.9. *Each polynomial $f \neq 0$ has a unique representation*

$$f = \sum_{i=1}^{d} f_i$$

where each f_i is homogeneous polynomial of degree i. The maximal part f_d, denoted by $\mathrm{LF}(f)$*, is called leading form of f.*

Theorem 2.10. *([2]) An ideal I is homogeneous if and only if for each $f \in I$, the homogeneous component f_i of f are in I as well.*

2.2 Properties of Homogeneous Bases

Definition 2.11. *Let $I \subset \mathcal{P}$ be an ideal.*

(i) We denoted by $LF(I)$ the set of leading forms of elements of I. Thus,

$$LF(I) = \Big\{LF(f) : f \in I\Big\}.$$

(ii)We denoted by $\langle LF(I)\rangle$ the ideal generated by $LF(I)$.

If we are given a finite generating set for I, say $I = \langle f_1, \ldots, f_s\rangle$, then the ideal $\langle LF(f_1), \ldots, LF(f_s)\rangle$ and $\langle LF(I)\rangle$ can be different.

Example 2.12. *Let $I = \langle f_1, f_2\rangle$, where $f_1 = x^2y - xy^2 + x^2$ and $f_2 = y^3 - xy^2 - 2x^2 + y$. Then $f = yf_1 + xf_2 = x^2y - 2x^3 + xy$. Hence $LF(f) = x^2y - 2x^3 \in \langle LF(I)\rangle$ but $LF(f) \notin \langle LF(f_1), LF(f_2)\rangle$.*

Now, we can give definition of H-basis similar to Grobner Basis.

Definition 2.13. *Let $I = \langle f_1, f_2, \ldots, f_s \rangle$ be an ideal. $H = \{f_1, f_2, \ldots, f_s\}$ is called a homogenous basis (H-basis) for I , if $\langle LF(I) \rangle = \langle LF(f_1), LF(f_2), \ldots, LF(f_s) \rangle$.*

Proposition 2.14. *[2] Let $g(x_1 \ldots x_n) \in \mathcal{P}$ be a polynomial of total degree d.*

(i) Let $g = \sum_{i=0}^{d} g_i$ be the expansion of g as the sum of its homogeneous components, where , g_i has total degree i. Then

$$g^h(x_0, \ldots, x_n) = \sum_{i=0}^{d} g_i(x_1, \cdots, x_n) x_0^{d-i}$$

is a homogeneous polynomial of total degree d in $k[x_0, \ldots, x_n]$. We will call g^h the homogenization of g.

(ii) The homogenization of g can be computed using the formula

$$g^h = x_0^d . g\left(\frac{x_1}{x_0}, \cdots, \frac{x_n}{x_0}\right).$$

(iii) Dehomogenizing g^h yields g. That is, $g^h(1, x_1, \ldots, x_n) = g(x_1, \ldots, x_n)$.

(iv) Let $F(x_0, \ldots, x_n)$ be a homogeneous polynomial and let x_0^e be the highest power of x_0 dividing F. If $f = F(1, x_1, \ldots, x_n)$ is a dehomogenization of F, then

$$F = x_0^e f^h.$$

Definition 2.15. *Let I be an ideal in $\mathcal{P}$. We define the homogenization of I to be the ideal*

$$I^h = \langle f^h : f \in I \rangle \subset k[x_0, \ldots, x_n]$$

where f^h is the homogenization of f.

Theorem 2.16. *[9]*

Let $\{f_1, f_2, \cdots, f_s\}$ and $I = \langle F \rangle$. Then the following conditions are equivalent.

(i) *F is a homogeneous basis of I.*

(ii) *For every* $f \in I$ *there exist polynomials* $a_1, a_2, \ldots, a_s$ *such that*

$$f = a_1 f_1 + \cdots + a_s f_s \;\; where \;\; deg(f) = max\{deg(a_i f_i)\}$$

(iii) $I^h = \langle f_1^h, f_2^h, \ldots, f_s^h \rangle$

CHAPTER 3

COMPUTING HOMOGENEOUS BASES

3.1 Subspaces of Matrices

Let A be an $m \times n$ matrix. We say A is in quasi row reduced echelon form if

(i) The first non-zero entry in any row is the number 1, these are called pivots.

(ii) A pivot is the only non-zero entry in its column.

It is easy to see that a quasi row reduced echelon form of any matrix can be obtained by the following row operations.

(a) Multiplying a row by a non-zero constant.

(b) Adding a multiple of a row to another row.

Definition 3.1. *Given an $m \times n$ matrix A with entries in a field k.*

(a) The subspace of k^m spanned by the row vectors of A is called the row space of A.

(b) The subspace of k^n spanned by the column vectors of A is called the column space of A.

(c) The nullspace of A is the set of all solutions to the homogeneous equation $Ax = 0$.

Theorem 3.2. *Let A be an $m \times n$ matrix.*

(i) The row space is not affected by row operations. Once the matrix is in quasi row reduced echelon form, the nonzero rows are a basis for the row space.

(ii) The dependence relations between the column vectors are not affected by row operations. It is easy to see that the independent columns of the quasi row reduced echelon form are precisely the columns with pivots. Hence corresponding columns of the matrix A form a basis for the column space of A.

(iii) Let $x = (x_1, x_2, \cdots, x_n)$. Let U be a quasi row reduced echelon form of A . Call the variable x_i basic varible if i-th column of U has pivot element. Otherwise call the variable x_i free variable . For any free variable x_i , let v_i be the solution of $Ux = 0$ such that all free variables are exactly 0, except for x_i which is 1. The set of vectors v_i's form a basis for the nullspace of A.

Example 3.3. *Let*

$$A = \begin{pmatrix} 1 & 0 & -2 & 1 & 0 \\ 2 & 1 & 0 & -1 & 1 \\ 0 & 1 & 4 & -3 & 1 \\ 1 & 0 & 2 & 0 & -2 \\ 0 & -2 & 0 & -1 & 2 \end{pmatrix}.$$

After the row operations quasi row reduced echelon form of A is

$$U = \begin{pmatrix} 1 & 0 & 0 & 0 & -\frac{1}{5} \\ 0 & 1 & 0 & 0 & -\frac{1}{5} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -\frac{9}{10} \\ 0 & 0 & 0 & 1 & -\frac{8}{5} \end{pmatrix}.$$

By Theorem 3.2, the basis of row space of A is

$$\left\{\left(1,0,0,0,-\frac{1}{5}\right),\left(0,1,0,0,-\frac{1}{5}\right),\left(0,0,1,0,-\frac{9}{10}\right),\left(0,0,0,1,-\frac{8}{5}\right)\right\}.$$

The basis of column space of A is

$$\left\{(1,2,0,1,0)^t,(0,1,1,0,-2)^t,(-2,0,4,2,0)^t,(1,-1,-3,0,-1)^t\right\}.$$

The basis of nullspace of A is

$$\left\{\left(\frac{1}{5},\frac{1}{5},\frac{9}{10},\frac{8}{5},1\right)^t\right\}.$$

Now we will give a wellknown theorem about subspaces of a matrix.

Theorem 3.4. *Let A be an $m \times n$ matrix*

(i) Dimension of the row space of A and the column space of A are equal. This number is called the rank of A. The dimension of nullspace of A is called the nullity of A.

(ii) $rank(A) + nullity(A) = n$.

Corollary 3.5. *Let A be an $m \times n$ matrix and U be quasi row reduced echelon form of A. For each column containing pivot element,there exists a column vector in column space of A such that each entry above the pivot element is zero. Furthermore these vectors form a basis for the column space of A.*

Proof. Columns in A which correspond to the columns in the quasi row reduced echelon form of A containing pivot elements is a basis for column space of A. Applying elementary column operations to these columns, the desired column vectors can be found easily. □

Example 3.6. *By using the matrix A in the above example, the basis of column space* *form the following matrix*

$$\begin{pmatrix} 1 & 0 & -2 & 1 \\ 2 & 1 & 0 & -1 \\ 0 & 1 & 4 & -3 \\ 1 & 0 & 2 & 0 \\ 0 & -2 & 0 & -1 \end{pmatrix}.$$

Applying the elementary column operations on this matrix we obtain

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & -2 & 2 & 1 \end{pmatrix}.$$

Hence the desired basis of the column space is

$$\left\{(1, 2, 0, 1, 0)^t, (0, 1, 1, 0, -2)^t, (0, 0, 0, 1, 2)^t, (0, 0, 0, 0, 1)^t\right\}.$$

3.2 Homogeneous Bases and Vector Spaces

In this section we define some vector spaces related to H-bases. We use the notation given in [8].

Definition 3.7. $\mathcal{P}_d^H = \{p \in \mathcal{P} \mid p$ *is a polynomial of degree* $d\}$.

The space of all polynomials of degree at most d can now be written as

$$\mathcal{P}_d = \bigoplus_{k=0}^{d} \mathcal{P}_k^H.$$

Lemma 3.8. *Let $\{g_1, \ldots, g_s\}$ be an H-basis of an ideal $I \subseteq \mathcal{P}$. Then the vector space $I \cap \mathcal{P}_d$ is generated by $x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} g_j$ with $i_1 + \cdots + i_n + deg(g_j) \le d$.*

Proof. Let $f \in I \cap \mathcal{P}$. Since $\{g_1, \ldots, g_s\}$ is a H-basis of I, there exist polynomials $h_1, \ldots, h_s$ such that

$$f = h_1 g_1 + \cdots + h_s g_s$$

where $deg(f) = d \ge deg(h_i g_i)$ $i = 1, \ldots, s$. Therefore it is clear that f is $\mathbb{K}$-linear combinations in the form $x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n} g_j$ with $i_1 + \cdots + i_n + deg(g_j) \le d$. □

Definition 3.9. *For given polynomials $f_1, \ldots, f_s$ and $d \in \mathbb{N}$, we define the vector space*

$$V_d(f_1, \cdots, f_s) = \left\{ \sum_{i=1}^{s} h_i LF(f_i) \mid h_i \in \mathcal{P}^H_{d-deg(f_i)}, i = 1, \ldots, s \right\}.$$

Clearly, $V_d = (f_1, \cdots, f_s)$ is a subspace of $\mathcal{P}^H_d$. For convenience, we also define for ideals I

$$V_d(I) = \left\{ LF(f) \mid f \in I, deg(f) = d \text{ or } f = 0 \right\}.$$

This is also a subspace of $\mathcal{P}^H_d$.

We let $W_d(f_1, \cdots, f_s)$ be the vector space such that

$$\mathcal{P}^H_d = V_d(f_1, \ldots, f_s) \bigoplus W_d(f_1, \ldots, f_s).$$

Definition 3.10. *If $f_1, \ldots, f_s \in \mathcal{P}$ and $d \in \mathbb{N}$, then let*

$$A_d(f_1, \ldots, f_s) = \left\{ x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mathrm{LF}(f_i) \mid \alpha_1 + \cdots + \alpha_n + \deg(f_i) = d, i = 1, \ldots, s \right\}.$$

Clearly $A_d(f_1, \ldots, f_s)$ spans the vector space $V_d(f_1, \ldots, f_s)$.

Definition 3.11. *Define the following order* $<_d$ *in* $A_d(f_1, \ldots, f_s)$*:*

$$x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mathrm{LF}(f_i) <_d x_1^{\beta_1} \cdots x_n^{\beta_n} \mathrm{LF}(f_j)$$

if $x_1^{\alpha_1} \cdots x_n^{\alpha_n} <_{lex} x_1^{\beta_1} \cdots x_n^{\beta_n}$ *or if* $x_1^{\alpha_1} \cdots x_n^{\alpha_n} = x_1^{\beta_1} \cdots x_n^{\beta_n}$ *and* $i > j$*.*

Definition 3.12. *For given polynomials* $g_1, \ldots, g_m \in \mathcal{P}$ *let*

$$Syz(g_1, \ldots, g_m) = \left\{ (h_1, \ldots, h_m) \in \mathcal{P}^m \mid \sum_{i=1}^{m} h_i g_i = 0 \right\}$$

be the module of syzygies with respect to $g_1, \ldots, g_m$*. For every* $d \in \mathbb{N}$ *let*

$$S_d(g_1, \cdots, g_m) = \left\{ (h_1, \cdots, h_m) \in \bigotimes \mathcal{P}^H_{d-deg(g_i)} \mid \sum_{i=1}^{m} h_i LF(g_i) = 0 \right\}.$$

Lemma 3.13. *[8] Let* $I = \langle f_1, \ldots, f_s \rangle \subseteq \mathcal{P}$ *be given . Then the following four statements hold.*

(i) $LF(I) = \bigoplus_{d \in \mathbb{N}} V_d(f_1, \cdots, f_s)$

(ii) $Syz(LF(f_1), \ldots, LF(f_s)) = \bigoplus_{d \in \mathbb{N}} S_d(f_1, \ldots, f_s)$

(iii) $dim(V_d(f_1, \ldots, f_s)) = dim(P^H_d) - dim(W_d(f_1, \ldots, f_s)$*, for all* $d \in \mathbb{N}$

(iv) $dim(S_d(f_1, \ldots, f_s)) = \sum_{i=1}^{s} dim(\mathcal{P}^H_{d-deg(f_i)}) - dim(V_d(f_1, \ldots, f_s)$

In the next section, we study the Hilbert function. We emphasize to find the Hilbert function of a monomial ideal. The details of the computation of Hilbert functions and Hilbert polynomials can be found in [4].

3.3 Hilbert Functions and Hilbert Polynomials

Definition 3.14. *Let* $I \subseteq \mathcal{P}$ *be an ideal. The mapping*

$$H_I : \mathbb{N} \mapsto \mathbb{N}$$

given by

$$H_I(d) = \dim(W_d(I))$$

is called the (homogeneous) Hilbert function of I.

Since the monomials of degree d is a basis for $\mathcal{P}_d^H$, the following result easily follows.

Proposition 3.15. *If I is a monomial ideal, then $H_I(d)$ is the number of monomials not in I of total degree d.*

The following well known result is fundamental for the computation of the Hilbert polynomial of any ideal.

Proposition 3.16. *[2] Let $>$ be a monomial order on $\mathcal{P}$. Then the monomial ideal $\langle LT(I)\rangle$ has some Hilbert function as I.*

Definition 3.17. *Let $I \subset \mathcal{P}$ be an ideal and H_I be its Hilbert function. The Hilbert-Poincare series HP_I of I is defined by*

$$HP_I(t) = \sum_{m \in \mathbb{N}} H_I(m) t^m.$$

Theorem 3.18. *[4] Hilbert-Poincare series of any ideal I can be written in the form*

$$HP_I(t) = \frac{Q(t)}{(1-t)^r} \text{ for some } Q(t) \in Z[t].$$

Definition 3.19. *With the notations of Theorem 3.18, we cancel all common factors in the numerator and denominator of $HP_I(t)$, then we obtain*

$$HP_I(t) = \frac{G(t)}{(1-t)^s}, \;\; 0 \leq s \leq r, \;\; G(t) = \sum_{m=0}^{d} g_m t^m \in Z[t]$$

such that $g_d = 0$ and $G(1) \neq 0$, that is, s is the pole order of $HP_I(t)$ at $t = 1$.

(i) *The polynomials $Q(t)$ and $G(t)$,given above, are called the first and the second Hilbert series of I respectively.*

(ii) *Let d be the degree of the second Hilbert series $G(t)$, and let s be the pole order of the Hilbert-Poincare series $HP_I(t)$ at $t = 1$, then*

$$P_I = \sum_{m=0}^{d} g_m \binom{s-1+n-m}{s-1} \in Q[n]$$

is called the Hilbert polynomial of I $\left(\text{with } \binom{n}{k} = 0 \text{ for } k < 0 \right)$.

Corollary 3.20. *For $n \geq d$, $P_I(n) = H_I(n)$. Moreover P_I is a polynomial in n of degree $s-1$ and the coefficient of n^{s-1} is $G(1)$.*

In the remaining of this section, we give an algorithm to computing the Hilbert polynomial of a monomial ideal.

Example 3.21. *Notice that $H_{\mathbb{K}[x_1,\ldots,x_r]} = P_{\mathbb{K}[x_1,\ldots,x_r]} = \binom{n+r-1}{r-1}$. Hence*

$$HP_{\mathbb{K}[x_1,\ldots,x_r]} = \sum_{m=0}^{\infty} \binom{r-1-m}{r-1} t^m = \frac{1}{(1-t)^r}.$$

Lemma 3.22. *[4] Let $I \subseteq \mathcal{P}$ and let $f \in \mathcal{P}$ be the homogeneous polynomial of degree d. Then*

$$HP_I(t) = HP_{\langle I,f\rangle}(t) + t^d HP_{I:\langle f\rangle}(t).$$

Since the Hilbert polynomial and the Hilbert-Poincare series determine each other, it suffices to study and compute Hilbert-Poincare series. Example 3.21 and Lemma 3.22 suggest the following algorithm compute the Hilbert-Poincare series of a monomial ideal, more precisely, it computes the polynomial $Q(t)$.

Algorithm 3.23. *(MonomialHilbertPoincare(I)) [4]*

$Input : I = \langle m_1, \cdots, m_p \rangle \subset \mathcal{P}$, m_i *monomials.*

$Output$: *A polynomial* $Q(t)$ *such that* $Q(t) \backslash (1-t)^n$ *is HP series of* I.

$Choose\ \ S = \{x^{\alpha_1}, \cdots, x^{\alpha_s}\}$ *a minimal set of generators of* I

$If\ \ S = \{0\}$ *then return* 1.

$If\ \ S = \{1\}$ *then return* 0.

If *all element of* S *have degree* 1 *then return* $(1-t)^s$.

$Choose\ \ 1 \leq i \leq s$ *such that* $deg(x^{\alpha_i}) > 1$

$Choose\ \ 1 \leq j \leq n$ *such that* $x_j \mid x^{\alpha_i}$.

$Return\ \ MonHilbertPoincare\langle I, x_j\rangle + t.MonHilbertPoincare(I : x_j)$

3.4 Computing Syzygy Modules of Leading Forms

The dimension of the finite dimensional spaces considered in Lemma 3.13 can easily be described using the Hilbert function.

Lemma 3.24. *[8] Let* $F = \{f_1, \ldots, f_s\}$ *be an H-basis of the ideal* I. *Then the following four statements hold for all* $d \in \mathbb{N}$

(i) $dim(V_d(I)) = \binom{d+n-1}{n-1} - H_I(d)$

(ii) $dim(S_d(f_1, \cdots, f_s)) = H_I(d) - \binom{d+n-1}{n-1} + \sum_{i=1}^{s} \binom{d+n-1-deg(f_i)}{n-1}$

(iii) $dim(I \cap \mathcal{P}_d) = \binom{d+n}{n} - \sum_{k=0}^{d} H_I(k)$

(iv) $dim(Syz(LF(f_1), \cdots, LF(f_s)) = \sum_{k=0}^{d} H_I(k) - \binom{d+n}{n} + \sum_{i=1}^{s} \binom{d+n-deg(f_i)}{n}$

Here we declare the binomial $\binom{a}{b}$ *to be* 0 *for all* a *with* $-a \in \mathbb{N}$.

Lemma 3.25. *Suppose that* $G = \{g_1, \cdots, g_t\}$ *is a generating set for an ideal* I *but* $J = \langle LT(g_1), \cdots, LT(g_t)\rangle \subsetneq \langle LT(I)\rangle$ *for a monomial order. Then,* $H_I(d) \leq H_J(d)$ *for any* $d \in \mathbb{N}$.

Proof. Since $J = \langle LT(g_1), \cdots, LT(g_t)\rangle \subsetneq \langle LT(I)\rangle$ for any $d \in \mathbb{N}$, we may have a monomial $x^\alpha \in \langle LT(I)\rangle$ but $x^\alpha \notin J$. Recall that for a monomial ideal J , $H_J(d)$ is the number of monomial not in J of total degree d. Hence $H_I(d) = H_{\langle LT(I)\rangle}(d) \leq H_J(d)$. □

Definition 3.26. *Let $B(d)$ be the set of lexicographically ordered monomials of degree d. It is well known that $B(d)$ is a basis for $\mathcal{P}_d^H$. For given polynomials $f_1, \ldots, f_s$ be polynomials, define the matrix $M(d)$ whose columns are coordinate vectors of $A_d(f_1, \ldots, f_s)$ with respect to $B(d)$. Furthermore, assume that columns of $M(d)$ are decreasingly ordered with $<_d$.*

Proposition 3.27. *Let $\widetilde{M}(d)$ be quasi row reduced form of the matrix $M(d)$ given by above definition.*

(i) Let $\{(i_1, j_1), \ldots, (i_r, j_r)\}$ be the set of indices for the pivot elements of $\widetilde{M}(d)$. Then $U(d) = \{A_d^{j_1}, \ldots, A_d^{j_r}\}$ is a basis for $V_d(f_1, \ldots, f_s)$. Here $A_d^{j_k}$ means the j_k-th element of $A_d(f_1, \ldots, f_s)$.

(ii) Let $W(d) = A_d(f_1, \ldots, f_s) \setminus U(d)$. For each $f = x^\alpha f_i \in W(d)$ there exists a syzygy $\mathbf{s}_f \in S_d(f_1, \ldots, f_s)$ such that

$$\mathbf{s}_f = f - \sum_{k=1}^{r} c_k A_d^{j_k}, \quad c_k \in \mathbb{K}.$$

Here $f <_d A_d^{j_k}$ for $k = 1, \ldots, r$. Furthermore these syzygies from a basis for $S_d(f_1, \ldots, f_s)$.

(iii) If $x_1^{\alpha_1} \cdots x_n^{\alpha_n} = B(d)_{i_k}$ for some $1 \leq k \leq r$, then there exists a polynomial $f \in V_d(f_1, \ldots, f_s)$ such that $\mathrm{LM}(f) = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ with respect to lexicographic order. Here $B(d)_{i_k}$ means the i_k-th element of $B(d)$.

Proof. Part (i) and (ii) easily follow from Theorem 3.2.

(iii) Let N be a matrix whose columns are coordinate vectors of $U(d)$ relative to basis $B(d)$. Let $\widetilde{N}$ be a column reduced form of N such that $\{(i_1, 1), \ldots, (i_r, r)\}$ be the set of indices of pivot elements of $\widetilde{N}$. Consider the polynomials whose coordinate vectors are columns of $\widetilde{N}$ relative to basis $B(d)$. These are the polynomials we looking for. □

Proposition 3.28. *For given polynomials $f_1, \ldots, f_s$ and $d \in \mathbb{N}$, let $U(d) \subseteq A_d(f_1, \ldots, f_s)$ be a basis for $V_d(f_1, \ldots, f_s)$ and $W(d) = A_d(f_1, \ldots, f_s) \setminus U(d)$. Define the set*

$$\mathcal{S}_e = \{x_1^{\beta_1} \cdots x_n^{\beta_n} \mathbf{s}_f \mid \mathbf{s}_f \in S_d(f_1, \ldots, f_s) \text{ and } \beta_1 + \cdots \beta_n = e - d\}$$

for some $e > d$. Then $\mathcal{S}_e$ is a linearly independent set.

Proof. Let X be the increasingly ordered set of elements of A_e with respect to $<_e$. Notice that $x^\alpha LF(f_i) <_d x^\beta LF(f_j)$ implies $x^\gamma x^\alpha LF(f_i) <_e x^\gamma x^\beta LF(f_j)$. By construction of $\mathcal{S}_e$, the first non zero entry of the coordinate vector of every $\mathbf{s}_e \in \mathcal{S}_e$ relative to X is 1 and the positions of these 1's are different for each $\mathbf{s}_e \in \mathcal{S}_e$. Hence the set $\mathcal{S}_e$ is a linearly independent set. □

Corollary 3.29. *Define the ideals $J_i = \langle x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid x_1^{\alpha_1} \cdots x_n^{\alpha_n} f_i \in W(d) \rangle \quad i = 1, \ldots, s$. Then*

$$dim(\mathcal{S}_e) = \sum_{i=1}^{s} \binom{n + e - deg(f_i) - 1}{n - 1} - \sum_{i=1}^{s} H_{J_i}(e - deg(f_i)).$$

Proof. The result is easily follows from definition of Hilbert function and Proposition 3.28. □

Algorithm 3.30. (Syzygy Computation)

Input: *An s-tuple polynomials $(f_1, \ldots, f_s)$ with $deg(f_1) < \cdots < deg(f_s)$.*

Output: *A generating set for $Syz((LF(f_1), \ldots, LF(f_s))$.*

$d := deg(f_2)$

$J := \{\} \subset \mathcal{P}, \; Syz := \{\} \subset \mathcal{P}^s, \; f(e) := 0, \; g(e) := 1$

WHILE $f(e) < g(e)$

  FORM $M(d)$

  COMPUTE $\widetilde{M}(d)$

  FIND $U(d)$ *and* $W(d) := A_d(f_1, \ldots, f_s) \setminus U(d)$

  $J := J + \langle B(d)_{i_k} \mid 1 \leq k \leq r \rangle$

$Syz := Syz + \langle \mathbf{s}_f \mid f \in W(d)\rangle$

$J_i := \langle x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mid x_1^{\alpha_1} \cdots x_n^{\alpha_n} f_i \in W(d)\rangle$

$f(e) := -\sum_{i=1}^{s} H_{J_i}(e - deg(f_i))$

$g(e) := H_J(e) - \binom{e+n-1}{n-1}$

d:=d+1

ENDWHILE

RETURN Syz

Example 3.31. *Since the aim is to find a basis for syzygy module of leading forms, we will take an homogeneous ideal so that leading forms are same as the polynomials. Consider the ideal* $I = \langle f_1 = y^2 + yz, f_2 = xz + z^2, f_3 = x^2y + y^3\rangle$.

For $d = 2$, *the matrix* $M(2)$ *is*

	f_1	f_2
x^2	0	0
xy	0	0
xz	0	1
y^2	1	0
yz	1	0
z^2	0	1

The quasi reduced matrix $\widetilde{M}(2)$ *is*

	f_1	f_2
x^2	0	0
xy	0	0
xz	0	1
y^2	1	0
yz	0	0
z^2	0	0

$W(d) := \{\}$. *So there is no syzgy.*

$J = \langle xz, y^2 \rangle$

For $d = 3$, *the matrix* $M(3)$ *is*

	xf_1	xf_2	yf_1	yf_2	zf_1	zf_2	f_3
x^3	0	0	0	0	0	0	0
x^2y	0	0	0	0	0	0	1
x^2z	0	1	0	0	0	0	0
xy^2	1	0	0	0	0	0	0
xyz	1	0	0	1	0	0	0
xz^2	0	1	0	0	0	1	0
y^3	0	0	1	0	0	0	1
y^2z	0	0	1	0	1	0	0
yz^2	0	0	0	1	1	0	0
z^3	0	0	0	0	0	1	0

The quasi reduced matrix $\widetilde{M}(3)$ *is*

	xf_1	xf_2	yf_1	yf_2	zf_1	zf_2	f_3
x^3	0	0	0	0	0	0	0
x^2y	0	0	0	0	0	0	1
x^2z	0	1	0	0	0	0	0
xy^2	1	0	0	0	0	0	0
xyz	0	0	0	1	0	0	0
xz^2	0	0	0	0	0	1	0
y^3	0	0	1	0	0	0	0
y^2z	0	0	0	0	1	0	0
yz^2	0	0	0	0	0	0	0
z^3	0	0	0	0	0	0	0

$W(d) := \{\}$. So there is no syzygy.

$J = \langle xz, y^2, x^2y \rangle$

For $d = 4$, the matrix $M(4)$ is

$$M(4) = \left(\begin{array}{c|c} M(3) & \\ & N(4) \\ 0 & \end{array} \right)$$

where $N(4)$ is

	y^2f_1	y^2f_2	yzf_1	yzf_2	yf_3	z^2f_1	z^2f_2	zf_3
x^4	0	0	0	0	0	0	0	0
x^3y	0	0	0	0	0	0	0	0
x^3z	0	0	0	0	0	0	0	0
x^2y^2	0	0	0	0	1	0	0	0
x^2yz	0	0	0	0	0	0	0	1
x^2z^2	0	0	0	0	0	0	0	0
xy^3	0	0	0	0	0	0	0	0
xy^2z	0	1	0	0	0	0	0	0
xyz^2	0	0	0	1	0	0	0	0
xz^3	0	0	0	0	0	0	1	0
y^4	1	0	0	0	1	0	0	0
y^3z	1	0	1	0	0	0	0	1
y^2z^2	0	1	1	0	0	1	0	0
yz^3	0	0	0	1	0	1	0	0
z^4	0	0	0	0	0	0	1	0

The quasi reduced matrix $\widetilde{M}(4)$ *is*

$$\widetilde{M}(4) = \left(\begin{array}{c|c} \widetilde{M}(3) & \\ & \widetilde{N}(4) \\ 0 & \end{array} \right)$$

where $\widetilde{N}(4)$ *is*

	y^2f_1	y^2f_2	yzf_1	yzf_2	yf_3	z^2f_1	z^2f_2	zf_3
x^4	0	0	0	0	0	0	0	0
x^3y	0	0	0	0	0	0	0	0
x^3z	0	0	0	0	0	0	0	0
x^2y^2	0	0	0	0	0	0	0	1
x^2yz	0	0	0	0	0	0	0	0
x^2z^2	0	0	0	0	0	0	0	0
xy^3	0	0	0	0	0	0	0	0
xy^2z	0	0	0	0	0	-1	0	0
xyz^2	0	1	0	0	0	1	0	0
xz^3	0	0	0	0	0	0	1	0
y^4	1	0	0	0	0	0	0	1
y^3z	0	0	1	0	0	0	0	0
y^2z^2	0	0	0	1	0	1	0	0
yz^3	0	0	0	0	1	0	0	-1
z^4	0	0	0	0	0	0	0	0

$J = \langle xz, y^2, x^2y, yz^3\rangle$

$W(4) := \{z^2 f_1, zf_3\}$ *So there are two syzygies.*

$Syz = \{(-x^2 - y^2, 0, y + z), (z^2 + xz, -y^2 - yz, 0)\}$

$J_1 = \langle z^2\rangle, J_2 = \{\}, J_3 = \langle z\rangle$

$$\begin{aligned} HP_J(t) &= \sum_{e=0}^{\infty}\binom{e+2}{2}t^e - 2\sum_{e=0}^{\infty}\binom{e+2}{2}t^{e+2} - \sum_{e=0}^{\infty}\binom{e+2}{2}t^{e+3} \\ &\quad +2\sum_{e=0}^{\infty}\binom{e+2}{2}t^{e+4} + \sum_{e=0}^{\infty}\binom{e+2}{2}t^{e+5} - \sum_{e=0}^{\infty}\binom{e+2}{2}t^{e+6} \\ &= \sum_{e=0}^{\infty}\left[\binom{e+2}{2} - 2\binom{e}{2} - \binom{e-1}{2} + 2\binom{e-2}{2} + \binom{e-3}{2} - \binom{e-4}{2}\right]t^e \\ &= \sum_{e=0}^{\infty} 2t^e \end{aligned}$$

So $H_J(e) = 2$

$g(e) = H_J(e) - \binom{e+n-1}{n-1}$

Hence $g(e) = -\frac{1}{2}e^2 - \frac{3}{2}e + 1$

$$\begin{aligned} HP_{J_1}(t) &= \sum_{e=0}^{\infty}\binom{e+2}{2}t^e - \sum_{e=0}^{\infty}\binom{e+2}{2}t^{e+2} \\ &= \sum_{e=0}^{\infty}\left[\binom{e+2}{2} - \binom{e}{2}\right]t^e \\ &= \sum_{e=0}^{\infty}(1+2e)t^e \end{aligned}$$

So $H_{J_1}(e) = 1 + 2e$. *Then* $H_{J_1}(e-2) = 2e - 3$.

Since $J_2 = \{\}, H_{J_2}(e) = \binom{e+2}{2}$.

This implies $H_{J_2}(e-2) = \binom{e}{2} = \frac{e^2}{2} - \frac{e}{2}$.

$$
\begin{aligned}
HP_{J_3}(t) &= \sum_{e=0}^{\infty} \binom{e+2}{2} t^e - \sum_{e=0}^{\infty} \binom{e+2}{2} t^{e+1} \\
&= \sum_{e=0}^{\infty} \left[\binom{e+2}{2} - \binom{e+1}{2} \right] t^e \\
&= \sum_{e=0}^{\infty} (1+e) t^e
\end{aligned}
$$

So $H_{J_3}(e) = 1 + e$. *Then* $H_{J_3}(e-3) = e - 2$.

$f(e) = -\sum_{i=1}^{3} H_{J_i}(e - deg(f_i)) = -\frac{e^2}{2} - \frac{5e}{2} + 5$

For $d = 5$ *the matrix* $M(5)$ *is*

$$
M(5) = \left(\begin{array}{c|c} M(4) & \\ & N(5) \\ 0 & \end{array} \right)
$$

where $N(5)$ *is*

	y^3f_1	y^3f_2	y^2zf_1	y^2zf_2	y^2f_3	yz^2f_1	yz^2f_2	yzf_3	z^3f_1	z^3f_2	z^2f_3
x^5	0	0	0	0	0	0	0	0	0	0	0
x^4y	0	0	0	0	0	0	0	0	0	0	0
x^4z	0	0	0	0	0	0	0	0	0	0	0
x^3y^2	0	0	0	0	0	0	0	0	0	0	0
x^3yz	0	0	0	0	0	0	0	0	0	0	0
x^3z^2	0	0	0	0	0	0	0	0	0	0	0
x^2y^3	0	0	0	0	1	0	0	0	0	0	0
x^2y^2z	0	0	0	0	0	0	0	1	0	0	0
x^2yz^2	0	0	0	0	0	0	0	0	0	0	1
x^2z^3	0	0	0	0	0	0	0	0	0	0	0
xy^4	0	0	0	0	0	0	0	0	0	0	0
xy^3z	0	1	0	0	0	0	0	0	0	0	0
xy^2z^2	0	0	0	1	0	0	0	0	0	0	0
xyz^3	0	0	0	0	0	0	1	0	0	0	0
xz^4	0	0	0	0	0	0	0	0	0	1	0
y^5	1	0	0	0	1	0	0	0	0	0	0
y^4z	1	0	1	0	0	0	0	1	0	0	0
y^3z^2	0	1	1	0	0	1	0	0	0	0	1
y^2z^3	0	0	0	1	0	1	0	0	1	0	0
yz^4	0	0	0	0	0	0	1	0	1	0	0
z^5	0	0	0	0	0	0	0	0	0	1	0

The quasi reduced matrix $\widetilde{M}(5)$ *is*

$$\widetilde{M}(5) = \left(\begin{array}{c|c} \widetilde{M}(4) & \\ & \widetilde{N}(5) \\ 0 & \end{array} \right)$$

where $\widetilde{N}(5)$ *is*

	y^3f_1	y^3f_2	y^2zf_1	y^2zf_2	y^2f_3	yz^2f_1	yz^2f_2	yzf_3	z^3f_1	z^3f_2	z^2f_3
x^5	0	0	0	0	0	0	0	0	0	0	0
x^4y	0	0	0	0	0	0	0	0	0	0	0
x^4z	0	0	0	0	0	0	0	0	0	0	0
x^3y^2	0	0	0	0	-1	0	0	1	0	0	-1
x^3yz	0	0	0	0	1	0	0	-1	0	0	1
x^3z^2	0	0	0	0	0	0	0	0	0	0	0
x^2y^3	0	0	0	0	1	0	0	0	0	0	0
x^2y^2z	0	0	0	0	-1	0	0	1	1	0	0
x^2yz^2	0	0	0	0	0	0	0	0	-1	0	0
x^2z^3	0	0	0	0	0	0	0	0	0	0	0
xy^4	0	0	0	0	-1	0	0	1	0	0	-1
xy^3z	0	0	0	0	0	-1	0	0	0	0	0
xy^2z^2	0	0	0	0	0	0	0	0	-1	0	0
xyz^3	0	0	0	0	1	0	0	-1	0	0	1
xz^4	0	0	0	0	0	0	0	0	0	1	0
y^5	1	0	0	0	1	0	0	0	0	0	0
y^4z	0	0	1	0	-1	0	0	1	0	0	0
y^3z^2	0	1	0	0	1	1	0	-1	0	0	1
y^2z^3	0	0	0	1	0	1	0	0	1	0	0
yz^4	0	0	0	0	0	0	1	0	1	0	0
z^5	0	0	0	0	0	0	0	0	0	0	0

$J = \langle xz, y^2, x^2y, yz^3\rangle$

$W(5) := xW(4) \cup yW(4) \cup zW(4) \cup \{y^2f_3\}$. *So there is one more syzygy.*

$Syz = \{(z^2+xz, -y^2-yz, 0), (-x^2-y^2, 0, y+z), (x^3-x^2y+x^2z+xy^2-y^3, -x^2y-y^3, y^2-xy)\}$

$J_1 = \langle z^2\rangle, J_2 = \{\}, J_3 = \langle z, y^2\rangle$ *Since* J, J_1, J_2 *are the same* $H_J(e), H_{J_1}(e), H_{J_2}(e)$ *do not* *change. Hence* $g(e) = -\frac{e^2}{2} - \frac{3e}{2} + 1$

$$\begin{aligned} HP_{J_3}(t) &= \sum_{e=0}^{\infty}\binom{e+2}{2}t^e - \sum_{e=0}^{\infty}\binom{e+2}{2}t^{e+1} - \sum_{e=0}^{\infty}\binom{e+2}{2}t^{e+2} + \sum_{e=0}^{\infty}\binom{e+2}{2}t^{e+3} \\ &= \sum_{e=0}^{\infty}\left[\binom{e+2}{2} - \binom{e+1}{2} - \binom{e}{2} + \binom{e-1}{2}\right]t^e \\ &= \sum_{e=0}^{\infty} 2t^e \end{aligned}$$

So $H_{J_3}(e) = 2$. *Then* $H_{J_3}(e-2) = 2$

$f(e) = -\sum_{i=1}^{3} H_{J_i}(e - deg(f_i)) = -\frac{e^2}{2} - \frac{3e}{2} + 1.$

Since $f(e) = g(e)$, *the algorithm terminates. Hence*

$$Syz = \{(z^2+xz, -y^2-yz, 0), (-x^2-y^2, 0, y+z), (x^3-x^2y+x^2z+xy^2-y^3, -x^2y-y^3, y^2-xy)\}$$

is a basis for $Syz(f_1, f_2, f_3)$.

3.5 An Algorithm for Computing Homogeneous Bases

First we need a technical notion.

Definition 3.32. *For given* $f, f_1, \ldots, f_m \in \mathcal{P}$ *we say* f *reduces to* $\widetilde{f}$ *modulo* $\mathcal{F} = \{f_1, \ldots, f_m\}$ *if*

$$\widetilde{f} = f - \sum_{i=1}^{m} g_i f_i, \quad \deg(\widetilde{f}) < \deg(f)$$

holds with polynomials g_i *satisfying* $\deg(g_i) \leq \deg(f) + \deg(f_i)$, $i = 1, \ldots, m$. *In this case we write*

$$f \xrightarrow{*} \widetilde{f}$$

By $\xrightarrow{*}$ *we denote the transitive closure of the binary relation* $\rightarrow$. *We also say* f *reduces modulo* $\mathcal{F}$ *to* g *if* $\xrightarrow{*} g$.

The algorithm is based on the following results. Different versions of this theorem can be found in [8] and [9].

Theorem 3.33. *Let $\mathcal{F} = \{f_1, \ldots, f_s\} \subseteq \mathcal{P}$ be a basis of the ideal I. Then the following conditions are equivalent.*

(i) $\mathcal{F}$ is an H-basis.

(ii) If $(g_1, \ldots, g_m) \in S_d(f_1, \ldots, f_s)$, then

$$p = \sum_{i=1}^{s} g_i f_i \xrightarrow{*} 0$$

The following algorithm is described by Moller and Sauer in [8]. It depends on the computation of syzygy basis of leading forms. The reduction step can be obtained using only linear algebraic methods. Combining this algorithm with our method of computing syzygy basis of leading forms described in previous section, one can obtain H-basis of an ideal using only linear algebraic techniques. This is the only known method of computation of an H-basis without using any Grobner Basis.

Algorithm 3.34. *Given a set of polynomials $\mathcal{F} = \{f_1, \ldots, f_s\}$ generating an ideal I. First compute*

$$Syz(LF(f_1), \ldots, LF(f_s)).$$

Then for each basis element $(g_1, \ldots, g_s)$ the polynomial p given in Theorem 3.33 has to be reduced modulo $\mathcal{F}$ as far as possible. If reduction process produce a non zero polynomial f, then the set $\mathcal{F}$ enlarged by f. Then for the new set $\mathcal{F}$ a basis of syzygy modules of leading forms will be computed. The process continue until no more enlargement of $\mathcal{F}$ is necessary.

Example 3.35. *Let $\mathcal{F} = \{f_1 = x^2 - y, f_2 = x^3 - z\}$ and I be the ideal generated by $\mathcal{F}$.*

Let $g_1 = LF(f_1) = x^2$, $g_2 = LF(f_2) = x^3$.

Syzygy basis of $\{g_1, g_2\}$ is $\{(x, -1)\}$.

Then $p = xf_1 - f_2 = -xy + z \rightarrow p$.

So $f_3 = p, g_3 = LF(p)$ and $\mathcal{F} = \mathcal{F} \cup \{f_3\}$.

Syzygy basis of $\{g_1, g_2, g_3\}$ *is* $\{(x, -1, 0), (y, 0, x)\}$.

Then $p = yf_1 + xf_3 = xz - y^2 \rightarrow p$

So $f_4 = h, g_4 = LF(h) = xz - y^2$ *and* $\mathcal{F} = \mathcal{F} \cup \{f_4\}$

Syzygy basis of $\{g_1, g_2, g_3, g_4\}$ *is* $\{(x, -1, 0, 0), (y, 0, x, 0), (-z, 0, -y, x)\}$.

Then $h = -zf_1 - yf_3 + xf_4 = 0$.

Hence $\{f_1 = x^2 - y, f_2 = x^3 - z, f_3 = -xy + z, f_4 = xz - y^2\}$ *is an H-basis for I.*

CHAPTER 4

APPLICATIONS OF HOMOGENEOUS BASES

4.1 Solving System of Polynomial Equations

For a given set of polynomials $\mathcal{F} = \{f_1, \ldots, f_s\}$, the vector spaces $V_d(f_1 \ldots, f_s)$ and $W_d(f_1 \ldots, f_s)$ are given by Definition 3.9 . In fact, if an inner product in $P = k[x_1, \ldots, x_n]$ is defined as the scalar product of coefficient vectors, then $W_d(f_1 \ldots, f_s)$ becomes the orthogonal complement of $V_d(f_1 \ldots, f_s)$.

Example 4.1. *Let $f_1 = x^2 - 2xy + 3x$ and $f_2 = -xy + y^2 + 2x - 3y + 1$. If we order monomials in P_2 as $\{x^2, xy, y^2, x, y, 1\}$, then $f_1 = (1, -2, 0, 3, 0, 0)$ and $f_2 = (0, -1, 1, 2, -3, 1)$. So $\langle f_1, f_2 \rangle = (-2)(-1) + (3)(2) = 8$.*

Definition 4.2. *Let $\mathcal{F} = \{f_1, \ldots, f_s\} \subset \mathcal{P}$ and $\varphi_d = \varphi_d(f_1, \ldots, f_s)$ denote the orthogonal projection of $P_d^{(H)}$ and $W_d(f_1 \ldots, f_s)$, $d \in N_0$. We define recursively a reduction. We say $f \in \mathcal{P}$ reduces fully to $\widetilde{f}$ modulo $\mathcal{F}$, if either there are polynomials $g_i \in P^{(H)}_{deg(f)-deg(f_i)}$, $i = 1, \ldots, s$, such that*

$$\widetilde{f} = f - \sum_{i=1}^{s} g_i f_i, \quad LF(\widetilde{f}) = \varphi_{deg(f)}(f)$$

or if $LF(f) = LF(\widetilde{f})$ and $f - LF(f)$ reduces fully modulo F to $\widetilde{f} - LF(\widetilde{f})$.

Each polynomial$f = \widetilde{f_0} \circ f$ degree d can be fully reduced modulo an arbitrary finite polynomial set F. At first step, we reduce $f = \widetilde{f_0}$ fully modulo F to a polynomial $W_d + \widetilde{f_1}$ with $W_d = \varphi_d(\widetilde{f_0}) \in W_d(f_1 \ldots, f_s)$ and $\widetilde{f_1} \in P_{d-1}$. Then we reduce $\widetilde{f_1}$ to

$W_{d-1} + \widetilde{f_2}$ with $w_{d-1} \in W_{d-1}(f_1 \dots, f_s)$ and $\widetilde{f_2} \in P_{d-2}$ etc.This procedure terminates after d reduction steps.

Example 4.3. *Let $\mathcal{F} = \{f_1 = x^2 + y^2 - 2x + 1, f_2 = 2xy - y^2 + y\}$ and $f = x^3 + 2xy^2 + x^2 - y^2 + x$. So $LF(f_1) = x^2 + y^2$ and $LF(f_2) = 2xy - y^2$.*

$x^3 + 2xy^2 = xLF(f_1) - xLF(f_2) - 2x^2y \Rightarrow w_3 = -2x^2y$

$\widetilde{f_1} = f - xf_1 + xf_2 - w_3 = 3x^2 + xy - y^2$

$3x^2 + xy - y^2 = 3LF(f_1) + 4LF(f_2) - 7xy \Rightarrow w_2 = -7xy$

$\widetilde{f_2} = \widetilde{f_1} - 3f_1 - 4f_2 - w_2 = 6x - 4y - 3 \Rightarrow w_1 = 6x - 4y - 3.$

Lemma 4.4. *[6] Let $F_i = \{f_1, \dots, f_s\} \subset P$ and $f \in P_d$. Then there are polynomials g_i with $deg(g_i) \leq deg(f) - deg(f_i)$, $i = 1, \dots, s$ and homogeneous $w_k \in W_k(f_1, \dots, f_s)$ for $k = 0, \dots, d$ such that*

$$f = w_d + w_{d-1} + \cdots + w_0 + \sum_{i=1}^{s} g_i f_i.$$

If F is an H-basis, the $w_0, \dots, w_d$ are uniquely determined by f for every $f \in \mathcal{P}$.

Definition 4.5. *Let $\mathcal{F} = \{f_1, \dots, f_s\} \subset \mathcal{P}$. For $f \in \mathcal{P}$ of degree d let*

$$f = \sum_{k=0}^{d} w_k + \sum_{i=1}^{s} g_i f_i$$

as in above lemma. Then $\sum_{k=0}^{d} w_k$ is called a completely reduced form of f with respect to F. If $\mathcal{F}$ is an H-basis, then the completely reduced form of f is also called normal form of f, $NF(f, \mathcal{F})$ for short.

Consider the set $\{f_1, \dots, f_m\} \subset \mathcal{P}$ with only finitely many common roots in algebraically closed field $\mathbb{K}$. Let $I = \langle f_1, \dots, f_m \rangle$. So its variety $V(I)$ consists of only finitely many points. For an $f \in \mathcal{P}/I$ denoting $[f]$ the equivalence class

$$[f] = \left\{ g \in \mathcal{P} \mid g - f \in I \right\}$$

Since we can add elements of $\mathcal{P}/I$ and multiply by constant (the coset $[c]$ for $c \in \mathbb{K}$), $\mathcal{P}/I$ also has the structure of a vector space over the field $\mathbb{K}$.

Definition 4.6. *Given a polynomial $f \in \mathcal{P}$, we can define a map $m_f : \mathcal{P}/I \mapsto \mathcal{P}/I$ by*

$$m_f\big([g]\big) = [f][g] = [fg] \in \mathcal{P}/I.$$

Proposition 4.7. *Let $f \in \mathcal{P}$. Then*

(i) The map m_f is a linear operator.

(ii) For $f, g \in \mathcal{P}$, $m_f = m_g$ if and only if $f - g \in I$.

Proof.

(i) Let $f, g, h \in \mathcal{P}$ and $c \in \mathbb{K}$.

$$m_f\big(c[g] + [h]\big) = [f]\big(c[g] + [h]\big) = c[f][g] + [f][h] = cm_f\big([g]\big) + m_f\big([h]\big)$$

(ii) If $m_f = m_g$, then

$$[f] = [f][1] = m_f\big([1]\big) = m_g\big([1]\big) = [g][1]$$

So $f - g \in I$. Conversely, if $f - g \in I$, then $[f] = [g]$, so $m_f = m_g$.

□

Proposition 4.8. *Let $f, g \in \mathcal{P}$. Then*

(i) $m_{f+g} = m_f + m_g$

(ii) $m_{f.g} = m_f(m_g)$.

Proof. Let $h \in \mathcal{P}$.

(i) $m_{f+g}[h] = [f+g][h] = [f][h] + [g][h] = m_f[h] + m_g[h]$.

(ii) $m_{f.g} = [fg][h] = [f][g][h] = [f]m_g[h] = m_f[m_g[h]]$.

□

Corollary 4.9. *Let $h \in \mathbb{K}[t]$ and $f \in P$. Then*

$$m_{h(f)} = h(m_f).$$

Lemma 4.10. *Let $S = \{p_1, \ldots, p_m\}$ be a finite subset of C^n. There exist polynomials $g_i \in C[x_1, \ldots, x_n]$, $i = 1, \ldots, m$, such that*

$$g_i(p_j) = \begin{cases} 0 & \text{if } i \neq j, \text{ and} \\ 1 & \text{if } i = j \end{cases}$$

Now we recall some concepts from linear algebra.

Definition 4.11. *Let A be an $n \times n$ matrix. A scalar λ is an eigenvalue of A if there is a nonzero column vector v such that $Av = \lambda v$. The the vector v is then an eigenvector of A corresponding to λ. For a scalar t, if we expand the determinant*

$$det(A - tI),$$

we obtain a polynomial expression $p(t)$ of degree n. That is, $p(t) = det(A - tI)$ which is called the characteristic polynomial of the matrix A. The eigenvalues of A are precisely the solution of the characteristic equation $p(t) = 0$.

Theorem 4.12. ***(Cayley-Hamilton Theorem)*** *Every square matrix A is a root of its characteristic polynomial.*

Theorem 4.13. *[3] Let $I \subset \mathcal{P}$ be a zero dimensional ideal and let $f \in \mathcal{P}$. Then, for $\lambda \in \mathbb{K}$, the followings are equivalent:*

(i) λ is an eigenvalue of the representation matrix of m_f.

(ii) λ is a value of the function f on $V(I)$.

Proof. $(i) \Longrightarrow (ii)$ Let λ be an eigenvalue of the representation matrix of m_f. Then there is a corresponding eigenvector $[z] \neq [0] \in \mathcal{P}/I$ such that $[f][z] = \lambda[z]$. This is equivalent to $[f-\lambda][z] = [0]$. Using contradiction method, suppose that λ is not a value of the function f on $V(I)$. That is, if $V(I) = \{P_1, \ldots, P_m\}$ then $f(P_i) \neq \lambda$ for all $i = 1, \ldots, m$. Let $g = f - \lambda$, so that $g(P_i) \neq 0$ for all i. By Lemma 4.10, there exists polynomial g_i such that

$$g_i(p_j) = \begin{cases} 0 & \text{if } i \neq j, \text{ and} \\ 1 & \text{if } i = j \end{cases}$$

Consider the polynomial

$$\widehat{g} = \sum_{i=1}^{m} \frac{1}{g(P_i)} g_i$$

It follows that $\widehat{g}(P_i)g(P_i) = 1$ for all i. Hence $1 - \widehat{g}g \in I\big(V(I)\big)$. By the Hilbert's Nullstellensatz, $(1-\widehat{g}g)^l \in I$ for some $l \geq 1$. Using Binomial Theorem,

$$\begin{aligned} (1-\widehat{g}g)^l &= \sum_{j=0}^{l} 1^l(-\widehat{g}g)^{l-j} \\ &= 1 - \sum_{j=0}^{l-1}(-\widehat{g}g)^{l-j} \\ &= 1 - \widehat{g}g \end{aligned}$$

for some $\widehat{g} \in \mathcal{P}$. Therefore $1 - \widehat{g}g \in I$ implies $[\widehat{g}g] = [1]$. So $[g]$ has inverse in $\mathcal{P}/I$. Then

$$[g][z] = [f-\lambda][z] = [0]$$

Multiply by $[\widehat{g}]$, $[z] = [0]$ which is a contradiction. $(ii) \Longrightarrow (i)$ Let $\lambda = f(p)$ for some $p \in V(I)$ and let $q(t)$ be characteristic polynomial of representation matrix of m_f. By

Cayley-Hamilton Theorem $q(m_f) = [0]$. By Corollary 4.9

$$m_{q(f)} = q(m_f) = [0].$$

Hence $q([f]) = [0]$ this implies $q(f) \in I$. Since $p \in V(I)$, $q\big(f(p)\big) = q(\lambda) = 0$. That means λ is an eigenvalue. □

When we apply Theorem 4.13 , with $f = x_i$, we obtain the following corollary.

Corollary 4.14. *Let $I \subseteq \mathcal{P}$ be a zero dimensional ideal. Then $\lambda \in \mathbb{K}$ is an eigenvalue of the representation matrix of m_{x_i} if and only if λ is x_i-coordinate of some point $P \in V(I)$. Hence, if $I = \langle f_1, \ldots, f_s\rangle$ be a zero dimensional ideal, we can solve the polynomial system*

$$f_1 = 0$$

$$f_s = 0$$

as follows: First we find a basis of $\mathcal{P}/I$ as collection of bases of W_d's. Secondly, find the representation matrix of m_{x_i} with respect to this basis for $i = 1, \ldots, n$. Then, the eigenvalues of these matrix gives us the x_i-coordinates of points of $V(I)$.

Example 4.15. *We try to solve the following system*

$$\begin{aligned} x^2 + y^2 + z^2 &= 4 \\ x^2 + 2y^2 &= 5 \\ xz &= 1 \end{aligned}$$

Let $I = \langle f_1 = x^2 + y^2 + z^2 - 4, f_2 = x^2 + 2y^2 - 5, f_3 = xz - 1\rangle$. It is easy to see that $\{f_1, f_2, f_3\}$ is an H-basis for I. Then the basis of W_0 is 1 and the basis of W_1 is $\{x, y, z\}$.

Let us find the basis of W_2. A spanning of V_2 is

$$\Big\{g_1 = LF(f_1) = x^2 + y^2 + z^2, g_2 = LF(f_2) = x^2 + 2y^2, g_3 = LF(f_3) = xz\Big\}.$$

These are the corresponding vectors $\Big\{(1,0,0,1,0,1),(1,0,0,2,0,0),(0,0,1,0,0,0)\Big\}$. *Let* $(v_1, v_2, v_3, v_4, v_5, v_6)$ *be the element of* W_2. *Then*

$$\begin{aligned}
\langle(1,0,0,1,0,1),(v_1,v_2,v_3,v_4,v_5,v_6)\rangle &= v_1 + v_4 + v_6 = 0\\
\langle(1,0,0,2,0,0),(v_1,v_2,v_3,v_4,v_5,v_6)\rangle &= v_1 + 2v_4 = 0\\
\langle(0,0,1,0,0,0),(v_1,v_2,v_3,v_4,v_5,v_6)\rangle &= v_3 = 0
\end{aligned}$$

solving the system we found $v_1 = -2v_4, v_6 = v_4, v_3 = 0$. *Taking* v_2, v_4, v_5 *as free variables we found the basis* $\Big\{(0,1,0,0,0,0),(0,0,0,0,1,0),(-2,0,0,1,0,1)\Big\}$. *So a basis of* W_2 *in* $\mathcal{P}_2^H$ *is* $\{xy, yz, -2x^2 + y^2 + z^2\}$. *A spanning of* V_3 *is*

$$\{xg_1, xg_2, xg_3, yg_1, yg_2, yg_3, zg_1, zg_2, zg_3\}.$$

Their corresponding vectors are

$$\begin{aligned}
u_1 &= (1,0,0,1,0,1,0,0,0,0)\\
u_2 &= (1,0,0,2,0,0,0,0,0,0)\\
u_3 &= (0,0,1,0,0,0,0,0,0,0)\\
u_4 &= (0,1,0,0,0,0,1,0,1,0)\\
u_5 &= (0,1,0,0,0,0,2,0,0,0)\\
u_6 &= (0,0,0,0,1,0,0,0,0,0)\\
u_7 &= (0,0,1,0,0,0,0,1,0,1)\\
u_8 &= (0,0,1,0,0,0,0,2,0,0)\\
u_9 &= (0,0,0,0,0,1,0,0,0,0)
\end{aligned}$$

Let $v = (v_1, \ldots, v_{10}) \in W_3$*. Then we get this system* $\langle u_i, v\rangle = 0$ *for* $i = 1, \ldots, 10$*. Solving system we found the basis* $\{(0, -2, 0, 0, 0, 0, 1, 0, 1, 0)\}$*. So the corresponding basis of* W_3 *is* $\{-2x^2y + y^3 + yz^2\}$*. One can easily see that* $V_d = P_d^H$ *for* $d > 3$*. Hence a basis for* W *is*

$$\{h_1 = 1, h_2 = x, h_3 = y, h_4 = z, h_5 = xy, h_6 = yz, h_7 = -2x^2+y^2+z^2, h_8 = -2x^2y+y^3+yz^2\}$$

Now, we will find the representation matrix of m_x *with respect to this basis.*

$x.1 = x = h_2,$

$x.x = x^2 = \frac{1}{3}f_1 + \frac{4}{3}h_1 - \frac{1}{3}h_7,$

$x.y = xy = h_5,$

$x.z = xz = f_3 + h_1,$

$x.xy = x^2y = \frac{1}{3}yf_1 + \frac{4}{3}h_3 - \frac{1}{3}h_8,$

$x.yz = xyz = yf_3 + h_3,$

$x.(-2x^2 + y^2 + z^2) = -2x^3 + xy^2 + xz^2 = -5xf_1 + 3xf_2 + 6zf_3 - 5h_2 + 6h_4,$ *and*

$x.(-2x^y + y^3 + yz^2) = -2x^3y + xy^3 + xyz^2 = -5xyf_1 + 3xyf_2 + 6yzf_3 - 5h_5 + 6h_6.$

Therefore

$$[m_x] = \begin{pmatrix} 0 & \frac{4}{3} & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & -5 & 0 \\ 0 & 0 & 0 & 0 & \frac{4}{3} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 \\ 0 & -\frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{3} & 0 & 0 & 0 \end{pmatrix}.$$

After some calculations, one can find the eigenvalues of $[m_x]$ *are*

$$\{-\sqrt{2}, -\sqrt{2}, \sqrt{2}, \sqrt{2}, -1, -1, 1, 1\}.$$

Similarly we find

$$[m_y] = \begin{pmatrix} 0 & 0 & \frac{11}{6} & 0 & 0 & 0 & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 1 & -\frac{1}{2} & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \frac{5}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & \\ 0 & 0 & \frac{1}{6} & 0 & 0 & 0 & 0 & \frac{5}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

and corresponding the eigenvalues of $[m_y]$ *are*

$$\{-\sqrt{2}, -\sqrt{2}, \sqrt{2}, \sqrt{2}, -\sqrt{\frac{3}{2}}, -\sqrt{\frac{3}{2}}, \sqrt{\frac{3}{2}}, \sqrt{\frac{3}{2}}\}.$$

Finally,

$$[m_z] = \begin{pmatrix} 0 & 1 & 0 & \frac{5}{6} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 1 & \frac{5}{6} & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & \frac{1}{6} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{6} & 0 & 0 \end{pmatrix}.$$

and the eigenvalues of $[m_z]$ *are*

$$\{-1,-1,1,1,-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\}.$$

After compare the coordinates, the solution set of the system is

$$\{(1,\pm\sqrt{2},1),(-1,\pm\sqrt{2},-1),(\sqrt{2},\pm\sqrt{\frac{3}{2}},\frac{1}{\sqrt{2}}),(-\sqrt{2},\pm\sqrt{\frac{3}{2}},-\frac{1}{\sqrt{2}})\}.$$

4.2 Interpolation Problem

Lemma 4.16. *Assume that* I *is a zero dimensional ideal and that the values* $f(p)$ *are distinct for* $p \in V(I)$. *We denote a basis of vector space* $\mathcal{P}/I$ *by* $B = \{h_1, \ldots, h_m\}$.

(i) If $p \in V(I)$, *Lemma 4.10 gives us* g *such that* $g(p) = 1$ *and* $g(p') = 0$ *for all* $p' \neq p$ *in* $V(I)$. *Then* $[g] \in \mathcal{P}/I$ *is an eigenvector of representation matrix of* m_f *and that the corresponding eigenspace has dimension* 1. *Furthermore all eigenspaces of* $[m_f]$ *are of this form.*

(ii) If $v = (v_1, \ldots, v_m)$ *is an eigenvector of* $[m_f]$ *corresponding the eigenvalue* $f(p)$, *then the polynomial*

$$\widetilde{g} = v_1h_1 + \cdots + v_mh_m$$

satisfies $\widetilde{g}(p) \neq 0$ *and* $\widetilde{g}(p') = 0$ *for* $p' \neq p$ *in* $V(I)$.

(iii) The polynomial g *of part (i) is*

$$g = \frac{1}{\widetilde{g}(p)}\widetilde{g}.$$

(iv) Given $V(I) = \{p_1, \ldots, p_m\}$ *and the corresponding eigenvectors of of* m_f, *we get polynomials* $g_1, \ldots, g_m$ *such that* $g_i(p_j) = 1$ *if* $i = j$ *and* 0 *otherwise.*

Definition 4.17. *The interpolation problem asks to find a polynomial* h *which takes preassigned values* $\lambda_1, \ldots, \lambda_m$ *at the points* $p_1, \ldots, p_m$. *This means* $h(p_i) = \lambda_i$ *for all* i.

Theorem 4.18. *Assume that I is a zero dimensional radical ideal such that $V(I) = \{p_1, \ldots, p_m\}$. By Lemma 4.16, there exist polynomials g_i for $i = 1, \ldots, m\}$ such that*

$$g_i(p_j) = \begin{cases} 0 & \text{if } i \neq j, \text{ and} \\ 1 & \text{if } i = j \end{cases}$$

The polynomial h such that $h(p_i) = \lambda_i$ for $i = 1, \ldots, m$ is given by

$$h = \lambda_1 g_1 + \cdots + \lambda_m h_m.$$

Hence to solve interpolation problem, first we need a polynomial f such that $f(p_i)$ is distinct for each $i = 1, \ldots, m$. Then we have to find eigenvectors of m_f and form the polynomial h as in the previous theorem.

Example 4.19. *Let us take the result of Example 4.15 as our starting point. So assume*

$$P = \Big\{p_1 = (1, \sqrt{2}, 1), p_2 = (1, -\sqrt{2}, 1), p_3 = (-1, \sqrt{2}, -1),$$

$$p_4 = (-1, -\sqrt{2}, -1), p_5 = (\sqrt{2}, \sqrt{\frac{3}{2}}, \frac{1}{\sqrt{2}}), p_6 = (\sqrt{2}, -\sqrt{\frac{3}{2}}, \frac{1}{\sqrt{2}}),$$

$$p_7 = (-\sqrt{2}, \sqrt{\frac{3}{2}}, -\frac{1}{\sqrt{2}}), p_8 = (-\sqrt{2}, -\sqrt{\frac{3}{2}}, -\frac{1}{\sqrt{2}})\Big\}.$$

We try to find a polynomial h such that

$$h(p_1) = \frac{1}{\sqrt{2}}, h(p_2) = -\frac{1}{\sqrt{2}}, h(p_3) = 1, h(p_4) = -1,$$

$$h(p_5) = -1, h(p_6) = 1, h(p_7) = -\frac{1}{\sqrt{2}}, h(p_8) = \frac{1}{\sqrt{2}}$$

Let $f(x, y, z) = x + y + z$. It easy to see that f takes different values for each p_i. If $I = \langle f_1 = x^2 + y^2 + z^2 - 4, f_2 = x^2 + 2y^2 - 5, f_3 = xz - 1\rangle$, clearly $V(I) = P$. We *found that Example 4.15 a basis for $\mathcal{P}/I$ is*

$$\{h_1 = 1, h_2 = x, h_3 = y, h_4 = z, h_5 = xy, h_6 = yz, h_7 = -2x^2+y^2+z^2, h_8 = -2x^2y+y^3+yz^2\}.$$

Now, we will find the representation matrix of m_f with respect to this basis.

$1.f = x + y + z = h_2 + h_3 + h_4,$

$x.f = x^2 + xy + xz = \frac{7}{3}h_1 + h_5 - \frac{1}{3}h_7,$

$y.f = xy + y^2 + yz = \frac{11}{6}h_1 + h_5 + h_6 + \frac{1}{6}h_7,$

$z.f = xz + yz + z^2 = \frac{11}{6}h_1 + h_6 + \frac{1}{6}h_7,$

$xy.f = x^2y + xy^2 + xyz = h_2 + \frac{7}{3}h_3 + h_4 - \frac{1}{3}h_8,$

$yz.f = xyz + y^2z + yz^2 = -\frac{1}{2}h_2 + \frac{11}{6}h_3 + \frac{5}{2}h_4 + \frac{1}{6}h_8,$

$(-2x^2 + y^2 + z^2).f = -8h_2 + 10h_4 + h_8,$

$(-2x^2y + y^3 + yz^2).f = \frac{1}{6}h_1 - 8h_5 + 10h_6 + \frac{5}{6}h_7$

Therefore

$$[m_f] = \begin{pmatrix} 0 & \frac{7}{3} & \frac{11}{6} & \frac{11}{6} & 0 & 0 & 0 & \frac{1}{3} \\ 1 & 0 & 0 & 0 & 1 & -\frac{1}{2} & -8 & 0 \\ 1 & 0 & 0 & 0 & \frac{7}{3} & \frac{11}{6} & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & \frac{5}{2} & 10 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & -8 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 10 \\ 0 & -\frac{1}{3} & \frac{1}{6} & \frac{1}{6} & 0 & 0 & 0 & \frac{5}{3} \\ 0 & 0 & 0 & 0 & -\frac{1}{3} & \frac{1}{6} & 1 & 0 \end{pmatrix}$$

After some calculations, one can find the eigenvectors of $[m_f]$ are

$$v_1 = (-2\sqrt{2}, -3\sqrt{2}, 2, 6\sqrt{2}, 3, -6, -\sqrt{2}, 1)^T,$$

$$v_2 = (2\sqrt{2}, -3\sqrt{2}, 2, 6\sqrt{2}, -3, 6, \sqrt{2}, 1)^T,$$

$$v_3 = (\frac{\sqrt{6}}{2}, -3\sqrt{3}, -1, 3\sqrt{3}, 3\sqrt{2}, -3\sqrt{2}, -\frac{\sqrt{6}}{2}, 1)^T,$$

$$v_4 = (-\frac{\sqrt{6}}{2}, -3\sqrt{3}, -1, 3\sqrt{3}, -3\sqrt{2}, 3\sqrt{2}, \frac{\sqrt{6}}{2}, 1)^T,$$

$$v_5 = (\frac{\sqrt{6}}{2}, 3\sqrt{3}, -1, -3\sqrt{3}, -3\sqrt{2}, 3\sqrt{2}, -\frac{\sqrt{6}}{2}, 1)^T,$$

$$v_6 = (-\frac{\sqrt{6}}{2}, 3\sqrt{3}, -1, -3\sqrt{3}, 3\sqrt{2}, -3\sqrt{2}, \frac{\sqrt{6}}{2}, 1)^T,$$

$$v_7 = (2\sqrt{2}, 3\sqrt{2}, 2, -6\sqrt{2}, 3, -6, \sqrt{2}, 1)^T,$$

$$v_8 = (-2\sqrt{2}, 3\sqrt{2}, 2, -6\sqrt{2}, -3, 6, -\sqrt{2}, 1)^T.$$

Then

$$\frac{1}{\sqrt{2}}v_1 - \frac{1}{\sqrt{2}}v_2 + v_3 - v_4 - v_5 + v_6 - \frac{1}{\sqrt{2}}v7 + \frac{1}{\sqrt{2}}v8 = \begin{pmatrix} -8 \\ 0 \\ 0 \\ 0 \\ 12\sqrt{2} \\ -12\sqrt{2} \\ -4 \\ 0 \end{pmatrix}$$

Hence

$$h = -8h_1 + 12\sqrt{2}h_5 - 12\sqrt{2}h_6 - 4h_7 = 8x^2 - 4y^2 - 4z^2 - 12\sqrt{2}yz + 12\sqrt{2}xy - 8$$

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