THE REPUBLIC OF TURKEY

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A TEST OF BLACK-LITTERMAN PORTFOLIO OPTIZATION ; EVIDENCES FROM BIST

Master's Thesis

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ABSTRACT

A TEST OF BLACK-LITTERMAN PORTFOLIO OPTIMIZATION;
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Mirzazadeh Barijough

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Harry Markowitz published his Nobel wining article Portfolio Selection in 1952 which is usually seen as genesis of Modern Portfolio Theory. Previous to his works investors were selecting asset portfolios just by their feelings on stock markets return, typically thought of an optimal portfolio as the one who maximized expected return. Markowitz developed a quantitative method to help investment managers to find out optimum weight of assets in portfolio by considering Expected Return and Risk of an investment. This classic Mean-Variance optimal portfolio selection is the foundation of Modern Portfolio Theory. Although Markowitz approach is very popular among investors and fundamentally significant, however since mean-variance optimization in very sensitive to expected returns, and expected returns are very difficult to estimate, the resulting portfolios are unbalanced in most of the cases. It may be the case that an investor wants to impose his or her views depending on a present news. Fisher Black & Robert Litterman were studying on a model to combine historical data and investor's point of view. Their research published by Goldman Sachs & Company in 1991 as Black-Litterman model.

This study tests these two approaches of asset allocation. First, a detailed description of and CAPM and pitfalls of Mean-Variance model are given. Next we discuss the needs to better model and an overview of Black & Litterman model will be given. Finally, we compare performance of these two models with an empirical test in Istanbul Stock & Exchange (BIST) stocks. Views vector for Black-Litterman model estimated by EGARCH-mean equation in univariate context.

 Keywords: Portfolio Optimzation, Mean-Variance, Black-Litterman, EGARCH, BIST

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ABBREVIATION

MV :   Mean-Variance
BL :   Black-Litterman
GARCH :   General Autoregressive Conditional Heteroskedasticity
EGARCH :   Exponential GARCH

SYMBOLS

Return of asset i	:	r_i
Variance of asset i	:	σ_i^2
Standard deviation – Volatility of asset i	:	σ_i
Covariance between assets i and j	:	$Cov(i,j)$
Correlation coefficient of assets i and j	:	$\rho_{i,j}$
Risk Aversion	:	δ
Risk-Free Rate	:	r_f

1. INTRODUCTION

One of the major concerns in investment literature is to find out the best way to allocate the assets. All investors concerned about how to select optimal portfolio that fulfill the investment objectives over the investment horizon. Asset allocation is a complex process for couple of reasons. Great number of opportunities to invest-in and inability to prognosticate the future are obstacles of this process. Aside from the many different investment opportunities that are available, nowadays information technology and world wide connections make it easy to invest whether in domestic opportunities or in an international project. For instance it is possible to invest European Markets and Emerging countries simultaneously. Investment is always a risky concept. Initial Investment is made for certain amount of money, but it is never certain about value of return in future. Furthermore, it is impossible to forecast future financial and economic events with certainty. These two problems make it difficult to have optimum and certain asset allocation.

Fundamentally there are two methods in portfolio selection, heuristic and quantitative. In heuristic method, asset allocation is made just with investor feeling and point of views about future performance of the investment that he or she collected from the news and media. This type rarely relies on a model. On the other hand, quantitative approaches apply a mathematical model in portfolio selection process. The model evaluates the investments and determines which one should select in process.

Due to the scarcity of resources as one the basic rules of the economics, all economic and also financial decisions are made in the context of trade-off. The main quantitative model provided by Harry Markowitz (1952). He recommends that in asset allocation process should not only look at the possible pay-off of the investment, but also take into account how certain one is that this payoff will actually be acquired. Markowitz identified the trade-off facing the investor: risk versus expected return. The investment decision is not necessarily which securities to buy, but how to divide the wealth amongst securities. In that article and subsequent works, Markowitz extended the techniques of linear programming to develop the critical line algorithm. The critical line algorithm identifies all feasible portfolios that minimize risk, as measured by variance or standard deviation, for a given level of expected return or maximize expected return

for a given level of risk. When graphed in standard deviation versus expected return space, these portfolios form the efficient frontier. The efficient frontier represents the trade-off between risk and expected return faced by an investor when forming his or her portfolio. The intuition behind the efficient frontier represents well-diversified portfolios. By this reason diversification plays an important role for achieving risk reduction in portfolio concept. Therefore, mean-variance model gives precise analysis of mathematical meaning to the common proverb "Don't put all of your eggs in one basket." Markowitz developed mean-variance analysis in the context of selecting a portfolio of common stocks. Over the last decades, mean-variance analysis enormously used in applied asset allocation. By formulating agued issues he proposed the Mean-Variance optimization process. Mean-variance model requires not only knowledge of the expected return and standard deviation on each asset, but also the correlation of returns for each and every pair of assets. Whereas a stock portfolio selection problem might involve hundred of stocks and then thousands of correlations, an asset allocation problem typically involves a bunch of asset classes i.e. stocks, bonds, real state and derivatives. Furthermore, the opportunity to reduce total portfolio risk comes from the lack of correlation across assets.

One might expect mean-variance model currently plays a dominant role in assets allocation process. However, this is not happening completely. Although this model inspired a rich field of science and used by many, it has couple of weak points. Firstly, the utility maximization process of Markowitz is highly sensitive to the input data set. Small changes in the estimated returns or volatilities can result in drastic changes in the final allocation. Secondly, the investors may have their estimations about future returns in hand and want to impose them in the optimization process. An investor may have different subjective views for some of the assets in addition to estimations coming from a quantitative model. He or she wants to merge the feelings with the estimated return came from quantitative model.

For more explanation, the input values for this model are estimated and the optimization procedure assumes that they are true specifications of assets. However, future returns are random variables and their true values differ from their expected values. Commonly, mean-variance optimization process results in extreme short sale positions or minus weights. However if portfolio weights are bounded between zero and one, majority of

assets take the zero weights and only a few will be incorporated into the optimal portfolio. Furthermore, if the parameters estimated correctly, the resulting portfolio weights obviously lead to the highest preference level. In contrast, if the parameters deviate from the forecast, the poorly diversified portfolio could achieve a poor preference level. By the same way, the optimal weight vector is very sensitive towards input parameters. Small changes in expected returns can result in large variations of portfolio weights.

These deficiencies encountered when using the Mean-Variance optimization in the real investment situation, motivated Black Fisher and Litterman Robert to develop better models on asset allocation. Black and Litterman while analyzing investment portfolios in Goldman Sachs, provided a model to combine Heuristic approach & Quantitative approach (1990). The first publication on the model was in 1990, and subsequently in 1991, 1992. Their proposed model allows investor to incorporate his or her feelings and views on optimization process. The difference between two main models, is rather than applying investors expected returns of all assets into the formula of the Markowitz Model to get optimal portfolio directly, the Black-Litterman Model defines a view portfolio, which specifies investors expected returns of some assets and the degree of confidence in each view. Based on this view portfolio, market equilibrium returns are adjusted to express views of investors. Instead of the view portfolio, the adjusted market equilibrium returns -the new Combined Return in Black-Litterman formula- is applied into the Markowitz Model to get optimal portfolio. The core idea of their model is to use market equilibrium as a neutral reference and then adjust equilibrium values in accordance with an investor's views to get the optimal portfolio for the investor. They simply use the CAPM equilibrium as the initial reference point and blend this prior information with the subjective analyst views in accordance with the confidence level of the investor about these views.

The traditional manager may form views from news about performance of companies, markets, interest rates and currencies. A view could be that Food industry will outperform Automobile industry. More over the view may also be that the Turkish companies will outperform Swedish companies or Arçelik will outperform Vestel. Using these views without a quantitative model, may have limited help in portfolio

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construction. Usually traditional investors, with lack of mathematic background, are reluctant to use quantitative models, as they feel that techniques of mean-variance analysis and related procedures do not produce value-added effectively. In contrast, the quantitative investor just uses the extreme mathematical methods to capture more return from portfolio. The Black-Litterman model integrates these two approaches and allows traditional investors to express their views and forecasts to consider in a quantitative model, to produce optimum portfolio with both viewpoints.

Black and Litterman (1990) (1991) emphasized on two strengths of their approach: First, they believe that the subjective views of the investors can be easily incorporated in the portfolio construction process. Secondly, their mean-variance optimization does not generate unreasonable solutions, as the original mean-variance framework does. The former point emerges from the feature of the model that investors' subjective views are expressed as linear combinations of expected returns of assets, rather than as expected returns of individual assets. That is, the subjective view need not be an exact value of the expected return of an individual asset, but rather can be expressed as the expected return of two assets or more in relation to each other. This type of formulation is easier for investors to apply. The later point comes from the feature that the investors' subjective views are bended with an equilibrium model that tilts the portfolio weights away from the market capitalization weights based on the relative uncertainty in the investor's views.

Despite numerous advantages of Black-Litterman model, which overcome drawbacks of the original Mean-Variance optimization, specifications of inputs were less straightforward. In other words, the initial publications were written on intuition and proof of formulas on their proposed model. Scowcroft & Satchell (2000) and Idzorek (2004), published their studies on demystification and step-by-step guide to apply Black-Litterman model in real world portfolios.

The Black-Litterman model provides the flexibility of combining the market equilibrium with additional market views of the investor. In the Black-Litterman model, the user inputs any number of views or statements about the expected returns of arbitrary portfolios, and the model combines the views with equilibrium, producing both the set of expected returns of assets as well as the optimal portfolio weights. In practice,

Black-Litterman model requires two sets of data to input. The first set of comes from CAPM equilibrium expected returns and historical variance-covariance, same is used in Original Mean-Variance optimization. The second one and the most significant set used in application of that model is to construct a vector of views on assets' expected rate of return.

The analyst may have views on all of the assets or construct a vector of views for selected assets. Investment analysts track the stock's behavior, markets, general economy criteria, news etc. and apply the realities in fundamental and technical analysis to make prognostication on future returns. In contrary, the model also could be used with quantitative forecast of assets either. Initially, Beach and Orlov (2007) suggested using GARCH models to create inputs for Black-Litterman model. Martellini and Ziemann (2010) include non-normally distributed returns and consider fat tails, to apply the model in special financial instruments i.e. hedge funds and derivatives.

The purpose of the research is to test the performance of the portfolios obtained from the Original Mean-Variance and Black Litterman models over a 5 year period. For former model it uses utility function for building optimum portfolio weights. For later model, it applies reverse optimization to construct the initial equilibrium returns as a neutral starting point, then uses estimation of the variance matrix for views as described in Walters (2007).

One of the features of this study, is the use of EGARCH derived views as proxies for investor views for the Black-Litterman model. Similarly, as Beach and Orlov (2007), however, forecasting the return for each asset has done individually. EGARCH-mean forecasting applied for each of the stocks in univariate context. The benefit of employing this model is that more objective views are obtained, i.e. they are not dependent on the subjective projections of the portfolio manager. In addition, GARCH type models are able to capture characteristics of stock returns. The outcome of this paper should be interesting both for private portfolio managers and institutions participating in Turkish market, and who wants to use an objective methodology for predictions of future returns and volatilities.

This study investigates the performance of portfolios optimized with original MV and

BL model and explores the behavior of Black-Litterman's results for different valued of Factor Tau, which is used to scale the investors uncertainty in their prior estimate of the returns.

As outline of this thesis, the first chapter gives a theoretical background and notation that will be used. In second chapter it moves to the basis of portfolio selection with the model of Markowitz and the subsequently developed capital asset pricing model. Furthermore, it will discuss the focus of this thesis: the Black-Litterman model. Next chapter will review related recent literatures. Afterward data specification and methodology of this thesis will be described in forth chapter. Chapter five will discuss empirical findings and results. Eventually conclusion and further studies will be included in the last chapter.

2. THEORETICAL FRAMEWORK

The study of asset allocation process has its own vocabulary. The most important vocabulary and concepts and notation will be discussed in this chapter. In addition, background of two portfolio optimization models will be explained later.

2.1 ASSET PROPERTIES

Terms used in this part are derived from Dictionary of Finance and Banking. (2008)

2.1.1 Classes

An investor can choose from wide variety of different assets. These opportunities can be divided in classes of assets with the same characteristics. The most well known asset class is equity. Equity or stock is the ownership of a part of a company. Equity of public companies can be traded on a stock exchange. Another asset class is fixed-income securities, also known as bonds. A bond is debt investment in which an investor loans money to an entity that borrows the funds for a defined period of time at a fixed or variable interest rate. Bonds used by companies, municipalities, governments to finance variety of projects and activities. Bonds, as every investment, vary in the degree of risk attached to them. The length of the borrowing period and the entity that issues the bonds are important risk factors. Short-term government bonds are generally regarded as very safe investments and generally referred to risk-free investment. The final class under consideration is cash equivalents. It derives from investing in foreign currency, either to bet on a change in the exchange rate or to insure, or hedge, investments in that currency against changes in the exchange rate.

2.1.2 Expected Return And Risk

The motivation behind investment in assets is to get profit in future. The ratio of profit or value-added to initial investment is known as the rate of return. Return defined for a period in the past, but in portfolio selection investor interested in the future behavior of an asset. Markowitz (1959) agues the future or forecasted return as expected value of the return. If r_t known as the return of asset up to time t, Expected Return or $E(r)$ is shorthand of $E(r_{t+1}|I_t)$ which means the forecast of the return at time $t+1$ given all

information up to and including time t. The expected return is on of two important characteristics of an asset for mean-variance optimization. Risk is the next important property of an asset for evaluation in portfolio selection process. By definition, risk is the chance to loose on an investment. Markowitz (1952) defined the risk quantitatively by variance of return over a period. Variance measures deviation around a point, so in case of investment deviation of return around expected return. Therefore in mean variance optimization process for an asset, investor should consider its Expected Return and its Risk, measured in variance.

Expected Value: The expected value also known as population mean and defined as the average value of sample and generally denoted by μ.

$$E(x) = \mu = \frac{1}{n} \sum_{i=1}^{n} x_i$$

Variance : The variance is a measure of how much variable varies around expected value μ. Variance is denoted by σ^2.

$$\sigma^2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - \mu)^2$$

Standard Deviation: The standard deviation (σ), is square root of the variance, in finance it is often called volatility. Since standard deviation is directly related to normal distribution, it is more intuitive measure of variability than variance.

Covariance The covariance provides a measure of the degree to which returns on two risky assets move in tandem. A positive covariance means that asset returns move together. A negative covariance means returns move inversely. Covariance between asset i and j is denoted by $Cov(i,j)$.

$$Cov(i,j) = \frac{1}{n} \sum_{i,j=1}^{n} (x_i - \mu_i)(x_j - \mu_j)$$

Correlation Coefficient In finance, a measure of two securities returns determines the degree to which two variable's movements are associated. It varies from -1 to $+1$ and denoted by $\rho_{i,j}$. -1 shows perfect negative correlation and $+1$ indicates perfect positive correlation.

$$\rho_{i,j} = \frac{cov(i,j)}{\sigma_x \sigma_y}$$

Normal Distribution In finance, it is assumed that the distribution of the asset returns have normal distribution, e.g. by Black & Litterman (1991)

Figure 2.1 Normal Distribution
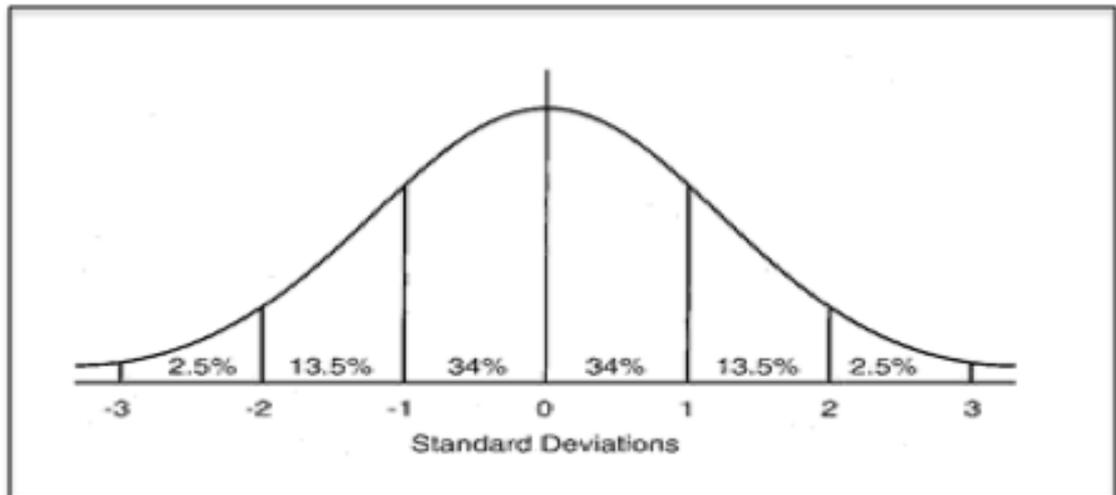
2.1.3 Portfolio of Assets

Portfolio consists a group of financial assets such as equities, bonds and cash equivalents held directly by investors. Proportion of these assets in portfolio called weight. A portfolio consisting of n assets, is represented mathematically by a vector $\mathbf{w} \in \mathbb{R}^n$. It has to sum up to one.

$$\sum_{i=1}^{n} w_i = 1$$

It could be possible to have negative weights in portfolio, since the investor can borrow money/stock or take short position in a specific asset. Borrowing makes sense if the investor forecasts price depreciation for an asset. The concept of risk and return in a portfolio can be demonstrated in mathematical way. By assuming r_i as return for asset i the expected return becomes $E(r_i)$ and σ_i^2 , $\sigma_{i,j}$ represent the variance of i and covariance between asset i and j respectively. For a portfolio that consists of n assets, the return of each asset in portfolio captured by the vector of returns. The vector of returns also has an expected value, $E(r) \in \mathbb{R}^n$. The covariance and the variance of the assets in the portfolio are represented in a symmetric covariance matrix $\Sigma \in \mathbb{R}^{n \times n}$ the diagonal entries of which are formed by the variance of the assets ($\sigma_{i,i} = \sigma_i^2$) as this is the covariance of an asset with itself. Return of portfolio is the weighted average of the asset returns.

The Expected return and covariance of a portfolio is determined:

$$E(r_p) = E\left(\sum_{i=1}^{n} w_i r_i\right) = \sum_{i=1}^{n} w_i E(r_i) = \mathbf{w}' E(r)$$

$$var(r_p) = var\left(\sum_{i=1}^{n} w_i r_i\right) = \sum_{i=1}^{n}\sum_{j=1}^{n} cov(w_i r_i, w_j r_j) = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \sigma_{i,j}$$

$$= \mathbf{w}' \sum \mathbf{w}$$

The intuition behind diversifying assets in a portfolio is similar to English proverb "do not put all your eggs in a basket" however to probe mathematically suppose we select a portfolio of n assets with equal expected returns but mutually uncorrelated to each other, $\sigma_{i,j} = 0$. The portfolio will be constructed with equal weighting scheme, $w_i = \frac{1}{n}$.

$$E(r_p) = \sum_{i=1}^{n} w_i E(r_i) = \frac{1}{n}\sum_{i=1}^{n} E(r) = E(r)$$

Thus, the expected return of a portfolio in this equation in independent of the number of assets in the portfolio. However, the variance of portfolio return depends on the number

of assets:

$$var(r_p) = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \sigma_{i,j} = \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} \sigma_{i,j} = \frac{1}{n^2}\sum_{i=1}^{n} \sigma^2 = \frac{\sigma^2}{n}$$

Obviously by increasing the number of uncorrelated assets in a portfolio, the volatility of portfolio gets closer to Zero:

$$\lim_{n\to\infty} var(r_p) = \lim_{n\to\infty}\left(\frac{\sigma^2}{n}\right) = 0$$

2.2 THE MARKOWITZ MEAN-VARIANCE OPTIMIZATION

In contemporary history of finance Markowitz's work plays a significant role in investment. Mean-Variance model is still basis for quantitative asset allocation. This optimiztion holds for Black-Litterman model, as well. Therefore, there would be an in-depth study of Mean-Variance model. In this chapter we are going to discuss model development and mathematics of Mean-Variance. Capital Asset Pricing Model can be used to determine expected return of an asset. CAPM will be discussed, too.

2.2.1 Model Evolution

To develop an investment model there should be an idea of the way which investor select a portfolio. Although majority of investors follow the news and economic conditions to form views and make prediction on future performance of markets, sectors and specific companies, however, these views alone is not enough to do asset allocation. The better way to select portfolio is to use quantitative models. Quantitative models can guide investors to asset allocation. In addition to its inputs, like other quantitative models portfolio selection model needs an objective function.

To make an **objective function** we should consider that investor aims to make positive return on the investment. Therefore, expected return function is an objective function. To have maximized expected return portfolio we can simply invest in a single asset portfolio with the highest expected return. However, it is against the concept of

diversification. When the performance of portfolio solely depends on the one asset this makes expected return very risky. On the other hand, monitoring and measuring risk in portfolio selection is other part of objective function. Diversification leads the portfolio to more steady expected return and acceptable risk. An investor that only takes on additional risk in trade-off between risk and return for additional expected return is called risk averse investor. Risk aversion is taken to best description of human investment behavior.

Markowitz Harry Markowitz (1952), identified the forecasted return with expected return and risk with variance of return. He went on to suggest that the above objective is the one to strive for and developed a mathematical model for portfolio selection. The objective, in terms of variance becomes to minimize the variance of return for a certain level of expected return. The expected value is often called the mean value. Therefore, this kind of optimization is Mean-Variance (MV) optimization. He defines a portfolio that minimizes variance for a certain level of expected return, or equivalently maximizes variance for a certain level of variance an efficient portfolio (Markowitz, 1987).

Utility Theory basically categorizes the preferences and formalizes the principle of risk aversion. A utility function is a function $u: Z \rightarrow \mathbb{R}$ it is a non-decreasing, continuous function that captures the investors preferences. An investor will prefer portfolio P_1 to P_2 if the expected utility of portfolio P_1 is greater than the expected utility of portfolio P_2. The specific utility function used varies among individuals, depending on their individual risk tolerance and their individual financial environment. The simplest utility function is a linear one u(x) = x. An investor using this utility function ranks portfolios by their expected values, risk does not play a role. The linear utility function is said to be risk neutral since there is no trade off between risk and expected return in the order of preferences. There is wide range of utility functions, however in practice certain standard types are popular. The most commonly used utility functions are Exponential $u(x) = -exp(-ax)$ with $a > 0$, Logarithmic $u(x) = \log(x)$, Power $u(x) = bx^b$ for $b, 1$ and $b \neq 0$, and the Quadratic function $u(x) = x - bx^2$ for $b > 0$. Utility score of a portfolio not only uses the risk-return characteristics, but also it should take the investor's risk aversion into account. For simplicity, many practitioners such as CFA

institute (2012) professionals use $U = E(r) - \frac{1}{2}\delta\sigma^2$, as a general utility function. In this equation δ stands for the risk aversion coefficient. Every unit of return is rewarded while every unit of volatility is penalized by the negative sign depending of the degree of risk aversion of the investor.

Efficient Portfolio A set of portfolios with returns that are maximized for a given level risk or vice versa given that there is not any other portfolios with a higher mean and no higher variance or less variance and no less mean. Markowitz (1987) defines Efficient Portfolio as A set of portfolios with returns that are maximized for a given level risk or vice versa given that there is not any other portfolios with a higher mean and no higher variance or less variance and no less mean.

Figure 2.2 : Efficient Frontier
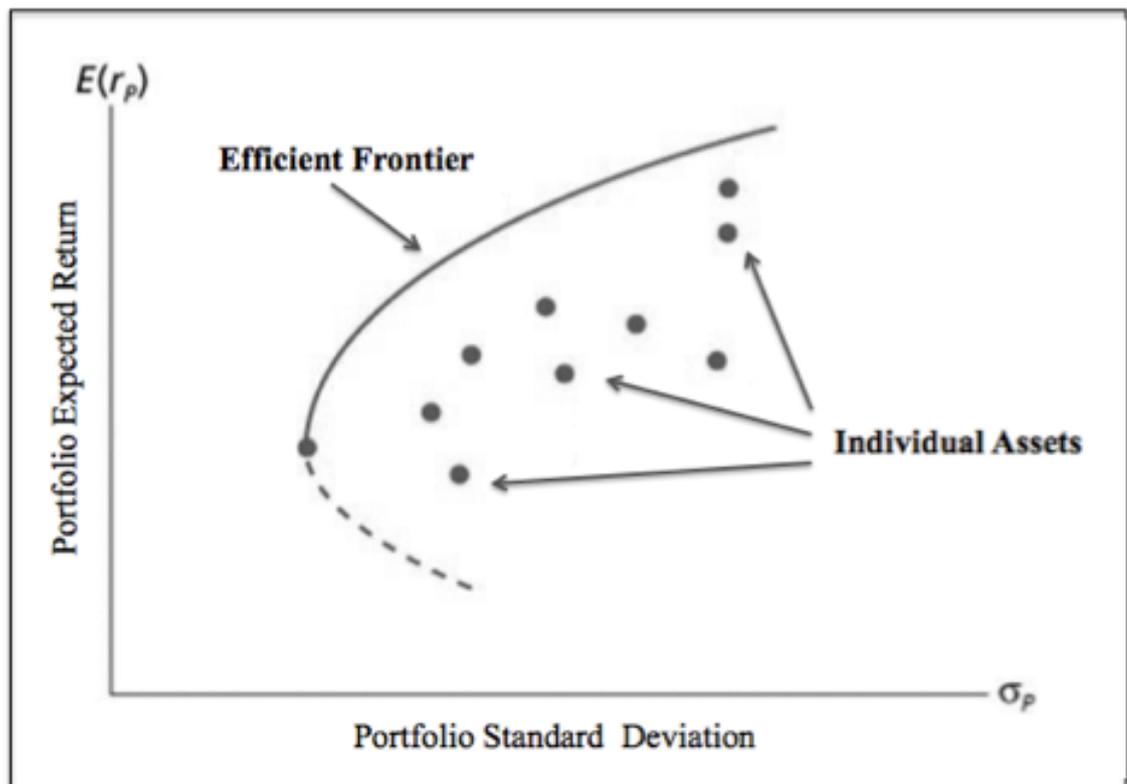
Following the construction of the efficient frontier, the favorite portfolio should be chosen based of investors risk aversion parameter, because different risk aversion levels

results in various indifference curves. That is to say, assorted investors may select different portfolios from the efficient frontier.

2.2.2 The Mathematics Of The Model

The main idea of mean-variance analysis has been explained in the previous parts. In general, the model should the model should minimize volatility of the portfolio for given level of expected return or maximize the expected return for a certain level of risk. In addition it must consider all weights add up to one. Hereby we review the general model of Markowitz Mean-Variance optimization problem.

$$Maximize \ \sum_{i=1}^{n} w_i r_i$$

$$Subject \ to \ \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \sigma_{ij} = \sigma$$

$$\sum_{i=1}^{n} w_i = 1$$

In practice, one may take utility function into account:

$$\underset{w}{\text{Max}} \ U = E\left(r_p\right) - \frac{1}{2} \delta \sigma_p^2$$

2.2.3 Drawbacks Of MV Optimization

2.2.3.1 Utility theory

The mean-variance criterion makes the exchange between risk and expected return explicit. The criterion states a preference for portfolios with a higher expected return relative to portfolios with a lower level of expected return (for the same level of risk). This seems a reasonable criterion for portfolio selection. However, care has to be taken in applying the criterion, since in some cases the criterion results in unlikely

preferences.

Hanoch and Levy (1969) analyzed this failure. They analyze preferences with the help of utility theory and subsequently compare these preferences with those obtained from mean-variance optimization. They conclude that in certain cases the preferences resulting from mean-variance optimization differ from those obtained by utility theory.

2.2.3.2 Normally distributed returns

Hanoch and Levy (1969) studied the question when the mean-variance criterion is a valid efficiency criterion for a risk avers investor. An efficiency criterion is said to be valid if it produces the same efficient set for all concave utility functions. The ranking of the elements in the efficient set still depends on the specific utility function. As Tobin (1958) already suspected that the mean-variance criterion is valid if and only if the distribution of the returns is of a two-parameter family. They concluded that the mean-variance criterion is optimal, when the distributions considered are all Gaussian normal. But the symmetric nature of this distribution seems to deny its usefulness as a good approximation to reality, for at least some types of risky portfolios. Even for symmetric distributions, the mean-variance criterion is not valid, when the distribution has more than two parameters.

2.2.3.3 Deficiencies of mean-variance optimization

The theoretical background of mean-variance optimization has been described, in which setting it makes good sense. However, when applying it to real live problems some flaws do arise.

In general diversification is thought of as a reasonable approach to spreading risk. Adding assets to a portfolio that are less than perfectly correlated to the assets already in the portfolio reduces the variance of the portfolio. Black and Litterman (1992) stated the Mean-variance optimization however, can result in portfolios with large long and short positions in only a few assets, which opposes the notion of diversification. If the parameters that are used in the optimization, like the vector of expected return and the covariance matrix, would be known with certainty, it would be reasonable to invest in such concentrated portfolios, but as the expected returns are just forecasts this seems a

very risky investment choice. Michaud (1989) concentrated portfolios are very counterintuitive, which is one of the reasons for the lack of popularity of using unconstrained mean-variance optimizers in making investment decisions

In A more practical problem is that the model requires input of expected return, variance and covariance of every asset under consideration. If an investor has 1000 assets in her portfolio; it becomes a very cumbersome task to give estimates for all the input parameters. There are solutions to this problem: historical data could be used to give an estimate of the expected return. But historical estimates are often bad predictors of future behavior (Black & Litterman, 1992). Another problem is how the investor should formulate her believes about future performance. Often an investor holds relative views on asset performance, for example that asset a will outperform asset b. Mean-variance analysis needs a specific estimate of the expected return of a single asset and cannot handle relative views.

Furthermore, The model is not robust. This implies that a small change in the values of the input parameters can cause a large change in the composition of the portfolio. The mean-variance model assumes that the input data is correct, without any estimation error. The model does not address this uncertainty and sets out to optimize the parameters as if they were certain. Michaud (1989) describes mean-variance optimizers even as estimation-error maximizer. Best and Grauer (1991) analyzed the behavior of the mean-variance optimizer under changes in the asset mean. They show that a small increase in an asset mean can cause a very different portfolio composition.

2.2.4 The Separation Theorem

Tobin (1958) proposed the separation theorem. He investigated the separation theorem based on presence of a risk free asset in portfolio selection process. The risk free asset makes it possible to draw a new efficient frontier that has a better risk-return balance. This is accomplished by forming a portfolio that consists of a combination of the risk-free asset and the tangency portfolio. In the presence of a risk free asset, the portfolio selection problem becomes a two-part problem. First the construction of an efficient frontier portfolio, and next the decision to combine this efficient portfolio to the desired risk-level by going long or short in the risk-free asset. The optimal allocation between

the efficient portfolio and the riskless asset depends on the investor's preference. The separation theorem plays an important role in the next development in modern portfolio theory, the development of the capital asset pricing model by Sharpe (1964), Lintner (1965), Mossin (1966) and Treynor (1961).

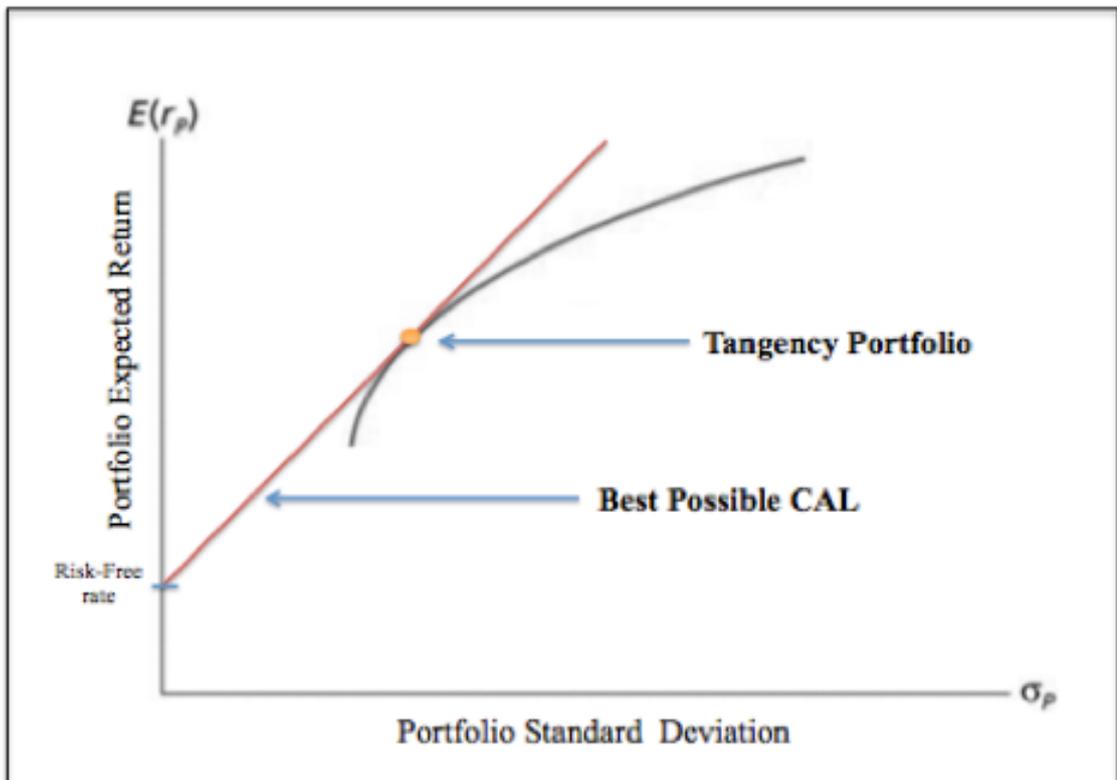$$E\left(r_{New\ Efficient}\right) = (1 - w_p)r_f + w_p\, r_p$$

$$\sigma^2_{New\ Efficient} = (1 - w_p)^2\sigma^2_{rf} + w^2_p\sigma^2_p + 2(1 - w_p)w_p cov\left(r_f, r_p\right)$$

$$\sigma^2_{rf} = cov\left(r_f, r_p\right) = 0$$

$$\sigma_{New\ Efficient} = w_p\sigma_p$$

Figuer 2.3 : Capital Allocation Line

Investors separate the investment and financing decision, leading to the name

Separation. All investors purchase the same portfolio of risky assets where the line from Risk-Free point touches the Efficient Frontier at the highest Expected Rate of Return possible. Because investors can borrow or lend at the risk-free rate, they move along the new line called, Capital Allocation Line. Alternatively, more risk-averse investors will lend funds and achieve a lower rate of return. And vice versa, less risk-averse investor will borrow fund and achieve a higher rate of return. But because of separation, each investor holds the same portfolio of risky assets.

2.3 CAPITAL ASSET PRICING MODEL

The Markowitz's study on portfolio selection became relevant with the publication of the Capital Asset Pricing Model (CAPM). William Sharpe (1964), John Lintner (1965), Jan Mossin (1966) and Jack Treynor (1961) worked separately on this theory. CAPM helps the mean-variance analysis of Markowitz to develop a model that can compute the expected return of an asset if equilibrium would exist in the market.

2.3.1 Assumptions

Luenberger (1998) states CAPM model is built based on some assumptions. Under these assumptions, equilibrium can be established in the market. This equilibrium is required to derive the pricing formula:

a) All investors use mean-variance analysis to select a portfolio.

b) All investors have homogeneous believes about the future return, variance and covariance of assets.

c) There is a unique risk free rate of borrowing and lending available for all investors.

2.3.2 Equilibrium

From Tobin (1958) separation theorem, it is known that everyone will invest in a single portfolio of risky assets. In addition, investors can borrow or lend at the risk free rate, to adjust the portfolio to the desired risk level. Since everyone uses the same means, variances and covariance, to determine the optimal portfolio, everyone will compile the same risky portfolio. Some investors will seek to avoid risk and will have a high percentage of the risk free asset in their portfolios. Other, who is more aggressive, will have a high percentage of the risky portfolio. However, every individual will form a

portfolio that is a mix of the risk free asset and the same risky portfolio.

If all investors purchase the same portfolio of risky assets, this portfolio leads to insight underlying the CAPM. As all investors share the same view, and at that moment there is only one optimal portfolio, it will result in a rising price of the assets in the optimal portfolio and hence a downward adjustment of the expected return. The opposite happens with the assets not in the portfolio. These price changes lead to a revision of the portfolios. And this goes on and on, until equilibrium is reached. In this equilibrium the optimal portfolio is the one that contains all assets proportional to their capitalization weights, that is the market portfolio. This means that the market portfolio is mean-variance efficient in equilibrium. The conclusion of the mean-variance approach and Tobin's separation theorem is that the optimal portfolio, in which everyone invests, must be the market portfolio ($\boldsymbol{w}_M$).

2.3.3 CAPM Formula

If the market portfolio M is mean-variance efficient, this equation holds for the expected return of an asset i satisfies the equilibrium:

$$E(r_i) - r_f = \frac{\sigma_{iM}}{\sigma_M^2}\left(E(r_M) - r_f\right) = \beta_i(E(r_M) - r_f)$$

Where β_i is a measure of sensitivity of an asset or in comparison to the market as a whole, CAPM allows risk to be divided in two parts. To develop this result the return (r_i) and variance (σ_i^2) of asset i is written as

$$E(r_i) = r_f + \beta_i\left(E(r_M) - r_f\right) + \epsilon_i$$

$$\sigma_i^2 = \beta_i^2\sigma_M^2 + Var(\epsilon_i)$$

Where (ϵ_i) is a random variable to indicate the uncertainty in the return and CAPM formula can be used to derive two results about it. From the formula for the expected value of (r_i) the first result follows: the expected value of ϵ_i must be zero. The second result follows by taking the correlation of the return of an asset with the return of the market portfolio r_M : from this it follows that the covariance of ϵ_i with the market portfolio is zero, Therefore $cov(\epsilon_i, \sigma_M) = 0$.

The first part $\beta_i^2 \sigma_M^2$ is called systematic risk. This is the risk associated with the market as a whole. This risk cannot be reduced by diversification because every asset with non-zero beta contains this risk. The second part, $var(\epsilon_i)$ is termed the unsystematic, idiosyncratic, or specific risk. This risk is uncorrelated with the market and can be reduced by diversification. It measured by beta, that is most important, since it directly combines with the systematic risk of other assets. A result of CAPM is that expected return depends on this beta. On other hand, strong assumptions of CAPM, make this model poor. For example, to reach the real market portfolio non-tradable assets should be included. In addition, since different investors may use various methods to estimate the expected returns they all may not have homogeneous expectations. However, CAPM is an effective equilibrium, if it can be combined with active methods such as Black-Litterman, it may become more functional in an investment decision process. CAPM can provide a solid reference path while the oscillations from this path can be captured by the active methods. Probably this is the main reason of preferring CAPM as the prior model in the original paper of Black and Litterman.

2.4 THE BLACK-LITTERMAN MODEL

The original mean-variance model was an innovative model in portfolio selection. It guides investors to allocate asset quantitatively in their portfolio. But as explained in the previous part, the model has its deficiencies. Investors who make use of this model may face many limitations. The mean-variance model unrealistically requires expected returns of all assets as input data. For investors, it is almost impossible to know expected returns with certainty. Even for portfolio managers, reliable return forecasts are only available for a small subset of assets. Drobetz & Köhler (2002) discuss while it is impossible to accurately predict the returns of all assets in practice, results of the Markowitz Model are very sensitive to small changes in expected returns. Small changes in expected returns can cause remarkable changes in the optimal portfolio weights. In addition, the MV model usually leads to extreme portfolio weights, which are unreasonable to be implemented in investors' portfolios. Black and Litterman (1990) describe when running the model without constraints, it almost always recommends portfolios with large negative weights in several assets; when optimizing a portfolio with constraints, the model gives a solution with zero weights in many of the

assets and therefore takes large positions in only a few of the assets. Moreover, Mankert (2006) states the mean-variance does not distinguish strongly held views from vague assumptions. Therefore, the optimal portfolio weights have no intuitive relation with the views investors actually wish to express.

2.4.1 Model Background

Initially Fischer Black and Robert Litterman, (1991) (1992) whilst working at Goldman Sachs Investment Company, initially proposed to improve the MV model. They ask some investors' perspective to develop a model that could be used to Goldman Sachs for portfolio selection. They made their portfolio selection model by considering invertors' feeling and views on the market. They have published some articles for their proposed model, However, none of them were describe the mathematics of the model precisely. Thereafter couple of researchers tried to clear the ambiguity of the model. Satchell and Scowcroft (2000) attempted to demystify BL's math model in their study. Lee (2000) made a short description of the model in the tactical asset allocation. Idzorek (2004) provides a comprehensive review of all the articles on the model and describes a new method for setting conditions.

Black and Litterman tried to build a more intuitive portfolio by computing a better estimate for the expected return vector. Then use this vector directly for better asset allocation in mean-variance optimizer. They identified two sources of information about the expected returns and they combined these two sources of information. The first source of information is obtained quantitatively, i.e. from CAPM if the market is in equilibrium. The CAPM returns form a frame to the modeling procedure, and use to modify extreme views of the second source of information. The second source of information is the point views and feelings of the investor. The investment manager has access to various information centers and may have different ideas about the market equilibrium. The main idea of BL model is to combine these to two sources of information, which results in a new vector of expected returns. This improved vector of expected returns leads to better asset allocation in portfolios.

The Black-Litterman original model (1991) uses an international index such as S&P 1200 as a proxy to reflect global benchmark in CAPM, to compute the equilibrium

returns in global context. In addition, investor has views on the expected return of assets. The BL-model allows investors to express their views in an absolute sense, as well as in a relative sense. For example asset X will have an expected return of r, or asset X will outperform asset Y. Expressing views in relative way is much closer to investor's feelings. Furthermore investor may not equally certain about every view. The model made it possible to considers a level of certainty to each view separately. The number of views of investor on about the asset is flexible. It ranges from no view at all, to as much views as there are assets to invest in. This makes the model much better to use. It is important to note that the process of specifying views about the assets in the investment universe is not a compulsory job. The investor may provide the model with no views, one view or one hundred views but is never obliged to provide a view for every asset class. This fact separates the Black-Litterman model from the classic mean-variance approach. Intuitively, it offers a great strength since it is implausible to assume that the investor can express a particular view for every asset class in the investment universe at all times.

The view specification process relies on two assumptions. Walters (2007) explains: first, that each view is unique and uncorrelated with the other views and second, that each view is fully invested so that the sum of weights in a view is zero if the view is relative or one if the view is absolute. Investors often focus only on a small part of the potential investment universe, choosing assets that they feel are undervalued, finding assets with positive momentum, or identifying relative value trades. In Black-Litterman model, it is only necessary to specify a view if the investor holds one. Different authors use different weighting schemes for the view matrix. Satchell and Scowcroft (2000) used an equal weighting scheme on the other hand. He and Litterman (1999) use a market capitalization-weight scheme. Idzorek (2004) argues that the method used by Satchell and Scowcroft may result in undesired and unnecessary tracking error. Instead, he suggest using the market capitalization scheme which means that the relative weighting of each asset in the P matrix will be proportional to that assets market capitalization divided with the whole market capitalization of either the outperforming asset or the underperforming asset. Consequently, views on small cap assets will receive a smaller relative weight than large market cap assets.

2.4.2 The Mathematics Of Model

Having discussed in previous parts, the Black-Litterman model assumes that there are two sources of information about the expected returns; market equilibrium and investor views. Both sources of information are assumed to be uncertain and are expressed in terms of probability distributions. The main challenge of this model for building one vector of expected returns is to merge these sets of information. Empirically, investors may combine the expected returns with their feelings heuristically. For example if investor has positive feelings regarding an asset, then simply increase the weight of the asset in portfolio, and vice versa for assets with negative outlook may decrease its share. Thereafter the amount of increase or decrease would be asked. Furthermore, assets may correlated: if one asset is expected to do well and therefore the weight is increased, then the weights of other positively correlated asset should also be increased. It would be very complicated to do this all by hand. Black and Litterman (1991) combined these two separate sources of information in a constructive manner and suggested two methods to accomplish this. First, the mixed estimation method of Theil (1971) which is related to the generalized least square method to estimate dependent parameters. Secondly, they suggest that the new vector of expected returns should be "assumed to have a probability distribution that is a product of two normal distributions".

In these studies, not only the mathematical method to compute the combined vector of expected returns is crudely described, but also the characteristics of the variables are debatable. Litterman (2003) shows that even the view is not related to the third asset, the corresponding weight is affected by the views, depending on the covariance structure among the assets. Although there are brief discussions about the estimation method and the variables, the full setup is not given in a clear and detailed manner. Intermediate steps and the derivations are also absent in the paper. Therefore, to keep simple, the mathematical procedure will be summarized in a different notation than the original papers.

Meucci (2010) describes Theil's mixed method in Black-Litterman application. For two sources of information they assign $\boldsymbol{\pi} = \delta\Sigma\mathbf{w}_M$ to equilibrium excess returns with the risk element of $\tau\Sigma$, and Q to the investor views, with the risk element of Ω. Investors derive these expected returns by a common factor $\mathbf{E}(\mathbf{r})$.

$$\boldsymbol{\pi} = I \,.\, \mathbf{E(r)} + u$$

$$Q = P.\,\mathbf{E(r)} + v$$

I is the identity matrix,

u is the error term with mean of zero and variance of $\tau\Sigma$

P is the matrix corresponding views

v is the error term with mean of zero and variance of Ω

$$\boldsymbol{\pi} \sim N(I \,.\, \mathbf{E(r)}\,;\tau\Sigma\,)$$

$$Q \sim (P.\,\mathbf{E(r)}\,;\ \Omega)$$

With Theil's mixed estimation, two equations are consolidated to estimate the common factor $E(R)$:

$$\begin{vmatrix}\boldsymbol{\pi}\\Q\end{vmatrix} = \begin{vmatrix}I\\P\end{vmatrix} . \mathbf{E(r)} + \begin{vmatrix}u\\v\end{vmatrix}$$

By least square method, the common factor E(R) here after μ_{BL} and its variance is calculated.

$$\mu_{BL} = [(\tau\Sigma)^{-1} + P'\Omega^{-1}P]^{-1}[(\tau\Sigma)^{-1}\boldsymbol{\pi} + P'\Omega^{-1}Q]$$

$$\Sigma_{BL} = [(\tau\Sigma)^{-1} + P'\Omega^{-1}P]^{-1}$$

The formula derivation omitted in this study , however it can be found in Idzorek's step by step guide (Idzorek, 2004).

μ_{BL} is the new(posterior) combined vector

τ is a scalar

Σ is the covariance matrix of excess returns

P is the matrix that identifies the asset involved in the views

Ω is a diagonal covariance matrix of error terms from the expressed views representing the uncertainty in each view

π is the implied equilibrium vector

Q is the vector of view in terms of relative or absolute changes

By applying the new combined return (μ_{BL}) into Markowitz model optimal portfolio weights are solved.

It is unclear what the parameters represent and how they should be specified. This makes model very confusing to use. Of the two approaches suggested by Black and Litterman the most widely used approach is the Bayesian one, especially after the publication of an article by Stephen Satchell and Alan Scowcroft (Scowcroft & Satchell, 2000) about the derivation of the BL formula. They use Bayes's Probability Density Function to merge the views of the investor with equilibrium expected returns.

They assumed investor has $k < n$ views, expressed as a linear relationship

$$P\mathbf{E}(\mathbf{r}) = Q + \epsilon$$

Where $P \in \mathbb{R}^{k \times n}, Q \in \mathbb{R}^k$, $\epsilon \in \mathbb{R}^k \sim N(0, \Omega)$ and $\Omega \in \mathbb{R}^{k \times k}$ is a diagonal covariance matrix. Here, P is the matrix that identifies the asset involved in the views, Q is the vector of view in terms of relative or absolute changes, ϵ is the uncertainty of the views, $\mathbf{E}(\mathbf{r})$ is an unknown vector and needs to be estimated from equilibrium. Bayes' Theorem is mathematically not very challenging. To apply the theorem to the problem at hand is less straightforward. Here to distinguish between notations, $\mathcal{P}$ represents the probability density distribution of Bayesian theorem. It is assumed that the investor forms his or her views using knowledge of the equilibrium expected returns π. Therefore, the equilibrium expected returns are considered the prior returns and these will be updated with the views of the investor. The posterior distribution combines both sources of information. Using Bayes' formula in this context yields:

$$\mathcal{P}(P\mathbf{E}(\mathbf{r})|\pi) = \frac{\mathcal{P}\big(\pi|P\mathbf{E}(\mathbf{r})\big)\mathcal{P}\big(P\mathbf{E}(\mathbf{r})\big)}{\mathcal{P}(\pi)}$$

To apply Bayesian theorem to the model they made to main assumption:

$$PE(\mathbf{r}) \sim N(Q, \Omega)$$

$$\boldsymbol{\pi}|PE(\mathbf{r}) \sim N(PE(\mathbf{r}), \tau\Sigma)$$

It assumed the investor views, and states that the expected returns are distributed normally around the assigned views. Second assumption states, given the expected returns, the CAPM equilibrium returns are distributed normally around the given returns. In Bayes' formula $PE(\mathbf{r})$ and $\boldsymbol{\pi}|PE(\mathbf{r})$ are known as the prior belief and the updating posterior returns respectively. Following the Bayesian solution, $\mathcal{P}(\boldsymbol{\pi})$ conducts like a constant in terms of the variable $\mathbf{E}(\mathbf{r})$ therefor cancelled out. The remaining expression is the main core of normal distribution, an can be find as:

$$\mathbf{E}(\mathbf{r})|\boldsymbol{\pi} \sim N(\mu_{BL}, \Sigma_{BL})$$

For calculated mean a variance as:

$$\mu_{BL} = [(\tau\Sigma)^{-1} + P'\Omega^{-1}P]^{-1}[(\tau\Sigma)^{-1}\boldsymbol{\pi} + P'\Omega^{-1}Q]$$

$$\Sigma_{BL} = [(\tau\Sigma)^{-1} + P'\Omega^{-1}P]^{-1}$$

The formula derivation proof could be found in demystification of the Black–Litterman (2000).

2.4.2 Expressing The Views

As discussed in previous part, the major challenge of Black and Litterman model is to incorporate the quantitative expected returns and views of the investor. Main part of the job is expressing views. Idzorek (2004) emphasized that the investor can express relative views, for example that asset A will outperform asset B by 2%. This manner of expressing is an important improvement of the BL-model over traditional mean-variance optimization, as this manner of expressing views is more intuitive than expressing absolute views.

Let us move to the mathematical description of the manner of expressing views and the view matrix P. An investor often holds views about performance of assets, asset classes or markets. The mathematical representation of these views needs to meet a few

characteristics. The views have to be specified relative to the vector of expected return E(r), the views have to be specified relative to each other and it has to be possible to express a level certainty in the view. These prerequisites lead to the following specification.

$$PE(r) = Q + \varepsilon \ where \ \varepsilon \sim N(0, \Omega),$$

$P \in \mathbb{R}^{k \times n}$ in known , $Q \in \mathbb{R}^k$ in known, $\varepsilon \in \mathbb{R}^k$ is and error vector with know variance $\Omega \in \mathbb{R}^{k \times k}$, $E(r) \in \mathbb{R}^n$ in unknown and needs to be estimated.

Assets that are under consideration can be specified in the matrix P, the vector Q expresses the relative change in performance and the vector of random variables ε expresses the uncertainty of the view. The vector ε is normally distributed with mean zero and variance Ω. That the mean is zero means that the investor does not have a standard bias against a certain set of assets. It is assumed that the views are mutually uncorrelated and therefore the covariance matrix Ω is diagonal. A variance of zero represents absolute certainty about the view. The vector $E(r)$ is the unknown expected return vector that needs to be estimated. What is often not noted is that Black and Litterman let the manner of formulating views in the matrix P completely free, they not did give any characteristics. Scowcroft & Satchell (2000) described a more general idea about expressing views on a portfolio of assets. Then the matrix P is considered as a series of portfolios and the vector Q holds the expected return of the corresponding portfolio. It is difficult for a person to estimate the expected return of a portfolio of assets. However, this more general definition does capture all manners of expressing views.

A portfolio could exist of one asset, which would correspond to expressing an absolute view on an asset; a portfolio could be zero-investment, this would correspond to expressing a relative view, and finally one has the possibility to express views on more than two assets. It is important to note that the vector Q denotes the forecasted relative performance of the assets.

An example can make this manner of expressing views more clear. There is four class of assets to invest:

I.Turkish Treasury Bonds

II. Real State Investment Trust (REIT)

III. Borsa Istanbul (BIST)

IV. International Bonds

V. Euro Bonds

Here investor could have view with maximum number of four. Suppose investor made relative and absolute views in this respect:

View 1 : REITS will have absolute return of 2.5%

View 2 : BIST will out perform REIT by 5 %

View 3: Turkish Bonds will outperform International Bonds and Euro Bonds By 3 %

This views can express in matrices in following way:

$$
\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & -0.5 & -0.5 \end{bmatrix} E(r) = \begin{bmatrix} 2.5\% \\ 5\% \\ 3\% \end{bmatrix} + \varepsilon
$$

Obviously, there is 3 rows and 5 columns in matrix P. Each row belongs to one view. The first view which is an absolute view, related asset is REIT, thus it tak e the value of 1 and other take zero. Second view expressed in relative form, the outperformer take the value of 1 and denominator takes -1. For the last view which is also stated in relative for the Turkish Treasury Bonds, take the value of 1, then International Bonds and Euro Bonds take the value of -0.5 equally. There is a significant point in dividing denominator assets. Here equally spreading is used. However, Market Capitalization weights could be used, as well. Matrix P, which is called the Link Matrix, links the views to corresponding assets.

In vector Q, there is two rows for each view. Therefore the first view 2.5% places in first row and 5% in second one respectively. So views will have a vector of $n \times 1$, n as the number of views. The next matrix in the link matrix which links the views to the asset. After constructing the P and Q matrices the remainder is just to put these value in BL combining formula.

2.4.2 The Factor Tau

The factor τ (Tau) is probably the single most confusing aspect of the Black-Litterman model. Many authors use different values, or just ignore it. Christodoulakis (2002) states originally it is used to specify the relation between the distribution of the asset returns and the distribution of the mean of the asset returns. Walters (2007) τ is used to scale the investors uncertainty in their prior estimate of the returns. There are several different approaches to calibrating it, or even including it described in the literature. Just to illustrate the difference of opinion, here is comments from three authors. He and Litterman (1991) set τ on 0.05. Satchell and Scowcroft (2000) state that many people use a value of τ around 1. Koch (2005) on the other hand takes a position somewhat in the middle and finds that values of $\tau = 0.3$ are reasonable. All these differing opinions require an investigation into the value of τ. Meucci (2010) proposes a formulation of the Black-Litterman model without τ. The main difference between the original reference model and other reference models is uncertainty. The posterior estimates include an updated covariance matrix. This model requires the investor to estimate an additional parameter τ which impacts the posterior covariance matrix as well as the estimated returns.

2.4.3 Following Studies On The Basic Model

He and Litterman (1999) proved a computationally more stable version of the posterior mean and covariance. Both the Original Reference Model approach used in Black and Litterman (1991) and He and Litterman (1999) and the Alternative Reference Model approach used in Satchell and Scowcroft (2000) is mathematically described in vague manner. Idzorek (2004) offers step-by-step instructions for practitioners to the Black-Litterman approach. Meucci (2010) discusses the original reference model and made comments on the first of two extensions made by Satchell and Scowcroft mentioned above and argues that Factor Tau should be set between 0 and 1 in practice rather than using the extension. Author described posterior distribution building on the formulas derived by He and Litterman (1999) and used in the original framework presents two puzzles, when views are stated with extreme confidence so that the variance of the views goes to infinity or zero, in other words investor confidence in views are zero or 100%. Walters (2007) presents a complete walkthrough and derivations of the Black-

Litterman approach as well as a description of its topics and parameters. The author offers some guidance regarding the choice between the two different reference models.

Having discussed the basic model and expressing views, couple of studies tried to obtain the views via quantitative models. Bewan and Winkelmann (1998) used two types of views: macro views and micro views. Macro views are generated by calibration process with constraints on information ratio. Micro views are assigned subjectively with three levels: High, medium, and low. They estimated the covariance matrix by the method mentioned in the study of Litterman and Winkelmann (1998). The study of Jones el at. (2007) uses the factor model of Carhart (1997), which is an extension of the three-factor model developed by Fama and French (1992). The authors also give a numerical illustration for the method they proposed. The study of Beach and Orlov (2007) is another example that attains the views by quantitative methods. They obtain the views via E-GARCH model. Their portfolio surpasses the market equilibrium weighted portfolio and the portfolio composed according to Markowitz mean-variance technique. Palomba (2006), applied Multivariate GARCH model is used to forecast security returns. These forecasted returns are used as the equilibrium returns, and this returns are combined with personal views. This paper differs from the most of the literature as it uses a time varying model to obtain equilibrium returns. Da Silva et al. (2009) discussed the utility optimization problem used for obtaining the optimal weights in active portfolio management. They stated that the utility maximization problem used in Black-Litterman method uses the unconstrained Sharpe ratio optimization. These studies will be reviewed by details in the next chapter.

3. LITERATURE REVIEW

Satchell and Scowcroft (2000) present a rather detailed mathematics of the Black-Litterman model using a Bayesian approach. In their paper, two worked examples using 11 and 15 asset classes respectively of international equity only to form an optimized portfolio. The authors postulate an assumption that makes their research stand out from Black and Litterman (1991) and He and Litterman (1999) in that they set. It was the first time researchers choose to eliminate tau from the framework, accepting the implication that investor uncertainty about the CAPM prior is not discounted. Furthermore, they set and an equally weighted vector of view weights throughout the first example and in the second example. This framework is combined with two views, one for each of the two worked examples. The first vie investor without hedging, whose home currency is Pound Sterling , believes that Swiss stocks will outperform German ones by with 0.5% per year, thus assuming a moderate relative view. The portfolio is constrained with respect to its beta the weights which sum to one. The portfolio has a Sharpe ratio of 0.16. The second example incorporates the belief that six hard currencies will outperform nine other European markets by 1.5%. The Sharpe ratio for this portfolio is 0.18 . The authors show that the Black-Litterman framework shifts the portfolio weights in an intuitive manner. They concluded that a combination of a neutral starting point, the CAPM prior and investor views is mixed in such a way in the Bayesian formulation that yields robust optimizer inputs. The authors proceed with an extension of the original model thus offering an alternative to the approach derived by Black and Litterman (1991) and He and Litterman (1999). This extension considered unknown and stochastic. This extension presents that the probability computations involve a multivariate t-distribution rather than the normal distribution typically assumed in the Black-Litterman framework. When the number of degrees of freedom is small, more weight will be put on the tails of the probability distribution. No numerical calculations using this model are offered the reader.

Lee (2000) presents a description of the Black-Litterman approach from a tactical asset allocation point of view. Author discusses the implied tactical trading rule of the Black-Litterman model and finds that the model has a different approach towards what risks

are important in making tactical bets compared to a total risk or total return framework. This is due to the fact that the Black-Litterman model considers low confidence views as more uncertain. In other words, some risks are more important with respect to tactical bets.

Idzorek (2004) provides an intuitive discussion about the model and numerical examples using eight asset classes including US bonds, International bonds, US large cap growth and value equities, US small cap growth and value equities, International developed equities and international emerging equities. Expected excess return vectors are estimated based on a historical average approach, CAPM relative the UBS Global Securities Markets Index approach and an implied equilibrium approach. It is shown that when risk aversion coefficient of approximately *3.07* is used, weights based on the implied equilibrium return vector equal the market capitalization weight. The author discusses that the historical average approach to estimating the vector of excess returns results in an extreme portfolio. Three different views, one absolute and two relative, are incorporated using the Black-Litterman model. The author investigates the differences between a market capitalization scheme and an equal weighting scheme in Link Matrix *P*. Author concluded that a market capitalization scheme is preferred since the equal weighting scheme may result in undesired and unnecessary tracking error. A new method for incorporating user-specified confidence level for investor views is described. With Idzorek's approach, the investor can determine the diagonal covariance matrix Ω by coupling implied confidence levels with a 0% to 100% user-specified confidence level in each view. The method removes the need for specifying and allows other types of information than the view portfolio variance that affect the confidence of each view statement.

Mankert (2006) describes the mathematical deviations of the Black-Litterman model using sampling theory rather than a Bayesian or Theil's Mixed Estimation approach. The theoretical and mathematical derivations are used to develop the model with respect to practical use. The quantitative approach is then combined with a discussion about behavioral finance and its implications for the Black-Litterman model. The author offers an extensive literature review and an in depth philosophical and mathematical description of the model.

Walters (2007) presents a complete walkthrough and derivations of the Black-Litterman approach as well as a description of its topics and parameters. The author also incorporates research from other authors parallel with the discussions about the mathematical parameters used in the model and explains them in their context. Furthermore, the practical steps that should be taken when using the Black-Litterman model as an asset allocation process is described following references. The author shows a replication of the numerical results from He and Litterman (1999) and Idzorek (2004). It is commented on the difficulties in reproducing some of the other important results from the prior research as insufficient data is supplied. The author follows with a description of the several extensions to the model including Lambda, which measures the impact of investor views on the posterior estimates and also confidence intervals. In addition, author offers some guidance regarding the choice between the two different reference models. The choice depends on whether the investor wants the ability to include the information contained in an updated covariance matrix as in the original reference model and by doing so, including and the need for specification of the same, or if he or she is willing to accept the simpler alternative reference model, thus excluding and the need for specifying it. The author presents a numerical example with unconstrained portfolios containing seven asset classes. The example includes a specification of two relative views and one absolute view. Both reference models are included so that the reader is provided a comparison between them. The main results are that equilibrium weights equal prior weights for the alternative reference model but not for the original reference model. This due to which is included in the original reference model reflects an uncertainty about the prior and since the investor is uncertain about the estimate some funds are withheld and placed in a riskless asset. Similarly, the investor is not fully invested in the posterior portfolio when using the original reference model as is the case when using the alternative reference model. The original reference model leads to larger movements from the equilibrium. Various methods of calibrating was discussed. Finally, it is concluded that most investors will do well using the alternative reference model.

Meucci (2010) discusses the original reference model and made comments on the extension made by Satchell and Scowcroft mentioned above and argues that Factor Tau

should be set between 0 and 1 in practice rather than using the extension. Author described posterior distribution building on the formulas derived by He and Litterman (1999) and used in the original framework presents two puzzles, when views are stated with extreme confidence so that the variance of the views goes to infinity or zero, in other words investor confidence in views are zero or 100%. The author argues that the original Black-Litterman model is constructed so that the posterior estimates become distorted. It is argued that when confidence in views is zero the posterior model should be the reference prior and when confidence in views is full the posterior should become the reference model conditioned on the specific views. Even though the original model yields estimates that are fully consistent mathematically, the posterior estimates might be counterintuitive. Therefore, the author suggests an alternative formulation called the market model in which is excluded entirely and where the posterior covariance is not updated. The alternative reference model does not include estimation, in which the mean of asset returns is no longer considered a random variable.

Becker and Gürtler (2010) generated the views for the Black-Litterman model with the help of analysts' forecasts in the basis of the dividend discount model as a future oriented valuation model. Authors examined four possibilities to compute expected returns with the Black-Litterman model. They determined confidences views in two ways. First, on the basis of number of analysts' forecasts and secondly by applying a Monte-Carlo simulation on the basis of distribution of analysts' forecasts. The effect of different views on portfolio weights analyzed. In their implementation of the Black-Litterman model, based on the number of analysts' forecasts outperforms all other strategies regarding the Sharpe ratio, in both constrained and unconstrained case.

Ojagverdiyeva & Prysyazhnyuk (2011) investigated the sensitivity of the weight vector obtained from the Black-Litterman model as response to the different methods of estimation of variance-covariance matrix of views. Authors showed the weights of optimal portfolio are not significantly sensitive to the variances of the views. For different values of Tau, they estimated omega in two ways; proportional to the variance covariance matrix of the historical excess returns, and omega which employed variances obtained from the EGARCH model estimation. There was no significant reaction to the omega estimation. The results showed that. Further, in all types of estimation methods,

the model is less extreme and has more intuitive and tolerate weights.

Bozdemir (2011) construct the vector of views with a quantitative method. Author used Autoregressive, AR(1) model to forecast future prices of 20 indices of Borsa Istanbul. He investigated the sensitivity of the weight vector constructed by Black-Litterman with respect to Factor Tau. The portfolios built by Black-Litterman performed worse than the market portfolio, in terms of the compound return, mean-variance ratio. As Tau gets larger, the difference between the two strategies becomes more visible.

Bertsimas et al. (2012) used techniques from inverse optimization to create reformulation of the Black-Litterman framework. They exploited the flexibility of specifying views and the ability to consider more general notions of risk to introduce on mean-variance inverse optimization (MV-IO) approach and a robust mean-variance inverse optimization (RMV-IO) approach. Computational evidence suggests that these approaches provide certain benefits over the traditional BL model, especially in scenarios where views are not known precisely.

Nordin (2012) tested the sensitivity of weight vector with respect to vector of views, risk aversion and factor Tau on portfolios made from MSCI, ACWI and IMI. Author concluded both Black-Litterman portfolios perform very well even with the crippling facts such as static views and a sub-par covariance estimate. Furthermore, both portfolios outperform the prior portfolio. Author shows that the Black-Litterman model offers generous calibration potential. He conclude the Black-Litterman has capability to produce high quality estimates far superior to the classical approach. Author states model is quite capable of creating estimates that in turn can be used to derive intuitive and high performing portfolios which also behave intuitively over time. While using the model, one should mind the sensitivity towards the covariance matrix and avoid using models, which gives poor risk estimates. Furthermore, the model offers significant tweaking opportunities.

Chincarini and Kim (2012) discuss three applications of the Black Litterman model that result in unnecessary costs to the investor. The first type creates a portfolio out of a prior and an estimate of the variance-covariance matrix, but fails to utilize the mean estimate. Not using the mean estimate amounts to ignoring a valuable piece of

information present in the data. Their conservative estimate of the loss from neglecting the mean is about a 1% reduction in expected annual returns. Although it is well known that means are not estimated as reliably as variance and covariances, ignoring the mean estimate cannot be an optimal solution. The second type creates a portfolio out of two conflicting models of security returns i.e CAPM. Authors find the magnitude of this loss to be around a 1% reduction in expected annual returns. One might justify the use of two models if the portfolio manager has no idea regarding which model is more likely to be true. Thus, it makes it hard to justify using two contradictory models of stock returns. The third application is the so-called reverse optimization technique. That is, practitioners often use the weights of an index and reverse optimize to obtain the implied expected returns of the market. Since the variance-covariance matrix are estimated with error, the implied expected returns of such a procedure will also be estimated with error. Authors quantify the magnitude of the errors associated with this technique. They found this error to be quite high, in some cases as high as 3.5% per month and much higher if the original benchmark was an equal-weighted benchmark.

Fischera and Seidla (2013) use a two-state regime-switching model with bull and bear markets, three different risk measures, volatility, modified value at risk, modified conditional value at risk, and three different return estimates; historical, CAPM and Black–Litterman and adjust return data for non-normality and serial correlation. The difference in the average monthly performance for the minimum risk portfolio between the best, with specification of regime-switching model, historical mean, modified VaR and the worst with is specification of single-regime model, historical mean, modified VaR, is about 0.2% and for the tangency portfolio between the best with specification of regime-switching model, CAPM return, modified CVaR and the worst with specification of single-regime model, CAPM return, volatility portfolio about 0.7%. Authors conclude the result that the non-normality in asset returns is better fitted by a regime-switching model than by different risk measures.

Mikaelian (2013) propose an implementation of Black-Litterman allocation approach with views based on time-varying risk premiums during different phases of business cycle. Views obtained by defining 5-phase business cycle taken from US economic history 1979–2012. He formulate facts on assets classes' co-movement during different

phases of business cycle and set simplistic rules for generating views based on mentioned facts. Macroeconomic indicators used to predict 5-phase business cycles. Author shows that Black-Litterman allocation has superior performance to almost all other allocation strategies during 1980–2011 years. Author concluded the minimum risk level the views are not significant, and the increase in the risk levels results in the increase of significance of views. Further at the highest risk level the portfolios not based on market capitalization show better results.

4. DATA SPECIFICATION AND MODEL

4.1 DATA SPECIFICATION

This study investigates the sensitivity of weight factor of Black-Litterman in respect to different Factor Tau values. Then it compares the corresponding portfolios with Original Mean-Variance and market capitalization weights. This paper will apply both optimization approaches to stock portfolios from Borsa Istanbul. The Borsa Istanbul, which is abbreviated to BIST founded in 1985. By the 2013 there are 371 companies listed on BIST with total market capitalization of 5.5 Billion Turkish Liras.

This study uses three main data sets: weekly and monthly closing prices and market capitalizations of 40 selected equity of Borsa Istanbul and government bonds as a proxy for risk-free rate. These stocks are high ranked in market capitalization in BIST listings and also have the historical price data available for studying period. All data obtained from Bloomberg terminal and covers the period from 1 Jan 2008 untill 31 Dec 2013.

The 40 used stocks include: Anadolu Efes Beverage, Aksa Energy, AltinYildiz Textile, Anadolu Hayat Insurance, Aselsan Electronics, Aslan Cement, Brisa Tyres, Carrefour Retails, Coca Cola Beverages, Cimsa Cement, Deniz Bank, Dogus Holdings, Enkai Constructions, Eregli Steels, Fenerbahce Sport Club, Ford Automobiles, Garanti Banks, Gubre Factories, Is Bank, Koc Holdings, Kipa Retails, KarDemir Steels, Nuh Cement, PetroKimya Holdings, Petrol Ofis, Sabanci Holdings, Selcuk Medicine, Sise Glass Industries, TAV Airports, Tuborg Beverages, Turkcell, Turkish Airlines, Tofas Turk Automobiles, Trakya Glass, Turk Telecom, Tupras Refineries, Ulker Food, YapiKredi Bank, YapiKredi Insurance.

The Bloomberg is one of the most famous and widely used financial data provider, which is used by financial research centers and professionals in the financial industry. It provides great possibilities to extract high quality data on virtually any financial asset traded in the world. Since Bloomberg is the premier choice of research centers, it supposed provide the most reliable and valid of the historical and real data sets to be high. As the calculations discussed in this study are correctly performed on extracted raw data from Bloomberg, the result data set can be trusted. In addition to data

reliability, using extracted from data Bloomberg, means that the results of this study can be replicated and analyzed without direct access to the containing data file of this study.

Sample period is 1 Jan 2008 to 31 Dec 2013, with 312 weekly and 72 monthly observed returns for each asset. The first five years is used exclusively to figure the historical return vectors and variance-covariance matrices. The test sample period is 1 Jan 2013 to 31 Dec 2013. The last period is assumed to make and compare portfolios made with MV and corresponding BL asset allocation models.

4.2 Methodology

Since the main goal is to compare the portfolios made by original mean-variance and Black-Litterman models, further analyzing the sensitivity of weight vector to various values of Factor Tau, these to model will be discussed separately in following parts.

4.2.1 The Original Mean-Variance

As agued in previous chapters, mean-variance model needs expected returns and variances of assets. Here average historical returns and historical variance-covariance matrix is used in optimization process. The utility function has two main constraints.

$$Max\ E(r_p) = \sum_{i=1}^{n} w_i r_i$$

$$Subject\ to\ \sigma_p^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \sigma_{ij} = \sigma$$

$$\sum_{i=1}^{n} w_i = 1$$

$$w_i \geq 0$$

Since in Borsa Istanbul, trader is not allowed to make short sells, this study uses constrained optimization with positive weights. Further, all weights should add-up to one.

Monthly returns in period of 1 Jan 2008 to 21 Dec 2012 is used to build variance-covariance matrix and historical return vector. These data putted in optimization to construct portfolio For January 2013. At 31 Jan 2013 since the data for real return for stock is available first the real return of portfolio calculated then, to build historical return vector and variance-covariance matrix, the return of Jan 2013 added to data base and portfolio for Feb 2013 will be optimized with new data. This procedure is repeated monthly until Dec 2013. After obtaining optimum weights from mean-variance approach for all months , optimum portfolios will be constructed with black-litterman approach.

4.2.2 The Black-Litterman Model

The Black-Litterman model uses two sources of information to blend in BL formulas. The first source is implied excess return called prior and the second source is views. The output of the formulas will be the BL Expected Excess Return and BL Variances. After obtaining these values, optimum weights could be calculated through a reverse optimization process. In BL model, the first set of information of returns set, namely the prior, contains the excess returns implied by CAPM and the historical variance-covariance matrix of these excess returns. The second information set, namely the views, consists of forecast for the asset returns using EGARCH estimation model, and the covariance matrix, which is same as the one in prior.

The main point in blending the two sets of information, the relative confidence between prior information and views may be assigned by Factor Tau. Since this study uses absolute view for all of the assets, the optimum portfolio is constructed with varying values of Tau to have a better comparison of the effect of prior set.

To put it briefly, the first step is to obtain the CAPM excess returns and historical variance-covariance matrix as the prior information set. Secondly, the EGARCH model is used to obtain the quantitative views; together with these views the historical covariance matrix will be the view information set. Thirdly, these information are blended via the BL formula with the desired confidence level. Fourth step is to use the

posterior information in the optimization process in order to find the optimum weights. Further, these vectors are used to calculate the monthly returns for 12 months, The last step is repeated 5 times for different Tau values.

4.2.2.1 The prior set

The implied excess returns is the main part of the prior set, to get these excess return the following utility maximization should be solved.

$$\max_{w} U = w'E(r) - \frac{1}{2}\delta w'\Sigma w$$

The analytical solution for this maximization equation is:

$$w = (\delta\Sigma)^{-1}E(r)$$

Since the implied excess return uses market weight's vector, the equation will be converted to this reverse optimization formula:

$$E(r) = \delta\Sigma w$$

The weight vector of CAPM is the market capitalization weights. For each asset the weight element can be calculate relative to total market capitalization. In this study 40 assets are used.

$$w_i = \frac{Matek\ Cap\ (i)}{\sum_{i=1}^{40} Market\ cap\ (i)}$$

Risk aversion coefficient is calculated by the following Formula. r_m is the average market return which can be obtained by matrix multipication of market cap vector and average return vector for each asset.

$$\delta = \frac{E(r_m) - r_f}{\sigma_m^2}$$

After figuring historical variance-covariance matrix, the CAPM implied excess returns can be found by utilizing the reverse optimization formula mentioned above.

4.2.2.2 The views

The intuition behind the original Black-LItterman model was to combine CAPM returns with investor's views, which allows analyst to combine any pair of information sets regarding to risk and return in asset universe. In basic model, a quantitative model was combined with qualitative views. Black and Litterman (1991) used the qualitative forecast of financial analyst as view's of investor. However a quantitative model could be implemented to generate vector of views. If both the prior information and the views obtained from quantitative sources, the blending is between two quantitative information sets, the posterior information will be a generated from two quantitative inputs. For this research, it was hard to find proper views and qualitative estimates for asset. Furthermore finding analyst reports are not only expensive to get, but also it is hard to find for a non-institutional researcher.

One of the features of this study is the use of EGARCH derived views as proxies for views of investor. GARCH type models are used to estimate the variance of the error terms as a function of past.

The benefit of employing this model is to obtain more objective views, in other word they are not dependent on the subjective projections of the portfolio manager. In addition, GARCH type models are able to capture characteristics of stock returns.

Engle (2002) emphasizes the most widely used GARCH specification asserts that the best predictor of the variance in the next period is a weighted average of the long run average variance, the variance predicted for this period, and the new information in this period that is captured by the most recent squared residual.

Brooks (2008) states using time dependent variance and covariance will enable the model to capture clustering effects in the data. It has been shown that variance in financial markets is high during certain periods and low during other. When the variances change over time it means that the time series has heteroskedasticity, it has a

changing volatility. Using GARCH also allows mean reverting i.e. if there is a long term means periods of high volatility mean reverting will decrease volatility over time and periods with low volatility will increase over time.

This study uses EGARCH proposed by Nelson (1991), as an extension of basic GARCH model. This paper computes parameter estimates over a rolling window of a fixed size through the sample. . For better estimation in time series, weekly prices used as high frequent input for EGARCH model. The historical data is split into the estimation sample of 260 weekly data and a prediction sample of 4 observations. Then we use the estimation sample and 4-step ahead predictions are made for the prediction sample. For each stock, after obtaining the forecasted prices for 4 consecutive weeks, the sum of logarithmic returns of four consecutive weeks is the total return of the upcoming month. To forecast future prices, the EGARCH model is applied for each of 40 stocks in this study using E-views (2007) software package.

In couple of stocks of Borsa Istanbul it was hard to find it significant, but the total results changed the idea. Brooks (2008) states EGARCH model provides positive variance estimates without using parameter restrictions and uses standardized shocks. EGARCH also captures the asymmetric response of variance to good and bad news. The EGARCH- Mean equation is:

$$r_t = \mu + \gamma \sigma_{t-1}^2 + \varepsilon_{t-1}$$

the variance equation is:

$$\ln\sigma_t^2 = \omega + \sum_{i=1}^{q} \beta_i \ln\sigma_{t-1}^2 + \sum_{i=1}^{p} \alpha_i \left|\frac{a_{t-1}}{\sigma_{t-1}}\right| + \sum_{i=1}^{r} \gamma_i \frac{a_{t-1}}{\sigma_{t-1}}$$

By considering EGARCH (1,1), the equation will be transformed to following:

$$\ln\sigma_t^2 = \omega + \beta\ln\sigma_{t-1}^2 + \alpha \left|\frac{a_{t-1}}{\sigma_{t-1}}\right| + \gamma \frac{a_{t-1}}{\sigma_{t-1}}$$

After forecasting return of next month for each stock, the view vector for can be figured for Black-Litterman formula.

Variance of view is another important part in blending formula. The forecasted returns play the role of investor's views in BL model. For the variance of forecasted return one could use estimated variances from EGARCH model. Idzorek (2004) uses the historical variance-covariance matrix to obtain variance of views. Cheung (2010) states the variance-covariance matrix of views is proportional to the historical variance-covariance matrix with a coefficient. Idzorek (2004) calculated Omega by following formula. The Link matrix P which defined already, multiplied to historical variance-covariance accompanied Factor Tau. Walters (2007) emphasized that Factor Tau is used to scale the investors uncertainty in their prior estimate of the returns.

$$\Omega = \tau P \Sigma P'$$

4.2.2.3 Mixing in BL formula

As discussed in theoretical framework, the Black-Litterman model in Bayesian manner it states that:

$$\mathbf{E(r)}|\boldsymbol{\pi} \sim N(\mu_{BL}, \Sigma_{BL})$$

$$\mu_{BL} = [(\tau\Sigma)^{-1} + P'\Omega^{-1}P]^{-1}[(\tau\Sigma)^{-1}\boldsymbol{\pi} + P'\Omega^{-1}Q]$$

$$\Sigma_{BL} = [(\tau\Sigma)^{-1} + P'\Omega^{-1}P]^{-1}$$

Applying this study's data to this formulas, these variables are defined as follows;
μ_{BL} Vector of excess returns (40×1)
τ A real number between 0 and 1 which justifies the relative confidence
Σ Historical variance-covariance matrix - prior (40×1)
P Link matrix (40×40)
Ω Variance-covariance matrix of views (40×40)

$\boldsymbol{\pi}$ Prior excess returns vector (40×1)

Q Views vector (40×1)

More details regarding to these variables will be discussed later.

BL model is round about to take a weighted average of prior returns and views considering the weights as the inverse of variances. The inversion of variance or risk referred as confidence.

For more clarity suppose there is just two returns. $\boldsymbol{\pi}$ and Q and the weights are $w_{\boldsymbol{\pi}}$ and w_Q respectively.

$$\mu_{BL} = \frac{\boldsymbol{\pi} w_{\boldsymbol{\pi}} + Q w_Q}{w_Q + w_Q}$$

$$\mu_{BL} = \frac{\boldsymbol{\pi} \left(1/Var_{\boldsymbol{\pi}}\right) + Q \left(1/Var_Q\right)}{\left(1/Var_{\boldsymbol{\pi}}\right) + \left(1/Var_Q\right)}$$

Since $Var_{\boldsymbol{\pi}} = \tau\Sigma$ and $Var_Q = \Omega$ after substitution and more simple algebra the equation finally transforms to:

$$\mu_{BL} = [(\Sigma)^{-1} + \Omega^{-1}]^{-1}[(\Sigma)^{-1}\boldsymbol{\pi} + \Omega^{-1}Q]$$

The link matrix P is an identity matrix equals to 1 in this example, however in multiple return cases it should assign to formula by :

$$\mu_{BL} = [(\tau\Sigma)^{-1} + P'\Omega^{-1}P]^{-1}[(\tau\Sigma)^{-1}\boldsymbol{\pi} + P'\Omega^{-1}Q]$$

This result is logical in this way. If the variance of the prior decreases, the weight of it increases to utilize the high level of certainty. On the contrary the more variance of prior, or the more uncertainty, the less weight of the prior, penalizing the uncertainty or risk. The same intuition applies to the views.

In expressing views, since next month's forecasted return is assigned in absolute manner to all stocks, the link matrix (P) is an identical (40×40). The Q vector is the forecasted values for each stock for one month. Omega Ω is the modified historical

variance-covariance (40×40) matrix for views. Omega is directly related to Factor Tau. This study uses the same variance-covariance matrix for the CAPM and the EGARCH estimated views. For modifying relative confidence for prior, model uses the factor Tau. As discussed in the beginning of this chapter, various methods are used for better estimation of Tau. Since EGARCH relatively is a good model for time series analysis, 4 values for Tau is assigned to calculations. Thus, 4 different values of Omega will be calculated for each month. As the amount of Tau raise, the dependence on first set of inputs -CAPM returns- decreases, therefore the optimization might come up with greater returns. After inserting all these values into BL formulas the outputs are posterior vector expected returns and posterior variance-covariance matrix.

Finding optimal posterior weights is to maximize utility function same as Mean-Variance model, however it uses the posterior returns and posterior variance-covariance.

$$\max_{w} U = w'\mu_{BL} - \frac{1}{2}\delta w'\Sigma_{BL}w$$

Since the main goal of this study is to investigates the sensitivity of weight vector and portfolio return in respect to Factor Tau, 4 values assigned to Tau to compare portfolios, this optimization should be done for posterior returns and posterior variance-covariance of each Tau.

As mentioned in previous parts, historical monthly and weekly data of 5 years are used to construct portfolios for each month of 2013. The portfolio weights will use real returns of 2013. For comparison the return on portfolio and Sharpe ratio will be monitored.

This study compares portfolios with Sharpe ratio, which proposed by Sharpe (1966). It is calculated by following formula:

$$Sharpe\ Ratio = \frac{\mu_p - r_f}{\sigma_p}$$

The Sharpe ratio is a statistical tool for comparing the risk-adjusted performance of investments over a given time period. The ratio is frequently used to rank mutual funds and other pooled funds. It can also be used to compare individual securities and

investment portfolios. The ratio measures how much an investment returned in excess of a risk-free investment per unit of risk taken.

5. EMPRIRCAL FINDINGS

Remembering the main motivation behind of this study, this paper tried to compare portfolio properties based on original mean-variance model and Black-Litterman optimization for on various values of τ. Since Factor Tau affects variance of views and confidence level of investor, the portfolios may perform miscellaneously. Specifically speaking, it investigates the sensitivity of portfolio weight in BL with respect to Factor Tau. As discussed in previous chapters, Black-Litterman model made significant progression in portfolio optimization. It let analyst to include his or her subjective views on assets into the equilibrium model to produce more reasonable results. The purpose is to see the effect of the variance-covariance matrix of views, Omega, on the composition of portfolios in Black-Litterman model and turn a comparison with original mean-variance optimized portfolios. Mathematically, this thesis examine the performance of the optimal portfolios built by Mean-Variance approach and Black-Litterman with different value of Factor Tau.

As an executive summary; this paper compose monthly portfolios according to historical monthly and weekly data. First data set is used to produce portfolios with original mean-variance optimization mentioned in methodology chapter. The second set is used to estimate a vector of views with EGARCH to be applied in Black-Litterman optimization method. For BL optimization paper uses rolling window approach to construct variance-covariance matrix for 12 month with 260 historical observations.

The EGARCH model estimates price for 4-step-head in price time series. The sum of return for four consecutive weeks considered as upcoming month's return and it take a part as absolute view of investor in Black-Litterman Model. In the next step, the raw data blended in Black-Litterman blending formula with different values of Factor Tau, which expresses the confidence in prior distribution. This paper compares the portfolio built by mentioned method with both absolute return and Sharpe ratio .

5.1 Portfolio Return

The following table presents a summary of monthly returns of 2013. The second column show the portfolio returns based on market capitalization weight. As seen,

couple of months, produce negative returns. Especially in June, July and August 2013 Turkish market went down. This is due to cross-country protests started on May, which was heavily affected the country's economy. Just on June 3rd, BIST experienced a loss of 10,5 percent in a single day. Similar story occurred in Dec 2013 due to corruption scandal, which aimed the ruling party. This event had an undeniable impact on national economy. However couple of stocks outperformed in mentioned months. (See Appendix I ; The Monthly return of each stock in 2013)

Table 5.1 Monthly Retuns

Monthly Return						
Month	Original MV	Market Weights	BL - Tau = 0,05	BL - Tau = 0,25	BL - Tau = 0,5	BL - Tau = 1
Jan-13	1,55%	1,49%	4,20%	4,67%	5,17%	5,79%
Feb-13	8,54%	1,84%	11,60%	12,35%	13,43%	15,28%
Mar-13	9,63%	8,70%	12,30%	12,85%	15,60%	17,05%
Apr-13	5,57%	-0,47%	7,36%	7,78%	8,33%	9,50%
May-13	7,42%	2,33%	10,80%	11,70%	12,45%	13,87%
Jun-13	-0,74%	-9,77%	3,44%	3,96%	4,38%	4,87%
Jul-13	-0,02%	-3,73%	2,74%	3,02%	3,28%	3,69%
Aug-13	-7,35%	-7,66%	3,52%	3,94%	4,28%	4,79%
Sep-13	10,13%	12,34%	18,60%	19,26%	20,32%	22,59%
Oct-13	7,14%	3,86%	7,83%	8,48%	9,17%	10,29%
Nov-13	4,74%	-1,86%	17,30%	18,87%	19,84%	21,39%
Dec-13	-5,93%	-8,83%	1,83%	2,05%	2,18%	2,97%

On the other hand, the first column shows the monthly return composed with mean-variance strategy. As seen on the table, in majority of periods Mean-Variance portfolios significantly outperformed in comparison to Market Capitalization Weighted approach. Only in Sep 2013 Market Cap approach performed better than MV by 2,21 percentage.

Rest of the table presents the monthly return of Black-Litterman approach for various values of Factor Tau. As discussed in previous parts Tau takes the values of 0,05 , 0,25, 0,5 and 1 respectively. Obviously, Black-Litterman optimization outperformed both Market Cap and MV approaches for all values of Tau. Further as the value of Tau increases, the corresponding portfolio performs better.

Table 5.2 Monthly Sharpe Ratio

Sharpe Ratio						
Month	Original MV	Market Weights	BL - Tau = 0.05	BL - Tau = 0.25	BL - Tau = 0.5	BL - Tau = 1
Jan-13	0,499	0,532	1,235	1,296	1,304	1,365
Feb-13	1,138	0,709	3,314	3,329	3,291	3,499
Mar-13	1,690	2,805	2,860	2,819	3,111	3,178
Apr-13	0,899	-0,204	2,831	2,823	2,748	2,927
May-13	1,726	0,729	2,842	2,905	2,810	2,926
Jun-13	-0,266	-2,571	1,186	1,288	1,295	1,346
Jul-13	-0,007	-1,285	0,637	0,663	0,654	0,688
Aug-13	-1,531	-2,252	1,067	1,127	1,112	1,163
Sep-13	3,196	2,938	4,326	4,226	4,053	4,211
Oct-13	1,676	1,170	2,373	2,424	2,383	2,499
Nov-13	1,636	-0,714	3,204	3,297	3,151	3,175
Dec-13	-1,796	-2,265	0,871	0,921	0,890	1,134

5.2 Sharpe Ratio

Last table represents an overview of sharpe ratio for corresponding optimization methods each moths of 2013. The second column discloses sharpe ratio based on market capitalization weights. As presented, couple of months got extreme negative returns. Mostly in June, July, August and December 2013, when the Turkish market was bearish. As discussed previously this is due to cross-country protests started on May, which was heavily affected the country's economy and corruption scandal, which targeted the ruling party.

Furthermore, the first column shows the monthly return constructed with Mean-Variance approach. Although, MV-made portfolios took negative scores in similar months accompanying market weight portfolios, the values are modified. Clearly, Market Capitalization Weighted approach is beaten by Mean-Variance.

The reminder designate the sharpe ratio value for portfolios composed by Black-Litterman approach for various values of Factor Tau. Recognizably, Black-Litterman optimization did not take negative values and performed better than both Market Cap and MV approaches for all values of Tau. However as the value of Tau increases, the sharpe ratio slightly fluctuates around.

The following figure illustrates the monthly return of portfolios, which composed using different types of optimization. As seen the return fluctuates over the months from -9,77 percent in June to 22,50 percent in September. Approximately, Mean-Variance returns rely between the market weight portfolios and the ones built by Black-Litterman approach. Only in October market weight performed better than Mean-Variance portfolio.
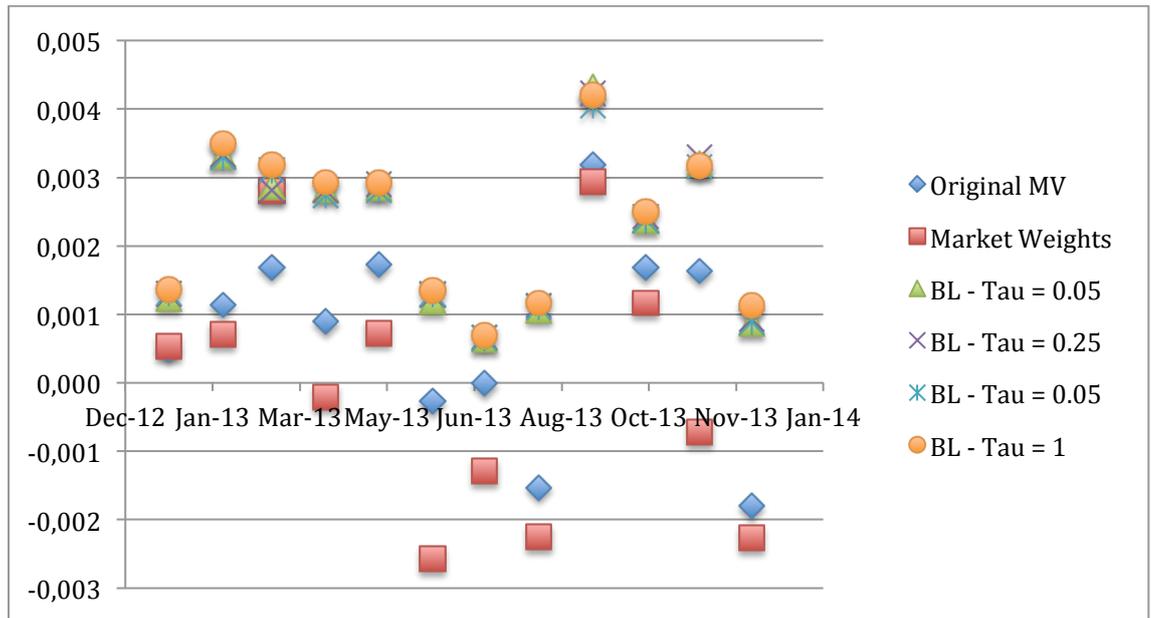
Portfolios composed with Black-Litterman performed better and do not result in negative returns. As the value of Factor Tau increases , the portfolio return gets larger. Worth to notice, in recession months, there is no significant difference between returns' of various Factor Tau values.

Figure 5.1 Monthly Return Of Each Strategy
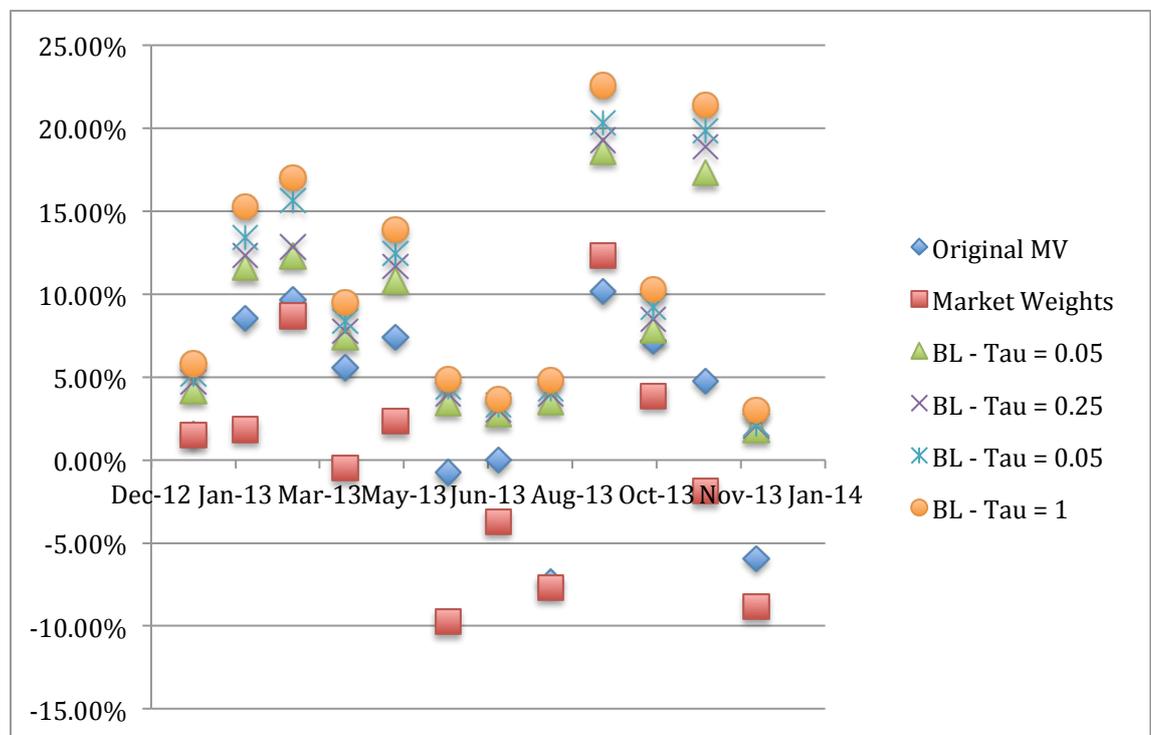
The next graph shows the Sharpe ratio scores in different months of 2013 for corresponding portfolio composition approach. The sharp ratio of market weight portfolios fluctuates over year. On the other hand, portfolios made for various values of Factor Tau, clearly there is no significant difference between in sharper ratios' of Black-Litterman approach.

Figure 1.2 Monthly Sharpe Ratio of Each Strategy

Sharpe ratios of portfolio composed with Mean-Variance optimization, fluctuates gradually between market weights and Black-Litterman methods.

6. CONCLUSION

Mean-Variance optimization has its origination to be the most popular method to set up portfolios, the popularity could be due to the understandable premise on which it sorts assets. The main logic in asset allocation is to balance risk and return, and to bear more risk if it company more ex return. Although the MV model allocates assets exactly on this way, it has couple of deficiencies. The MV-made portfolios are often very concentrated in only a few assets and do not reflect the views of the investor. In order to deal with these flaws efficiently, investors often constrain the MV model in such way that the possible portfolios lie in a bandwidth they are comfortable with.

Black-Litterman model tries to wipe out these sets of problems by making intuitive portfolios for investors. The model did its job that the BL model has become a very popular and many papers are written on the subject. However, the papers mainly try to explain the model, as the mathematics of the original model was not very clearly described. Especially the parameter Factor Tau is a source of confusion. It is used to scale the variance matrix of the equilibrium returns, but how the matrix should be scaled is unclear and what the value of the scaling parameter should be or on what it should depend was unclear, as well. The method solves the problem of specifying Factor Tau, but mathematically there is no consistent reason to specify these variables together.

The main purpose of the study was to determine the performance of the Black-Litterman model in comparison to mean-variance optimization. This study uses different values of Factor Tau to calibrate the optimization. The assets are completely selected from large market capitalization listing of Borsa Istanbul. This study used historical monthly and weekly data from 1 Jan 2008 to 31 Dec 2012 to construct monthly portfolios for 2013 then it compared the performance of monthly portfolios with absolute return and Sharpe ratio measures.

One of the features of this study is to implement a quantitative method to construct views of investor for Black-Litterman model. The views estimated via a quantitative model. Times series forecast plays an important role here. As the future prices estimated for 4 weeks with high precision, it is intuitive to blend CAPM returns with estimated returns in BL formula. Views vector of investor are forecasted by EGARCH-mean model. This study applied EGARCH estimation of returns in a univariate context. The results of this thesis are based on historical monthly and weekly prices of 5 years for

total of 40 stocks. These assets are selected as high market capitalization from BIST listing which have last 6 years data. Implicitly, it is assumed that all stocks are fully investible which means that an investor can, without any prohibition, trade an amount of each asset any running time of the stock market. The results of this study are into the comparison between performance of MV and BL models.

The portfolios compared over the 12 month period from January 2013 to December 2013. The performance will be measured via the Sharpe ratio and mean return values. While the MV model uses historical returns and variance–covariance matrix to build up optimal portfolios. The BL model produce portfolios from the BL optimized returns and the covariance matrix Σ. In BL optimization, the Factor Tau varied over, to track the sensitivity of the weight vector and determine the optimal value.

In review of mean returns of portfolios, it can safely be concluded in study period, portfolios composed by Black-Litterman beat both Original Mean-Varianc and market weight portfolios. Black-Litterman approach considered to produce very efficient portfolios due to its diversification. In fact when views are incorporated in BL model, the behavior of the portfolio gets better than Original Mean-Variance. In addition as illustrated in previous chapter, BL model uses future estimates to blend in formula and optimize portfolio with specific value of Facto Tau. As the value of Factor Tau increases the confidence of investor on implied excess return diminishes and the confidence of view vector increase. Thus Black-Litterman relies more and more on view vector.

Apparently EGARCH estimated views helped Black-Litterman to construct outperforming portfolios. Results showed a significant sensitivity of weight vector to Factor Tau. As the value of Tau factor increases the BL return performs better. One important observation in results is that in recession period of economy there is no significant difference in portfolio performance of various Factor Tau values, on the other hand, in when market goes up BL-made portfolios with higher level of Factor Tau outperformed other approaches'.

In Sharpe Ratio analysis of portfolios, obviously Black-Litterman model performed better than Original Mean-Variance and market capitalization weight methods. Furthermore, in sensitivity analysis of Sharpe ratio regarding to Factor Tau, there is no such significant difference for different values of factor Tau.

Due to the extreme allocations, the mean-variance portfolios are useless in practice and real investment, and over time the portfolios are instable and less-performed. However, Mean-Variance play the important role on initiating the Black-Litterman approach. The critical difference between two approaches is the way they use inputs. While the mean-variance approach uses the mean of past returns as a forecast of the future and a simple covariance matrix as a forecast of future risk, the Black-Litterman is slightly more advanced. As shown in this study, Black-Litterman approach tries to update the historical variance-covariance and expected return vector with future estimated of investor, which results in better optimization and better performance. Further the Factor Tau which is one the vague points of this, affects the return vector significantly. This is an important finding in itself, because it states that the performance the Black-Litterman is boosted with higher values of Tau in boom periods of total economy.

To sum up, this paper studied the performance of portfolio composed from 40 stocks of Borsa Istanbul with Mean-Variance and Black-Litterman approaches. For BL part estimations from EGARCH model is used as a proxy for investor views. Further, Black-Litterman portfolios, optimized with four different values for Factor Tau. As discussed, weight vector of BL model is significantly sensitive to Factor Tau. By slight increase in value corresponding portfolio performed better. Along other studies, He and Litterman (1991) set Tau= 0.05, Satchell and Scowcroft (2000) set Tau around 1. Koch (2005) on the other hand set Tau = 0.3, it can be concluded that, the EGARCH model fits the data well and captures all information needed for variances. BL optimization resulted in better portfolio performance with Tau=1, which is very extreme approach the in optimization.

For further studies it is recommended to apply a multivariate GARCH models to produce investor views, using parallel markets i.e. Gold, Oil and Foreign Exchange. It would better to consider international investment, where the inter-market correlation is at lower levels. Multivariate Models might forecast future return precisely.

The optimization in this study was constrained on positive weight. Investors may have ability to take short sell positions. Considering this, it would be interesting to see some modeling with unconstrained portfolio optimization. The portfolio return's sensitivity towards the Factor Tau pointed out in this study. It is suggested to review effect of other components on portfolio behavior and performance. Considering this, it would be

interesting to use the Black-Litterman approach with sophisticated methods of covariance estimation. Further studying the Black-Litterman model for non-normal distributed return i.e. skewed with fat tails, would help decision making in asset allocation to compose high-return portfolios.

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