DOKUZ EYLÜL UNIVERSITY
GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES

INFINITESIMAL PROPERTIES OF LIE GROUPS

by
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İZMİR

INFINITESIMAL PROPERTIES OF LIE GROUPS

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We have read the thesis entitled **"INFINITESIMAL PROPERTIES OF LIE GROUPS"** completed by **GÖKÇE ÇAKMAK** under supervision of **ASSOC. PROF. DR. İLHAN KARAKILIÇ** and we certify that in our opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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INFINITESIMAL PROPERTIES OF LIE GROUPS

ABSTRACT

In this study, important mathematical objects have been investigated for robotics called Lie groups and Lie algebras. The word "infinitesimal" means "near the identity". With the aid of infinitesimality, we have analyzed from smooth manifolds to two important maps which are defined from Lie algebra to Lie group: Exponential and Cayley maps. Specifically, we have investigated the special orthogonal group of order three as an example for infinitesimal generators, exponential map and Cayley transform which forms a basis for mobile robots. A general definition for exponential map has been given but because of the condition of skew-symmetry, particularly Cayley map is defined for the Lie algebras that is constituted by antisymmetric matrices.

Keywords: Lie group, Lie algebra, exponential map, infinitesimal generator, Cayley map.

LIE GRUPLARININ ÇOK KÜÇÜK ÖZELLİKLERİ

ÖZ

Bu çalışmada, robotik alanı için önemli olan matematiksel konular; Lie grup ve Lie cebiri araştırılmıştır. Burada "çok küçük" kelimesi "birim eleman civarında" anlamına gelmektedir. Çok küçük olma özelliğinin yardımıyla, düzgün manifoldlardan başlayarak, matris grupları üzerinde Lie cebirinden Lie grubuna tanımlı iki önemli dönüşüm olan üstel ve Cayley transformasyonlarına kadar incelenmiştir. Özellikle, robot hareketlerine de temel olacak şekilde özel ortogonal grup için çok küçük üreteçler bulunmuştur, üstel ve Cayley dönüşüm örneklendirmeleri yapılmıştır. Üstel fonksiyonlar genel olarak tanımlanmıştır, ancak ters simetri koşulundan dolayı Cayley dönüşümü özel olarak aykırı simetrik matrislerden oluşturulan Lie cebirler için verilmiştir.

Anahtar kelimeler : Lie grup, Lie cebir, üstel dönüşüm, çok küçük üreteç, Cayley dönüşüm.

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CHAPTER ONE
INTRODUCTION

The study of Lie group theory along with Lie algebras and their applications are the profound part of the mathematics. The topic had influenced and still influencing many disciplines of mathematics, physics and engineering. A wide range of mathematical topics such as group theory, complex calculus, differential geometry, algebraic geometry, partial differential equations and topology have taken part in the vast structure of the Lie theory (Howe, 1983). Its real life applications are mostly practiced in the area of robotics and kinematics as the essential tool of geometrical mechanics.

The topics including rigid body motions, rotations, mechanical constraints and deformations are all treated by the matrix Lie groups, specifically special orthogonal group and special Euclidean group. Because of the important implications of functionality, rather than vector spaces, Lie groups are preferred.

By the usefulness of geometrical mechanics and Lie group theory, there exist a broad written material in robotics about Lie groups such as (Selig, 2005) and (Murray et al., 1994).

From the very beginning in the 19th century, group theory is developed by famous mathematicians such as Marius Sophus Lie, Hermann Klaus Hugo Weyl, Wilhelm Karl Joseph Killing. Sophus Lie's main objective was to extend the applications of algebraic equations to the differential equations. To make it happen, in early phases, Lie used infinitesimal generators of monoparametric subgroups with matrix multiplication as operation, later known as Lie algebra.

This thesis is aiming to give a brief look into Lie theory. One of our targets is understand and apply the theory by solving examples regarding matrix groups. We will examine exponential map, infinitesimal generators, Cayley map and their positions in the Lie group theory.

In the first chapter, an abbreviated yet sufficient exposition has been carried out for the base of Lie group theory. Smooth manifolds are explained.

In the second chapter, Lie groups and Lie algebras are explained with the aid of matrix Lie groups in real numbers. By straightforward generalization of the left and right multiplication maps, adjoint maps are stated. Then by applying the idea of Lie brackets Lie algebra is defined and it has elaborated again by means of tangent space.

At the start of chapter three, the meaning of one-parameter subgroups are stated in detail. And also this chaper consists of exponential mapping which a result of monoparametric subgroups. With the help of exponential map, the origin of this topic that is infinitesimal transformations are explained and supported with an example from matrix groups. Mapping from the tangent space of the group to the group itself is elucidated in the latter section of this chapter followed by a calculation of the formula of Cayley transform with the matrix eigenvalues.

Chapter four consists of examples of matrix groups, particularly $GL(n,\mathbb{R})$, $SO(n,\mathbb{R})$, $SE(n,\mathbb{R})$. These groups are elaborated by stating their fundamental properties. Next, these groups are proven to be Lie groups and Lie algebras as well as their exponential maps are established.

1.1 Preliminaries

Before analyzing this topic, the initial question that should be asked is "What is Lie group, and how can we define it?". To explore this subject and it's applications, we need to look up some of the definitions such as manifolds, differentiable manifolds, matrix groups etc. so that by using these tools we can accomplish our task to provide the necessary information to make calculus on them. For more information on manifolds see (Tu, 2008) and (Lee, 2003).

Definition 1.1.1. An n-dimensional manifold $\mathcal{M}$ is a set that contains a collection of one-to-one functions (called "patches") $\varphi : U \to \mathcal{M}$ where U is an open set in $\mathbb{R}^n$ which satisfies the following properties;

- The images of the patches in the collection cover $\mathcal{M}$.
- For any patch φ and ψ in the collection, the compositions $\psi^{-1} \circ \varphi$ and $\varphi^{-1} \circ \psi$ are differentiable and defined on $\mathbb{R}^n$. (The patch ψ is defined on the open set V in $\mathbb{R}^n$)
- For any points $a, b \in \mathcal{M}$ that $a \neq b$, there are disjoint patches φ and ψ with $a \in \varphi(U)$ and $b \in \psi(V)$.

So, basically a manifold is the definition of an object that is covered by one-to-one, differentiable patches. For example, any open set of $\mathbb{R}^n$ is an n-dimensional manifold.

Next statement describes the key element in the Lie group explanation process: smooth manifold. In general, smooth (or differentiable) manifold is similar to the Euclidean space locally, with the addition of differentiability.

Definition 1.1.2. Let M be a topological space.

- A pair (φ, P) consisting of an open subset $P \subseteq M$ and a homeomorphism $\varphi : P \to \varphi(P) \subseteq \mathbb{R}^n$ of P onto an open subset of $\mathbb{R}^n$ is called an n-dimensional chart for P of M.
- If $\mathcal{P} = \{P_\alpha : \alpha \in A\}$ is an open covering of M and $\Gamma = \{\gamma_a : P_a \to R_a\}$ is a collection of charts, then Γ is called an atlas for M if, whenever $P_a \cap P_b \neq \emptyset$, the transition map

  $$\gamma_b \circ \gamma_a^{-1} : \gamma_a(P_a \cap P_b) \to \gamma_b(P_a \cap P_b) \tag{1.1}$$

  is a diffeomorphism. The atlas denoted by $(M, \mathcal{P}, \Gamma)$ is called a smooth manifold of dimension n.

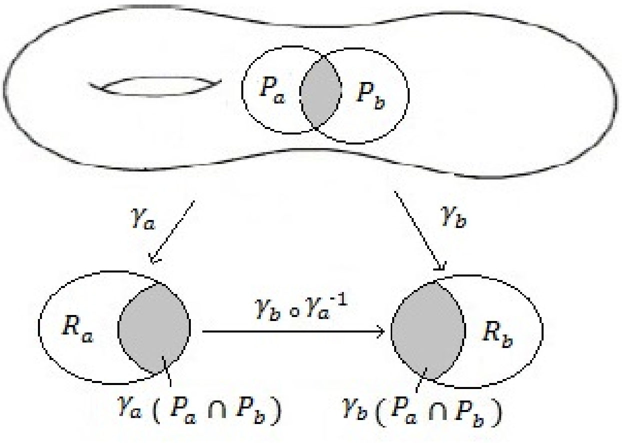

Figure 1.1 Transition map of a smooth manifold

For example; the set of $n \times n$ matrices with real entries $M(n, \mathbb{R})$ is a smooth manifold. Recall that vector spaces are smooth manifolds, and $M(n, \mathbb{R})$ is a vector space under scalar multiplication and matrix addition, thus $M(n, \mathbb{R})$ is a smoth manifold.

CHAPTER TWO
MATRIX LIE GROUPS AND LIE ALGEBRAS

2.1 Lie Groups and Matrix Lie Groups

The next step before answering the "initial question", is to elucidate the meaning of matrix groups. A general linear group $GL(n,\mathbb{R})$ is the group of invertible matrices in $M(n,\mathbb{R})$, the set of $n\times n$ matrices (Hilgert & Neeb, 2012).

Definition 2.1.1. A closed subspace and subgroup of $GL(n,\mathbb{R})$, say G, is a matrix group over $\mathbb{R}$ with degree n.

It can also be stated as "G is a matrix subgroup of $GL(n,\mathbb{R})$".

$GL(n,\mathbb{R})$, $SL(n,\mathbb{R})$, $SO(n,\mathbb{R})$ and $SE(n,\mathbb{R})$ are all examples of matrix groups, in fact all these matrix groups are subgroups of $GL(n,\mathbb{R})$.

Consider a closed subgroup of a matrix group G where $G \in GL(n,\mathbb{R})$, say H. For all $i \geq 0$, every sequence $\{A_i\} \in H$ has its limit in $H \subseteq G$. Thus, H is closed in general linear group and so we are able to write a new definition;

Definition 2.1.2. A closed subgroup of a matrix group is called a matrix subgroup.

For more information on matrix groups and their relation with the topic see (Baker, 2002). So the answer of the "initial question" that has been asked previously is;

Definition 2.1.3. Let G be a smooth manifold with continuous multiplication map $\text{mult} : G\times G \to G$, continuous inverse map $\text{inv} : G \to G$ and $G\times G$ as the product manifold. Then G is called a Lie group, if mult and inv are smooth maps.

This definition implies that; if the given group action is continuous then an algebraic group which is also a smooth manifold is called a Lie group.

Lie group theory can be used in kinematics and dynamics providing a new look (Paraskevopoulos & Natsiavas, 2013) and for a robotic approach see (Selig, 2004).

If a closed subgroup of a Lie group is also a submanifold for the group, then it is called a Lie subgroup.

Before giving some examples we need to clarify the definition of Lie group. For a group $\omega, \omega_1, \omega_2 \in \Omega$, mult and inv maps are defined as:

$$\mathrm{mult}(\omega_1, \omega_2) = \omega_1 \omega_2, \quad \mathrm{inv}(\omega) = \omega^{-1} \tag{2.1}$$

The benefit of using the condition of the smoothness of the maps is that it permits us to use the differential and integral calculus. The concept here is to use a local object, which we will be calling its Lie algebra, rather than a global one.

For a Lie group G with $g_1, g_2, g \in G$, the multiplication map has two different cases: left multiplication map and right multiplication map which are defined as:

$$L_{g_1} : G \to G \quad g \mapsto g_1 g \tag{2.2}$$

$$R_{g_2} : G \to G \quad g \mapsto g g_2 \tag{2.3}$$

The mappings are one-to-one, onto, continuous and also differentiable because $g_1 g$ and $g g_2$ are continuous functions of g. In addition, we have the conjugation map:

$$\gamma_{g_3} : G \to G \quad g \mapsto g_3 g g_3^{-1} \tag{2.4}$$

which fixes the identity element:

$$\gamma_g(e) = geg^{-1} = e \tag{2.5}$$

Observe that for elements $g_1, g_2, g \in G$;

$$L_{g_1}^{-1}(g) \;=\; g_1^{-1} g = L_{g_1^{-1}}(g) \tag{2.6}$$

$$R_{g_2}^{-1}(g) \;=\; g g_2^{-1} = R_{g_2^{-1}}(g) \tag{2.7}$$

For proof refer to (Baker, 2002). Since both of the maps and their inverses are smooth, L_g and R_g are diffeomorphisms. For the identity element $e \in G$;

$$\begin{aligned} L_g \circ R_g^{-1}(e) &= L_g(eg^{-1}) = geg^{-1} = \gamma_g(e) && (2.8) \\ R_g^{-1} \circ L_g(e) &= R_g^{-1}(ge) = geg^{-1} = \gamma_g(e) && (2.9) \end{aligned}$$

As $\gamma_g(e)$ is a composition of two smooth maps $\gamma_g(e)$ is a diffeomorphism, for scalars a, b and for $g_1, g_2, g \in G$;

$$\begin{aligned} \gamma_g(ag_1 + bg_2) &= g(ag_1 + bg_2)g^{-1} \\ &= agg_1g^{-1} + bgg_2g^{-1} \\ &= a\gamma_g(g_1) + b\gamma_g(g_2) && (2.10) \end{aligned}$$

Thus γ_g is a homomorphism. Recall that; a homomorphism from a group to the general linear group of a vector space is called a representation. Differentiation of the map γ_g at the identity gives;

$$(d\gamma_g)_e = T_eG \rightarrow T_eG \tag{2.11}$$

that is, $(d\gamma_g)$ maps the tangent space of G at e to itself. Hence we define;

Definition 2.1.4. The differential of the map γ_g at the identity of the group is called the adjoint map

$$Ad(g) = (d\gamma_g)_e : T_eG \rightarrow T_eG \tag{2.12}$$

The adjoint representation of the Lie group G is the homomorphism

$$Ad : G \rightarrow Aut(T_eG) \tag{2.13}$$

where $Aut(T_eG)$ is the automorphism of the tangent space to G at the identity.

By taking the differential of the map Ad, we evaluate a new map;

$$ad : T_eG \rightarrow End(T_eG) \tag{2.14}$$

which is a representation of T_eG. Denote T_eG as $\mathfrak{g}$. Then, adjoint representation can be rewritten as;

$$ad : \mathfrak{g} \to End(\mathfrak{g}) \tag{2.15}$$

By adopting the idea above; we have the necessary information to construct Lie bracket which is a consequential tool to expound the definiton of Lie algebra. Since we are specifically investigating matrix groups, we will evaluate the Lie bracket by assimilating the tool, known as the adjoint representation, in the sense of matrix groups.

For a matrix group G, analyze a matrix $X \in G$ sufficiently close to the identity. It can be evaluated as $X(r) = e^{rQ}$ for $Q \in \mathfrak{g}$ and r is a real parameter. We can approximate this matrix with exponential map by

$$X \approx I + rQ + O(r^2) \tag{2.16}$$

which yields its inverse as

$$X^{-1} \approx I - rQ + O(r^2) \tag{2.17}$$

where I is the identity of the group G. For any element $P \in \mathfrak{g}$, application of conjugation map reveals that;

$$\begin{aligned} \gamma_X(P) &= XPX^{-1} \\ &= (I + rQ)P(I - rQ) + O(r^2) \\ &= P + r(QP - PQ) + O(r^2) \end{aligned} \tag{2.18}$$

where $O(r^2)$ is the error term of the exponential function. Differentiating with respect to r and setting $r = 0$ provides the necessary information:

$$ad(Q)P = (d\gamma_X)_0 = QP - PQ \tag{2.19}$$

More formally; let $A, B \in \mathfrak{g}$ be two elements. The linear transformation $ad(A)B$ acts like a function with two distinct variables A and B which employs the idea of a map $T_eG \times T_eG \to T_eG$. Denote

$$[A, B] := ad(A)B \tag{2.20}$$

which leads to;

Definition 2.1.5. For any $A, B \in \mathfrak{g}$, the Lie bracket is defined by $[A, B] = AB - BA$.

We have another way to illustrate this definition. Consider $GL(n, \mathbb{R})$. The tangent space of $GL(n, \mathbb{R})$ near the identity can be represented as the endomorphism space of $\mathbb{R}^n$. For any given $A, B \in GL(n, \mathbb{R})$ at the identity, let $\alpha : I \to GL(n, \mathbb{R})$ be a path with the properties $\alpha(0) = e = I$ and $\alpha'(0) = A$ where $A \in GL(n, \mathbb{R})$, which yields

$$\begin{aligned} [A, B] = ad(A)B &= \frac{d}{dt}(Ad(\alpha(t))B)|_{t=0} \\ &= \frac{d}{dt}(\alpha(t)B\alpha^{-1}(t))|_{t=0} \\ &= \alpha'(0)B\alpha(0) + \alpha(0)B(-\alpha^{-1}(0)\alpha'(0)\alpha^{-1}(0)) \\ &= AB - BA \end{aligned} \tag{2.21}$$

The Lie bracket has important properties:

- It is a linear operation.

  For $J_1, J_2, K_1, K_2 \in \mathfrak{g}$, $j_1, j_2, k_1, k_2 \in \mathbb{R}$,

$$\begin{aligned} [j_1 J_1 + j_2 J_2, K] &= (j_1 J_1 + j_2 J_2)K - K(j_1 J_1 + j_2 J_2) \\ &= j_1(J_1 K - K J_1) + j_2(J_2 K - K J_2) \\ &= j_1[J_1, K] + j_2[J_2, K] \end{aligned} \tag{2.22}$$

  and

$$\begin{aligned} [J, k_1 K_1 + k_2 K_2] &= J(k_1 K_1 + k_2 K_2) - (k_1 K_1 + k_2 K_2)J \\ &= k_1(J K_1 - K_1 J) + k_2(J K_2 - K_2 J) \\ &= k_1[J, K_1] + k_2[J, K_2] \end{aligned} \tag{2.23}$$

- It is antisymmetric.

  For $J, K \in \mathfrak{g}$;

$$[J, K] = JK - KJ = -(KJ - JK) = -[K, J] \tag{2.24}$$

- It satisfies the Jacobi identity.

  For $J, K, L \in \mathfrak{g}$;

$$
\begin{aligned}
[J,[K,L]]+[K,[L,J]]+[L,[J,K]] &= [J,KL-LK]+[K,LJ-JL] \\
&\quad +[L,JK-KJ] \\
&= JKL-JLK-KLJ+LKJ \\
&\quad +KLJ-KJL-LJK+JLK \\
&\quad +LJK-LKJ-JKL+KJL \\
&= 0 \qquad (2.25)
\end{aligned}
$$

2.2 Lie Algebras

In this section, Lie algebras corresponding to Lie groups and some of the properties of it will be defined.

Definition 2.2.1. A Lie algebra over $\mathbb{R}$ consists of a vector space $\mathfrak{g}$ over a field $\mathbb{R}$, together with a $\mathbb{R}$-bilinear map, i.e., for $m_1, m_2, m, n_1, n_2, n \in \mathfrak{g}$ and $\alpha_1, \alpha_2, \alpha, \beta_1, \beta_2, \beta \in \mathbb{R}$

$$[\alpha_1 m_1 + \alpha_2 m_2, n] = \alpha_1 [m_1, n] + \alpha_2 [m_2, n] \qquad (2.26)$$

$$[m, \beta_1 n_1 + \beta_2 n_2] = \beta_1 [m, n_1] + \beta_2 [m, n_2] \qquad (2.27)$$

where $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is called the Lie bracket such that for $m, n, l \in \mathfrak{g}$,

$$[m,n] = -[n,m] \qquad (skew-symmetry) \qquad (2.28)$$

$$[m,[n,l]]+[n,[m,l]]+[l,[m,n]] = 0 \qquad (Jacobi\,identity) \qquad (2.29)$$

From another perspective, for any tangent vector $\mathcal{X}_e \in T_eG$, we can define a vector field $\mathcal{X}$ by

$$\mathcal{X}_g = (dL_g)(\mathcal{X}_e) \qquad (2.30)$$

Observe that for each $g, h \in G$;

$$\begin{aligned} (dL_g)(\mathcal{X}_h) &= (dL_g)(dL_h)(\mathcal{X}_e) \\ &= dL_g \circ dL_h(\mathcal{X}_e) \\ &= dL_{gh}(\mathcal{X}_e) \\ &= \mathcal{X}_{gh} \\ &= L_{g*}\mathcal{X}_h \end{aligned} \tag{2.31}$$

which displays that $\mathcal{X}$ is invariant under left translation. (Differentiation dL_g is L_{g*}.) More generally;

Definition 2.2.2. A smooth vector field $\mathcal{X}$ on a Lie group G is a left invariant vector field, if it satisfies the condition

$$(dL_g)(\mathcal{X}_h) = \mathcal{X}_{gh} \tag{2.32}$$

Our claim is, the set of all left invariant vector fields of a group with the Lie bracket operation is the Lie algebra of that group. Initially, to prove this let;

$$\mathfrak{g} = \{\mathcal{X} \in \Gamma^\infty(TG) \ : \ (dL_g)(\mathcal{X}_h) = \mathcal{X}_{gh} \ for \ g, h \in G\} \tag{2.33}$$

where $\Gamma^\infty(TG)$ is called "the set of smooth left invariant vector fields on G". We need to take the Lie bracket structure into account to reveal that $\mathfrak{g}$ is indeed the Lie algebra of the group G. For $\mathcal{X}, \mathcal{Y} \in \Gamma^\infty(TG)$, $g, h \in G$ we can state that the Lie bracket of two left invariant vector fields is a left invariant vector field.

$$\begin{aligned} dL_g[\mathcal{X}, \mathcal{Y}]_h &= [\mathcal{X}, \mathcal{Y}]_{gh} \\ &= \mathcal{X}_h(\mathcal{Y}_g) - \mathcal{Y}_h(\mathcal{X}_g) \\ &= \mathcal{X}_h(dL_g(\mathcal{Y})) - \mathcal{Y}_h(dL_g(\mathcal{X})) \\ &= \mathcal{X}_h\mathcal{Y} - \mathcal{Y}_h\mathcal{X} \\ &= [\mathcal{X}, \mathcal{Y}]_h \end{aligned} \tag{2.34}$$

This shows that $\mathfrak{g}$ is closed under the Lie bracket operation which leads to the statement: $\mathfrak{g}$ is a Lie algebra of G. Application of the last statement yields a new definition:

Definition 2.2.3. For a Lie algebra $\mathfrak{g}_1$, a Lie subalgebra of $\mathfrak{g}_1$, denoted by $\mathfrak{g}_2$, is a subspace of $\mathfrak{g}_1$ with Lie bracket operation $[\xi_1, \xi_2] \in \mathfrak{g}_2$ for $\xi_1, \xi_2 \in \mathfrak{g}_2$.

Theorem 2.2.4. *If a group G is a matrix subgroup of $GL(n, \mathbb{R})$, then the Lie algebra of the group G, $\mathfrak{g}$, is a Lie subalgebra of $\mathfrak{gl}(n, \mathbb{R})$.*

Proof: Let α_1, α_2 be two smooth curves with $\alpha_1(0) = \alpha_2(0) = I$, where I is the identity of the group $GL(n, \mathbb{R})$. Consider $\xi : domain(\alpha_1) \times domain(\alpha_2) \to G$ with

$$\xi(a, b) = \alpha_1(a)\alpha_2(b)\alpha_1^{-1}(a) \tag{2.35}$$

Differentiation with respect to the variable b and substitution of $b = 0$ gives;

$$\begin{aligned} \frac{d\xi(a,b)}{db}|_{b=0} &= \alpha_1(a)\alpha_2'(b)\alpha_1^{-1}(a)|_{b=0} \\ &= \alpha_1(a)\alpha_2'(0)\alpha_1^{-1}(a) \in \mathfrak{g} \end{aligned} \tag{2.36}$$

We also know that

$$\begin{aligned} \frac{d\alpha_1^{-1}(a)}{da} &= -\alpha_1^{-2}(a)\alpha_1'(a) \\ &= -\alpha_1^{-1}(a)\alpha_1'(a)\alpha_1^{-1}(a) \end{aligned} \tag{2.37}$$

Using the limit definition of differentiation and considering

$$\alpha^{-1}(a) = \alpha(-a) \tag{2.38}$$

yields

$$\begin{aligned} \frac{d[\alpha_1(a)\alpha_2'(0)\alpha_1^{-1}(a)]}{da} &= \lim_{a \to 0} \frac{\alpha_1(a)\alpha_2'(0)\alpha_1^{-1}(a) - \alpha_2'(0)}{a} \\ &= \alpha_1'(a)\alpha_2'(0)\alpha_1^{-1}(a)|_{a=0} \\ &- \alpha_1(a)\alpha_2'(0)\alpha_1^{-1}(a)\alpha_1'(a)\alpha_1^{-1}(a)|_{a=0} \\ &= \alpha_1'(0)\alpha_2'(0)\alpha_1^{-1}(0) - \alpha_1(0)\alpha_2'(0)\alpha_1'(0) \\ &= \alpha_1'(0)\alpha_2'(0) - \alpha_2'(0)\alpha_1'(0) \\ &= [\alpha_1'(0), \alpha_2'(0)] \end{aligned} \tag{2.39}$$

Since $\mathfrak{g}$ is closed, the limit exists and so $\beta'(0) = [\alpha_1'(0), \alpha_2'(0)] \in \mathfrak{g}$ which shows that $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{gl}(n, \mathbb{R})$.

CHAPTER THREE
CAYLEY AND EXPONENTIAL MAPS

3.1 One-Parameter Subgroups

Before expressing the exponential map, we will initially specify one-parameter subgroups which is also known as monoparametric subgroups.

Definition 3.1.1. For all $a, b \in \mathbb{R}$, define a smooth map $\phi : \mathbb{R} \to G$ of a Lie group with the addition action, such that

$$\phi(a+b) = \phi(a)\phi(b) \tag{3.1}$$

and

$$\phi(-a) = \phi^{-1}(a) \tag{3.2}$$

Then ϕ is called a one-parameter subgroup of G.

By the definiton of one-parameter subgroups we can state that these subgroups are Abelian and satisfies group homomorphism. In fact, this subgroup can be illustrated as differentiable homomorphism.

Differentiating equation (3.1) with respect to b and substituting $b = 0$ yields

$$\phi'(a) = \phi(a)\phi'(0) \tag{3.3}$$

This is a linear first order differential equation. Solving this equation for an element $a \in \mathbb{R}$ we acquire

$$\phi(a) = \phi(0)exp\{a\phi'(0)\} \tag{3.4}$$

Observe that;

$$\phi(0) = \phi(0+0) = \phi(0)\phi(0) \;\Rightarrow\; \phi^{-1}(0)\phi(0) = \phi^{-1}(0)\phi(0)\phi(0) \;\Rightarrow\; \phi(0) = e \tag{3.5}$$

where e is the identity of the group. Application of this condition yields;

$$\phi(a) = exp\{a\phi'(0)\} \tag{3.6}$$

We can also apply these equations to the case of matrix groups. Let Ψ be $n \times n$ real matrix and define the smooth map $\phi : \mathbb{R} \to M(n, \mathbb{R})$ with $\phi(0) = I$, where I is the $n \times n$ identity matrix. Expressing the definition with matrix indices yields;

$$\phi_{ij}(\psi + \varphi) = \sum_{i,j=1}^{n} \phi_{ik}(\psi)\phi_{kj}(\varphi) \tag{3.7}$$

Again if we solve $\phi'(\Psi) = \phi(\Psi)\phi'(0)$ considering the exponential of Ψ;

$$exp(\Psi) = e^{\Psi} := I + \Psi + \frac{\Psi^2}{2!} + \frac{\Psi^3}{3!} + ... \tag{3.8}$$

with the initial condition $\phi(0) = I$ yields;

$$\phi(\Psi) = \phi(0)exp\{\phi'(0)\Psi\} \Rightarrow \phi(\Psi) = exp\{\phi'(0)\Psi\} \tag{3.9}$$

This reveals that, the most general form of monoparametric (or one-parameter) subgroup of a Lie group contains the identity and because of the differential, it is expressed by the exponential map from the tangent space of the Lie group to the group itself. The points that are expressed in the last equation are uniquely determined at the identity.

Left and right translation maps are considered crucial in the field of one-parameter subgroups and so, quite beneficial for the exponential map. Using the left invariant vector field properties with $\sigma \in \mathbb{R}$ we have;

$$\begin{aligned} L_{\phi^*}\phi'(0) &= L_{\phi^*}\mathcal{X}_e \\ &= \mathcal{X}_{\phi e} \\ &= \mathcal{X}_{\phi} \\ &= \phi'(\sigma) \end{aligned} \tag{3.10}$$

where $\mathcal{X}_e$ is a tangent vector for the group at the identity and

$$\begin{aligned} R_{\phi^*}\phi'(0) &= R_{\phi^*}\mathcal{X}_e \\ &= \mathcal{X}_{e\phi} \\ &= \mathcal{X}_{\phi} \\ &= \phi'(\sigma) \end{aligned} \tag{3.11}$$

Combining the equations (3.10) and (3.11) we get;

$$\phi'(\sigma) = L_{\phi^*}\phi'(0) = R_{\phi^*}\phi'(0) \tag{3.12}$$

The integral curve which passes from e, that is the identity of the vector field over the group G, can be constructed by a left or right translation of the tangent vector at the identity and this integral curve is the one-parameter subgroup of the group G, whose tangent vector at the identity is $\phi'(0)$. In other words monoparametric subgroups, tangent vectors and left/right invariant vector fields have one-to-one correspondences with each other.

3.2 The Exponential Map

The exponential map is the connection between the Lie group and its Lie algebra. Let $\phi_{\mathcal{W}}$ be a one-parameter subgroup of a Lie group corresponding to $\mathcal{W}$, where $\mathcal{W}$ is in the tangent space of the group near the identity.

Definition 3.2.1. Let G be real Lie group with the tangent space $T_eG = \mathfrak{g}$ at the identity 1. The exponential map of the group G can be defined as

$$exp : \mathfrak{g} \to G, \quad \mathcal{W} \mapsto \phi_{\mathcal{W}}(1) \tag{3.13}$$

where $\phi_{\mathcal{W}}(1)$ is the unique one-parameter subgroup with tangent vector at 1.

We want to determine an association between $\phi_{\mathcal{W}}$ and $\phi_{p\mathcal{W}}$ so that we can use exponential map corresponding to $p\mathcal{W}$. Consider $g(r) = \phi_{\mathcal{W}}(pr)$. Differentiating with respect to r and substituting $r = 0$ yields

$$g'(r) = p\phi'_{\mathcal{W}}(pr) \;\Rightarrow\; g'(0) = p\phi'_{\mathcal{W}}(0) \tag{3.14}$$

because of the illustration of $\mathcal{W}$. Next, consider

Proposition 3.2.2. *For all $\mathcal{W}$ in the tangent space of the group G at the identity, there exists a unique map $\phi_{\mathcal{W}} : \mathbb{R} \to G$ such that $\phi'_{\mathcal{W}}(0) = \mathcal{W}$. This map is called monoparametric subgroup of G.*

As it was stated by the (Kirillov Jr., 2008) the proof of this proposition leads to

$$\phi'_{p\mathcal{W}}(0) = p\mathcal{W}. \tag{3.15}$$

By uniqueness of one-parameter subgroup $\phi'_{\mathcal{W}}(0) = \mathcal{W}$, we can state that

$$\phi_{\mathcal{W}}(pr) = \phi_{p\mathcal{W}}(r) \tag{3.16}$$

Rewriting the equation with $r = 1$ yields;

$$\phi_{\mathcal{W}}(p) = \phi_{p\mathcal{W}}(1) \tag{3.17}$$

Using definition (3.2.1.) we get;

$$\phi_{\mathcal{W}}(p) = \phi_{p\mathcal{W}}(1) = exp(p\mathcal{W}) \tag{3.18}$$

There is an immediate consequence of above statement:

Corollary 3.2.3. *For any real Lie group G with $T_eG = \mathfrak{g}$,*

$$exp((p + r)\mathcal{W}) = exp(p\mathcal{W})exp(r\mathcal{W}) \tag{3.19}$$

for any $p, r \in \mathbb{R}$.

If G is a matrix group then we can define the exponential map as $exp : \mathfrak{g} \to G$

$$exp(T) = \sum_{i=0}^{\infty} \frac{T^i}{i!} = I + T + \frac{T^2}{2} + \frac{T^3}{3!} + ... \tag{3.20}$$

where I is the $n \times n$ identity matrix and $T \in G$ which shows that for matrix groups, the exponential map has a correlation with matrix exponential. The Lie algebra of a matrix group can be identified with the help of exponential map;

Definition 3.2.4. For a matrix Lie group G, $\forall T \in G$ we denote $\mathfrak{g}$ as the Lie algebra of G when $e^{tT} \in G$ with $t \in \mathbb{R}$.

The differential of the exponential map at the identity will be of great help. Consider $(d\,exp)_0 : \mathfrak{g} \to \mathfrak{g}$. Combining the differential of the equation (3.18) at $a = 0$ and proposition (3.2.2.) yields

$$\begin{aligned} \frac{d}{da}(exp\, p\mathcal{W})|_{p=0} &= \phi'_{\mathcal{W}}(p)|_{p=0} \\ &= \phi'_{\mathcal{W}}(0) \\ &= \mathcal{W} \end{aligned} \tag{3.21}$$

Proposition 3.2.5. *Let* $\Omega_1, \Omega_2 \in \mathfrak{g} = M(n, \mathbb{R})$. *If* Ω_1 *and* Ω_2 *commute, then*

$$exp(\Omega_1 + \Omega_2) = exp(\Omega_1) exp(\Omega_2) \tag{3.22}$$

Proof: Observe that

$$\begin{aligned} exp(\Omega_1 + \Omega_2) &= \sum_{i=0}^{\infty} \frac{(\Omega_1 + \Omega_2)^i}{i!} \\ &= \sum_{i=0}^{\infty} \frac{1}{i!} \sum_{j=0}^{k} \binom{k}{j} \Omega_1^j \Omega_2^{k-j} \\ &= \sum_{i=0}^{\infty} \left(\sum_{j=0}^{k} \frac{1}{j!(k-j)!} \Omega_1^j \Omega_2^{k-j} \right) \\ &= \left(\sum_{i=0}^{\infty} \frac{\Omega_1^i}{i!} \right) \left(\sum_{j=0}^{\infty} \frac{\Omega_2^j}{j!} \right) \\ &= exp(\Omega_1) exp(\Omega_2) \end{aligned} \tag{3.23}$$

Proposition 3.2.6. *For* $\Omega \in M(n, \mathbb{R})$,

$$exp(\Omega)^{-1} = exp(-\Omega). \tag{3.24}$$

Proof: Using Proposition (3.2.5.) with $\Omega_1 = \Omega$ and $\Omega_2 = -\Omega$ yields

$$\begin{aligned} exp(\Omega)exp(-\Omega) &= exp(\Omega - \Omega) \\ &= exp(0) \\ &= I \end{aligned} \tag{3.25}$$

and thus

$$exp(\Omega)^{-1} = exp(-\Omega). \tag{3.26}$$

3.3 Infinitesimal Transformations

Sophus Lie is the first mathematician who worked generously on infinitesimal transformations, because Lie theory is mostly centered around of it. In fact the term "Lie algebra" is the algebra of infinitesimal transformations founded by H. Weyl. The main idea here is to deal with a transformation near the groups' identity rather than dealing with the whole group on each element.

The tangent space of a Lie group as defined by its one-parameter subgroups is called "infinitesimal description of a Lie group" or as Lie put it "infinitesimal group" (Arvanitoyeorgos, 2003). The set of infinitesimal generators of monoparametric subgroups of a continuous group is, in today's terms "Lie algebra" (Kosmann-Schwarzbach, 2010).

Let $\phi(a)$ be a monoparametric subgroup of a Lie group G. Its derivative can be written using the limit definition of differentiation:

$$\begin{aligned} \frac{d}{da}\phi(a) &= \lim_{n\to 0}\frac{\phi(a+n)-\phi(a)}{n} \\ &= \lim_{n\to 0}\frac{\phi(a)\phi(n)-\phi(a)}{n} \\ &= \lim_{n\to 0}\frac{\phi(a)(\phi(n)-e)}{n} \\ &= \phi(a)\lim_{n\to 0}\frac{\phi(n)-e}{n} \end{aligned} \tag{3.27}$$

where e is the identity of the group. Since the elements of the group are smooth, the limit at the last equation exists. If the limit is equal to a matrix, say Ω, we can solve it as differential equation;

$$\phi'(a) = \Omega\phi(a) \;\Rightarrow\; \phi(a) = e^{a\Omega} \tag{3.28}$$

Near the identity of the group it is equal to the Ω and Ω is called the infinitesimal generator for $\phi(a)$.

Example 3.3.1. Consider $SO(3,\mathbb{R})$.

$$\phi_1(\theta) = \begin{pmatrix} cos\theta & -sin\theta & 0 \\ sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.29}$$

is a monoparametric subgroup of $SO(3,\mathbb{R})$. Because;

$$\begin{aligned}
\phi_1(\theta+\gamma) &= \begin{pmatrix} cos(\theta+\gamma) & -sin(\theta+\gamma) & 0 \\ sin(\theta+\gamma) & cos(\theta+\gamma) & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} cos\theta cos\gamma - sin\theta sin\gamma & -cos\theta sin\gamma - sin\theta cos\gamma & 0 \\ cos\theta sin\gamma + sin\theta cos\gamma & cos\theta cos\gamma - sin\theta sin\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} cos\theta & -sin\theta & 0 \\ sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} cos\gamma & -sin\gamma & 0 \\ sin\gamma & cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \phi_1(\theta)\phi_1(\gamma)
\end{aligned} \tag{3.30}$$

and

$$\begin{aligned}
\phi_1(-\theta) &= \begin{pmatrix} cos(-\theta) & -sin(-\theta) & 0 \\ sin(-\theta) & cos(-\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} cos\theta & sin\theta & 0 \\ -sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} cos\theta & -sin\theta & 0 \\ sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} \\
&= \phi_1^{-1}(\theta) \qquad (3.31)
\end{aligned}$$

We want to determine the infinitesimal generator for the one-parameter subgroup $\phi_1(\theta)$. In order to find the matrix, we need to determine the differential of ϕ_1.

$$\begin{aligned}
\phi_1'(\theta) &= \begin{pmatrix} -sin\theta & -cos\theta & 0 \\ cos\theta & -sin\theta & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} cos\theta & -sin\theta & 0 \\ sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (3.32)
\end{aligned}$$

Denote

$$\Upsilon_1 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \qquad (3.33)$$

The infinitesimal generator for $\phi_1(\theta)$ is Υ_1. To determine all of the infinitesimal generators for $SO(3,\mathbb{R})$, we need to specify the groups monoparametric subgroups. Following the same steps as above we see that

$$\phi_2(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & cos\theta & sin\theta \\ 0 & -sin\theta & cos\theta \end{pmatrix} \qquad (3.34)$$

and

$$\phi_3(\theta) = \begin{pmatrix} cos\theta & 0 & sin\theta \\ 0 & 1 & 0 \\ -sin\theta & 0 & cos\theta \end{pmatrix} \tag{3.35}$$

are the remaining monoparametric subgroups of $SO(3,\mathbb{R})$ and regarding to these one-parameters, the infinitesimal generators are

$$\Upsilon_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \tag{3.36}$$

and

$$\Upsilon_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \tag{3.37}$$

respectively. Hence we state that the generators of the group $SO(3,\mathbb{R})$ are $\Upsilon_1, \Upsilon_2, \Upsilon_3$.

Bearing in mind that a Lie group element can be determined as a consequence of an exponential map, recall the equation (3.28). With the infinitesimal generators that have been acquired by one-parameter subgroups, one can find the elements of Lie group. For the compatibility with the Rodrigues' equation we have chosen the indices like below:

$$\begin{aligned} exp(\upsilon\Upsilon) &= exp(\upsilon_3\Upsilon_1 + \upsilon_1\Upsilon_2 + \upsilon_2\Upsilon_3) \\ &= exp\left(\upsilon_3\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \upsilon_1\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} + \upsilon_2\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}\right) \\ &= exp\left(\begin{pmatrix} 0 & -\upsilon_3 & 0 \\ \upsilon_3 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\upsilon_1 \\ 0 & \upsilon_1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & -\upsilon_2 \\ 0 & 0 & 0 \\ \upsilon_2 & 0 & 0 \end{pmatrix}\right) \\ &= exp\begin{pmatrix} 0 & -\upsilon_3 & \upsilon_2 \\ \upsilon_3 & 0 & -\upsilon_1 \\ -\upsilon_2 & \upsilon_1 & 0 \end{pmatrix} \end{aligned} \tag{3.38}$$

As expected, the matrix

$$\begin{pmatrix} 0 & -\upsilon_3 & \upsilon_2 \\ \upsilon_3 & 0 & -\upsilon_1 \\ -\upsilon_2 & \upsilon_1 & 0 \end{pmatrix} := M \tag{3.39}$$

is from $\mathfrak{so}(3,\mathbb{R})$ and by example (4.2.3) from chapter four, one can state the Rodrigues' formula

$$exp(M) = I_3 + \frac{sin\alpha}{\alpha} M + \frac{(1-cos\alpha)}{\alpha} M^2 \tag{3.40}$$

where $\alpha^2 = \upsilon_1^2 + \upsilon_2^2 + \upsilon_3^2$ and I_3 is the 3×3 identity matrix.

3.4 Cayley Maps

Cayley mappings are the functions from the Lie algebra of the group to the Lie group itself. We cannot state that "All Lie groups have Cayley mappings." because of skew-symmetry of the matrices needed.

For a rigid body in three dimensional space the rotation can be represented as

$$m = [\Omega]n \tag{3.41}$$

with respect to a fixed frame and $m, n \in \mathbb{R}^3$. Since this movement is only established by rotations and transformations, which are also known as rigid body transformations, matrix Ω must be orthogonal

$$\Omega^T \Omega = I \tag{3.42}$$

Orthogonality requires

$$(m-n)^T(m+n) = 0. \tag{3.43}$$

Observe that

$$\begin{aligned} m+n &= [\Omega]n+n=[\Omega+I]n && (3.44)\\ m-n &= [\Omega]n-n=[\Omega-I]n && (3.45)\\ m-n &= [\Omega-I][I]n \\ &= [\Omega-I][\Omega+I]^{-1}[\Omega+I]n \\ &= [\Omega-I][\Omega+I]^{-1}(m+n) && (3.46)\end{aligned}$$

Let

$$\Psi=[\Omega-I][\Omega+I]^{-1}. \qquad (3.47)$$

with indices $\Psi=[\psi_{ij}]$, $1\le i,j\le n$. Considering equations (3.43) and (3.46) for any vector $k\in\mathbb{R}^3$ yields

$$\begin{aligned} k^T\Psi k &= (\psi_{11}k_1+\psi_{12}k_2+...+\psi_{1n}k_n)k_1\\ &+ (\psi_{21}k_1+\psi_{22}k_2+...+\psi_{2n}k_n)k_2\\ &+ ...\\ &+ (\psi_{n1}k_1+\psi_{n2}k_2+...+\psi_{nn}k_n)k_n\\ &= \sum_{i,j=0}^{n}(\psi_{ij}+\psi_{ji})k_ik_j\\ &= 0 && (3.48)\end{aligned}$$

Thus,

$$\begin{cases}\psi_{ij}=-\psi_{ji} & \text{if } i\neq j\\ \psi_{ii}=0 & \text{if } i=j\end{cases} \qquad (3.49)$$

Hence the matrix Ψ is antisymmetric and

$$\begin{aligned} \Psi &= [\Omega-I][\Omega+I]^{-1} && (3.50)\\ \Psi[\Omega+I] &= [\Omega-I] \\ \Omega[I-\Psi] &= [I+\Psi] \\ \Omega &= [I+\Psi][I-\Psi]^{-1} && (3.51)\end{aligned}$$

Definition 3.4.1. A Cayley map for the group is a mapping from skew-symmetric matrices to the orthogonal matrices, in other words; if Ω is a skew-symmetric matrix and I is the identity matrix for the group, then

$$(I+\Omega)(I-\Omega)^{-1} \tag{3.52}$$

is orthogonal, titled as Cayley map (or transformation) of the matrix Ω and denoted as $Cay(\Omega)$.

Another way to look at the Cayley map is that this mapping is from tangent space of the group to the manifold.

We can illustrate Cayley map as $(I-\Omega)^{-1}(I+\Omega)$ because $(I+\Omega)$ and $(I-\Omega)^{-1}$ commute:

$$(I+\Omega)(I-\Omega) = I-\Omega^2 = (I-\Omega)(I+\Omega) \tag{3.53}$$

Multiplying with $(I-\Omega)^{-1}$ from the left hand side gives;

$$(I-\Omega)^{-1}(I+\Omega)(I-\Omega) = (I+\Omega) \tag{3.54}$$

This time, multiplying last equation with $(I-\Omega)^{-1}$ from the right hand side yields the required result:

$$(I-\Omega)^{-1}(I+\Omega) = (I+\Omega)(I-\Omega)^{-1}. \tag{3.55}$$

Every antisymmetric matrix ($\Omega^T = -\Omega$) represents an orthogonal matrix with the aid of Cayley transformation. Observe that;

$$\begin{aligned}
Cay(\Omega) &= (I+\Omega)(I-\Omega)^{-1} && (3.56)\\
[Cay(\Omega)]^T &= [(I+\Omega)(I-\Omega)^{-1}]^T \\
&= [(I-\Omega)^{-1}]^T(I+\Omega)^T \\
&= [(I-\Omega)^T]^{-1}(I^T+\Omega^T) \\
&= [(I^T-\Omega^T)]^{-1}(I-\Omega) \\
&= (I+\Omega)^{-1}(I-\Omega) && (3.57)
\end{aligned}$$

$$\begin{aligned} Cay(\Omega)[Cay(\Omega)]^T &= (I+\Omega)(I-\Omega)^{-1}(I+\Omega)^{-1}(I-\Omega) \\ &= (I-\Omega)^{-1}(I+\Omega)(I+\Omega)^{-1}(I-\Omega) \\ &= I \end{aligned} \tag{3.58}$$

For the special orthogonal group of order three, Cayley map can be regarded as a second order equation with respect to Ω. To evaluate it, consider a general 3×3 antisymmetric matrix:

$$\Omega = \begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix} \tag{3.59}$$

This matrix has three independent elements, namely ω_x, ω_y, ω_z. Characteristic equation can be found by solving $det(\Omega - \lambda I) = 0$ where λ is the eigenvalue and I is the 3×3 identity matrix.

$$|\Omega - \lambda I| = \begin{vmatrix} -\lambda & -\omega_z & \omega_y \\ \omega_z & -\lambda & -\omega_x \\ -\omega_y & \omega_x & -\lambda \end{vmatrix} = 0 \tag{3.60}$$

$$\lambda^3 + \lambda(\omega_x^2 + \omega_y^2 + \omega_z^2) = \lambda[\lambda^2 + (\omega_x^2 + \omega_y^2 + \omega_z^2)] = 0 \tag{3.61}$$

This equation states that the eigenvalues of the antisymmetric matrix is either equal to zero, $\lambda_1 = 0$, or equal to imaginary numbers $\lambda_{2,3} = \pm i\sqrt{\omega_x^2 + \omega_y^2 + \omega_z^2}$.

Letting $\tau^2 = -\lambda^2$ with

$$\lambda^2 = -(\omega_x^2 + \omega_y^2 + \omega_z^2) \tag{3.62}$$

we have

$$\begin{aligned}
\Omega^3 &= \begin{pmatrix} 0 & \omega_z(\omega_x^2+\omega_y^2+\omega_z^2) & -\omega_y(\omega_x^2+\omega_y^2+\omega_z^2) \\ -\omega_z(\omega_x^2+\omega_y^2+\omega_z^2) & 0 & \omega_x(\omega_x^2+\omega_y^2+\omega_z^2) \\ \omega_y(\omega_x^2+\omega_y^2+\omega_z^2) & -\omega_x(\omega_x^2+\omega_y^2+\omega_z^2) & 0 \end{pmatrix} \\
&= -(\omega_x^2+\omega_y^2+\omega_z^2)\begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix} \\
&= -\tau^2\Omega
\end{aligned} \tag{3.63}$$

Thus, every 3 × 3 skew-symmetric matrix complies with the equation:

$$\Omega^3 + \tau^2\Omega = 0 \tag{3.64}$$

Our aim is here is to write Cayley map in terms of matrices and their eigenvalues. Application of the above equation yields

$$\begin{aligned}
(I_3-\Omega)^{-1} &= I_3+\Omega+\Omega^2+\Omega^3+... \\
&= I_3+\Omega+\Omega^2-\tau^2\Omega-\tau^2\Omega^2+\tau^4\Omega+... \\
&= I_3+(1-\tau^2+\tau^4-...)\Omega+(1-\tau^2+\tau^4-...)\Omega^2 \\
&= I_3+\left(\sum_{i=0}^{\infty}(-1)^i\tau^{2i}\right)\Omega+\left(\sum_{i=0}^{\infty}(-1)^i\tau^{2i}\right)\Omega^2 \\
&= I_3+\frac{1}{\tau^2+1}\Omega+\frac{1}{\tau^2+1}\Omega^2
\end{aligned} \tag{3.65}$$

and so

$$\begin{aligned}
(I_3+\Omega)(I_3-\Omega)^{-1} &= (I_3+\Omega)\left(I_3+\frac{1}{\tau^2+1}\Omega+\frac{1}{\tau^2+1}\Omega^2\right) \\
&= I_3+\frac{1}{\tau^2+1}\Omega+\frac{1}{\tau^2+1}\Omega^2+\Omega+\frac{1}{\tau^2+1}\Omega^2+\frac{1}{\tau^2+1}\Omega^3 \\
&= I_3+\frac{2}{\tau^2+1}\Omega+\frac{2}{\tau^2+1}\Omega^2
\end{aligned} \tag{3.66}$$

As well as writing Cayley map in terms of eigenvalues and matrices, we can state its inverse with eigenvalues and matrices, or more accurately with its trace. To find the trace of Ω^2, calculate Ω^2:

$$\Omega^2 = \begin{pmatrix} -(\omega_y^2+\omega_z^2) & \omega_x\omega_y & \omega_x\omega_z \\ \omega_x\omega_y & -(\omega_x^2+\omega_z^2) & \omega_y\omega_z \\ \omega_x\omega_z & \omega_y\omega_z & -(\omega_x^2+\omega_y^2) \end{pmatrix} \tag{3.67}$$

Clearly

$$tr(\Omega^2) = 2\lambda^2 = -2\tau^2 \tag{3.68}$$

Consider the Cayley map and denote $Cay(\Omega) = \Psi$. Since Ω is antisymmetric, $tr(\Omega) = 0$. Employing this fact to Cayley map yields

$$\begin{aligned} tr(\Psi) &= tr(I_3) + \frac{2}{\tau^2+1}tr(\Omega) + \frac{2}{\tau^2+1}tr(\Omega^2) \\ &= 3 - \frac{4\tau^2}{\tau^2+1} \end{aligned} \tag{3.69}$$

Rewriting gives

$$\frac{1}{tr(\Psi)+1} = \frac{\tau^2+1}{4} \tag{3.70}$$

Taking the Cayley transformations transpose and substituting the antisymmetry property of the matrix Ω yields

$$\begin{aligned} \Psi^T &= I_3 + \frac{2}{\tau^2+1}\Omega^T + \frac{2}{\tau^2+1}(\Omega^2)^T \\ &= I_3 - \frac{2}{\tau^2+1}\Omega + \frac{2}{\tau^2+1}\Omega^2 \end{aligned} \tag{3.71}$$

Substituting equation (3.70) gives the required result:

$$\Psi - \Psi^T = \frac{4}{\tau^2+1}\Omega \tag{3.72}$$

$$\begin{aligned} \Omega &= (\Psi - \Psi^T)\frac{\tau^2+1}{4} \\ &= (\Psi - \Psi^T)\frac{1}{tr(\Psi)+1} \end{aligned} \tag{3.73}$$

We can interpret form equation (3.73) that the inverse of Cayley transform is not defined when $tr(\psi) = -1$, that is; since ψ is an orthogonal matrix, $tr(\psi) = 2cos\alpha + 1$

for the rotation angle α. The transform is not defined if $\alpha = \pm\pi$. The skew-symmetric matrices determine the rotation matrices when $\pm\pi$ is not included in the domain for Cayley inverse.

It can be concluded from above that Cayley maps state the relation between skew-symmetric matrices and orthogonal matrices similar to exponential mappings. In fact, there is a relation between two transformations see (Selig, 2007) and (Erkus & Karakilic, 2014).

CHAPTER FOUR
EXAMPLES OF MATRIX LIE GROUPS

Now, some detailed examples about matrix groups will be given. We will prove that they are Lie groups and will reveal their Lie algebras as well as their exponential maps. For more data about these groups and their role in kinematics and mobile robots see (McCarthy, 1990) and (Ebetiuc & Staab, 2005).

4.1 General Linear Group

General linear group of order n with components in $\mathbb{R}$ is the set of $n \times n$ invertible real matrices with the group operation is as matrix multiplication, that is;

$$GL(n,\mathbb{R}) = \{A \in M_{n\times n}(\mathbb{R}) \mid detA \neq 0\} \tag{4.1}$$

with

$$dim GL(n,\mathbb{R}) = n^2 \tag{4.2}$$

Some of the subgroups of $GL(n,\mathbb{R})$ are:

1. The special linear group:
   $SL(n,\mathbb{R}) = \{J \in GL(n,\mathbb{R}) | \ detJ = 1\}$
2. The special orthogonal group:
   $SO(n,\mathbb{R}) = \{J \in GL(n,\mathbb{R}) | \ detJ = 1 \text{ and } J^T J = I\}$ where I is the identity matrix of the group and J^T is the transpose of the matrix J.
3. The special unitary group:
   $SU(n,\mathbb{R}) = \{J \in GL(n,\mathbb{R}) | \ detJ = 1 \text{ and } J^* = J^{-1}\}$ where I is the identity matrix of the group, J^* is the adjoint matrix of J and J^{-1} is the inverse matrix of J.

Example 4.1.1. General linear group $GL(n,\mathbb{R})$ is a Lie group and its Lie algebra is $\mathfrak{gl}(n,\mathbb{R}) = \{A : A \in M(n,\mathbb{R})\}$.

We will show this step by step:

- $GL(n,\mathbb{R})$ is open in $M(n,\mathbb{R})$.

  Proof: The definition of determinant using cofactor expansion along the i^{th} row of a matrix $A \in M(n,\mathbb{R})$ can be written as;

  $$detA = a_{i1}A_{i1} + a_{i2}A_{i2} + ... + a_{in}A_{in} = \sum_{j=1}^{n}(-1)^{i+j}a_{ik}A_{ik} \tag{4.3}$$

  So, the determinant function $det : M(n,\mathbb{R}) \to \mathbb{R}$ is clearly continuous. The set $\{0\}$ is closed, the complement $\mathbb{R}\setminus\{0\}$ is an open subset of $\mathbb{R}$. Since $det^{-1}(\mathbb{R}\setminus\{0\}) = GL(n,\mathbb{R})$, i.e. the inverse image of an open set under a continuous map is again open, $GL(n,\mathbb{R})$ is an open subset of $M(n,\mathbb{R})$.

- $GL(n,\mathbb{R})$ is a smooth manifold.

  Proof: First step states that "$GL(n,\mathbb{R})$ is open subset of $M(n,\mathbb{R})$." Since open subset of a smooth manifold is again a smooth manifold, $GL(n,\mathbb{R})$ is a smooth manifold.

- $GL(n,\mathbb{R})$ is a Lie group.

  Proof: With the aid of first and second steps, to show that $GL(n,\mathbb{R})$ is a Lie group, we now only need to prove that

  $$\text{mult} : GL(n,\mathbb{R}) \times GL(n,\mathbb{R}) \to GL(n,\mathbb{R}) \tag{4.4}$$

  $$\text{inv} : GL(n,\mathbb{R}) \to GL(n,\mathbb{R}) \tag{4.5}$$

  maps are smooth. For any $A, B \in GL(n,\mathbb{R})$;

  $$(A,B) \mapsto \left(\sum_{k=1}^{n} a_{ik}b_{kj}\right)_{i,j=1,2,..,n} \tag{4.6}$$

  is clearly smooth because the entries of the right hand side are just polynomials of elements of A and B. For inversion map recall that for $A \in GL(n,\mathbb{R})$, since $detA \neq 0$ and A is invertible we can write

  $$A^{-1} = \frac{1}{detA}adj(A) \tag{4.7}$$

Rewriting the equation;

$$A^{-1} = \frac{1}{detA}(-1)^{i+j}det(A_{ij}) \tag{4.8}$$

where $A_{ij} \in GL(n-1,\mathbb{R})$ is a cofactor matrix whose entries are the remaining ones of the matrix A's i^{th} row and j^{th} column deleted. It shows that, A^{-1} is a polynomial with entries from the matrix A, i.e., *inv* map is also smooth.

- $GL(n,\mathbb{R})$'s Lie algebra is $\mathfrak{gl}(n,\mathbb{R})$.

  Proof: Claim: The Lie algebra of the general linear group is $M(n,\mathbb{R})$, the set of $n \times n$ matrices.

  Let A be any $n \times n$ real matrix, then clearly e^{tA} is invertible and real valued. Also if e^{tA} is real $\forall t \in \mathbb{R}$, then since;

$$\frac{d}{dt}(e^{tA})|_{t=t_0} = \lim_{\Delta t \to 0} \frac{e^{(t_0+\Delta t)A} - e^{t_0}}{\Delta t} = Ae^{t_0 A} \tag{4.9}$$

  for $t_0 = 0$ we get;

$$\frac{d}{dt}(e^{tA})|_{t=0} = \lim_{\Delta t \to 0} \frac{e^{\Delta t A} - I}{\Delta t} = A \tag{4.10}$$

  which is also real because inside of the limit is in $M(n,\mathbb{R})$. Hence;

$$\begin{aligned} \mathfrak{gl}(n,\mathbb{R}) &= \{A : A \in M(n,\mathbb{R})\} \\ &= \{A \in M(n,\mathbb{R}) : e^{tA} \in M(n,\mathbb{R}), \forall t \in \mathbb{R}\} \\ &= M(n,\mathbb{R}) \end{aligned} \tag{4.11}$$

Example 4.1.2. The exponential map of general linear group from $\mathfrak{gl}(n,\mathbb{R})$ to $GL(n,\mathbb{R})$ is defined as

$$exp(T) = \phi_T(1) = \sum_{i=0}^{\infty} \frac{T^i}{i!} \tag{4.12}$$

To show this we will use the exponential map definition related to the monoparametric subgroup

$$exp(T) = \phi_T(1) \tag{4.13}$$

Recall that for $T \in \mathfrak{gl}(n,\mathbb{R})$,

$$\phi_T(t) = \sum_{i=0}^{\infty} \frac{t^i T^i}{i!} \tag{4.14}$$

Clearly $\phi_T(0) = I$ and

$$\begin{aligned} \frac{d}{dt}\phi_T(t) &= \sum_{i=1}^{\infty} \frac{i t^{i-1} T^i}{i!} \\ &= \sum_{i=1}^{\infty} \frac{t^{i-1} T^i}{(i-1)!} \\ &= \left(\sum_{i=1}^{\infty} \frac{t^{i-1} T^{i-1}}{(i-1)!} \right) T \\ &= \left(\sum_{i=0}^{\infty} \frac{t^i T^i}{i!} \right) T \\ &= \phi_T(t) T \end{aligned} \tag{4.15}$$

Thus we can state that ϕ_T is in the set of left invariant vector field of $\mathcal{X}_T$. Hence

$$exp(T) = \phi_T(1) = \sum_{i=0}^{\infty} \frac{T^i}{i!} \tag{4.16}$$

is the exponential map of $GL(n,\mathbb{R})$.

4.2 Special Orthogonal Group

Elements of finite groups can be considered as operators that act on space such as translations, counterclockwise or clockwise rotations. To elaborate more we will check on some of the matrix groups.

Initially, for $SO(n,\mathbb{R})$, we will check the aspects of orthogonal matrix group. The isometry group $O(n,\mathbb{R})$ can be written as;

$$O(n,\mathbb{R}) = \{A \in GL(n,\mathbb{R}) \mid A^T A = I\} \tag{4.17}$$

where $A = [a_{ij}]$, $1 \leq i,j \leq n$ A^T is the transpose of A and I is the $n \times n$ identity matrix with;

$$dim\, O(n,\mathbb{R}) = \binom{n}{2} \tag{4.18}$$

Consider $det : O(n,\mathbb{R}) \to \mathbb{R}$. For $A \in O(n,\mathbb{R})$, applying the fact that determinant function is continuous, we can write $A^T A = I$ as

$$\begin{aligned} detI &= det(A^T A) \\ &= det(A^T)\,det(A) \\ &= (detA)^2 \end{aligned} \tag{4.19}$$

and since $detI = 1$, $detA = \pm 1$. Because of this expression, orthogonal matrix group can be written as

$$O(n,\mathbb{R}) = O^+(n,\mathbb{R}) \cup O^-(n,\mathbb{R}) \tag{4.20}$$

The two important disjoint subsets of $O(n,\mathbb{R})$ are;

$$O^-(n,\mathbb{R}) = \{A \in O(n,\mathbb{R}) | detA = -1\} \tag{4.21}$$

$$O^+(n,\mathbb{R}) = \{A \in O(n,\mathbb{R}) | detA = 1\} \tag{4.22}$$

Since $O^-(n,\mathbb{R}) \cap O^+(n,\mathbb{R}) = \emptyset$, $O(n,\mathbb{R})$ has two connected components, however $O(n,\mathbb{R})$ is not connected. Elements of $O^-(n,\mathbb{R})$ are called negative/indirect isometries or reflections. The other subgroup $O^+(n,\mathbb{R})$ which is known as special orthogonal group $SO(n,\mathbb{R})$, the group of positive/direct isometries because it is a subgroup of special Euclidean group. It's elements represent rotations and

$$dimSO(n,\mathbb{R}) = \binom{n}{2} \tag{4.23}$$

The groups $SO(2,\mathbb{R})$ and $SO(3,\mathbb{R})$ are crucial for our study, so we will elaborate these groups in more detail.

Special orthogonal group of order two is the set of all matrices of the form with the rotation angle θ :

$$A = \begin{pmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{pmatrix} \tag{4.24}$$

Dimension of this group is

$$dimSO(2,\mathbb{R}) = \binom{2}{2} = 1 \tag{4.25}$$

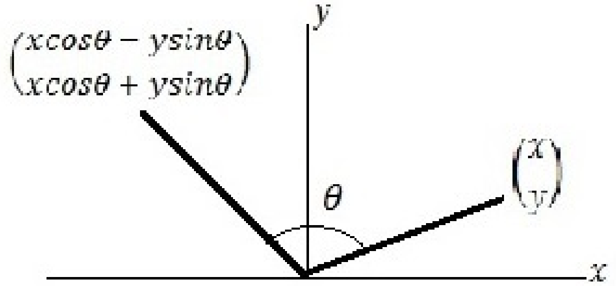

Figure 4.1 Rotation with the angle θ in $SO(2,\mathbb{R})$

Observe that, $detA = cos^2\theta + sin^2\theta = 1$ and

$$A^T A = \begin{pmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{pmatrix} \begin{pmatrix} cos\theta & sin\theta \\ -sin\theta & -cos\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I \tag{4.26}$$

as expected. This group can be illustrated as the rotations in the plane.

Special orthogonal group of order three is the set of all rotations with axes that passes through origin in three dimensonal real space. Dimension of this group is

$$dimSO(3,\mathbb{R}) = \binom{3}{2} = 3. \tag{4.27}$$

The group can be illustrated as

$$SO(3,\mathbb{R}) = \{A \in GL(3,\mathbb{R}) \mid A^T A = I,\ detA = 1\} \tag{4.28}$$

Rotations can be decomposed into three parts which can be referred as three axial rotations: X,Y and Z. These three rotation angles are known as Euler angles. Any given rotation can be written as the decomposition of these three angles.

With two reference frames we can construct three rotation elements corresponding to three Euler angles. Evaluation is as follows: Rotate the frame about the z-axis by an angle γ, then rotate about the new y-axis by angle β and then rotate it again about the x- axis by the angle α. Adopting this point of view yields the elementary rotations respectively to x,y and z-axes:

$$R_x(\alpha) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & cos\alpha & -sin\alpha \\ 0 & sin\alpha & cos\alpha \end{pmatrix} \tag{4.29}$$

$$R_y(\beta) = \begin{pmatrix} cos\beta & 0 & sin\beta \\ 0 & 1 & 0 \\ -sin\beta & 0 & cos\beta \end{pmatrix} \tag{4.30}$$

$$R_z(\gamma) = \begin{pmatrix} cos\gamma & -sin\gamma & 0 \\ sin\gamma & cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{4.31}$$

These Euler angles are also known as roll, pitch and yaw respectively. The rotations in $\mathbb{R}^3$ can be constructed from these three basic rotations except for singularities. For example $R_x(\alpha)R_y(\beta)R_z(\gamma)$ is equivalent to

$$\begin{pmatrix} cos\beta cos\gamma & -sin\gamma cos\beta & sin\beta \\ sin\alpha sin\beta cos\gamma + sin\gamma cos\alpha & -sin\alpha sin\beta sin\gamma + cos\gamma cos\alpha & -sin\alpha cos\beta \\ -cos\alpha sin\beta cos\gamma + sin\alpha sin\gamma & cos\alpha sin\beta sin\gamma + sin\alpha cos\gamma & cos\alpha cos\beta \end{pmatrix} \tag{4.32}$$

and for the value $\beta = \pi/2$ the matrix above becomes;

$$\begin{pmatrix} 0 & 0 & 1 \\ sin(\alpha+\gamma) & cos(\alpha+\gamma) & 0 \\ -cos(\alpha+\gamma) & sin(\alpha+\gamma) & 0 \end{pmatrix} \tag{4.33}$$

which creates limited movements because of three angles dropping to one, which means it has only one degree of freedom.

Our main purpose is to show that;

Example 4.2.1. $SO(n,\mathbb{R})$ is a Lie group and its Lie algebra is

$$\mathfrak{so}(n,\mathbb{R}) = \{\Omega \in M(n,\mathbb{R}) \mid \Omega^T + \Omega = 0,\ det\Omega = 1\} \tag{4.34}$$

To prove that $SO(n,\mathbb{R})$ is a Lie group we will apply a theorem (Baker, 2002) :

Theorem 4.2.2. *If H is a closed subgroup of G which is a Lie group, then H is a Lie subgroup of G.*

If $\Omega^T\Omega = I$ for $\Omega \in O(n,\mathbb{R})$, then $\Omega^T = \Omega^{-1}$. The mapping

$$\Phi : O(n,\mathbb{R}) \to M(n,\mathbb{R}), \quad \Omega \mapsto \Omega^{-1} - \Omega^T \tag{4.35}$$

is continuous because the entries of $\Omega^{-1} - \Omega^T$ are polynomials. Inverse image of the mapping

$$\Phi^{-1}(\Omega^{-1} - \Omega^T) = \Phi^{-1}(0) = O(n,\mathbb{R}) \tag{4.36}$$

reveals that $O(n,\mathbb{R})$ is a closed subgroup of $GL(n,\mathbb{R})$ which we've proven before that it is a Lie group. By the theorem above $O(n,\mathbb{R})$ is a Lie subgroup of $GL(n,\mathbb{R})$. Because of the fact that $SO(n,\mathbb{R})$ is a connected component of $O(n,\mathbb{R})$, $SO(n,\mathbb{R})$ is also a Lie group.

To evaluate the Lie algebra $\mathfrak{so}(n,\mathbb{R}) = \{\Omega \in M(n,\mathbb{R}) \mid \Omega^T + \Omega = 0,\ det\Omega = 1\}$, we need to prove that $\Omega \in \mathfrak{so}(n,\mathbb{R})$ if and only if $\Omega^T\Omega = 0$. Initially assume that $\Omega^T\Omega = 0$, then for all $\omega \in \mathbb{R}$ we have;

$$(e^{\omega\Omega})^T = e^{\omega\Omega^T} = e^{-\omega\Omega} = (e^{\omega\Omega})^{-1} \tag{4.37}$$

which confirms that $\Omega \in \mathfrak{so}(n,\mathbb{R})$. The main diagonal entries of the skew-symmetric matrix $-\Omega = \Omega^T$ are zero, in other words we can write $\Omega = [\omega_{ij}]$ as

$$\Omega = [\omega_{ij}] = \begin{cases} \omega_{ij} = -\omega_{ji}, & \text{if } i \neq j \\ 0, & \text{otherwise} \end{cases} \tag{4.38}$$

which leads to $tr(\Omega) = 0$.

Conversely, assume that $\Omega \in \mathfrak{so}(n,\mathbb{R})$. For all $\omega \in \mathbb{R}$, we have

$$e^{-\omega\Omega} = (e^{\omega\Omega})^{-1} = (e^{\omega\Omega})^T = e^{\omega\Omega^T} \tag{4.39}$$

Differentiating with respect to ω and letting $\omega = 0$ yields;

$$\frac{d}{dt}(e^{-\omega\Omega})|_{\omega=0} = \frac{d}{dt}(e^{\omega\Omega^T})|_{\omega=0} \quad \Rightarrow \quad -\Omega = \Omega^T \tag{4.40}$$

Hence we have determined that

$$\mathfrak{so}(n,\mathbb{R}) = \{\Omega \in M(n,\mathbb{R}) \mid \Omega^T + \Omega = 0,\ det\Omega = 1\} \tag{4.41}$$

is the Lie algebra of the Lie group $SO(n,\mathbb{R})$. For example, the elements of the Lie algebra of $SO(3,\mathbb{R})$ are 3×3 skew-symmetric matrices which are of the form

$$\begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix} \tag{4.42}$$

For $\mathfrak{so}(3,\mathbb{R})$,

$$A = \begin{pmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{pmatrix} \tag{4.43}$$

and

$$B = \begin{pmatrix} 0 & -b_z & b_y \\ b_z & 0 & -b_x \\ -b_y & b_x & 0 \end{pmatrix} \tag{4.44}$$

the Lie bracket structure can be stated as

$$\begin{aligned} [A,B] &= AB - BA \qquad (4.45) \\ &= \begin{pmatrix} 0 & a_yb_x - a_xb_y & a_zb_x - a_xb_z \\ a_xb_y - a_yb_x & 0 & a_zb_y - a_yb_z \\ a_xb_z - a_zb_x & a_yb_z - a_zb_y & 0 \end{pmatrix} \end{aligned}$$

We are investigating the exponential map for the special orthogonal group of degree 3, because it gives an unambiguous formula which is known as Rodrigues' rotation formula.

Example 4.2.3. The exponential map of $SO(3,\mathbb{R})$ is

$$e^M = I_3 + \frac{sin\alpha}{\alpha}M + \frac{(1-cos\alpha)}{\alpha}M^2 \tag{4.46}$$

where I_3 is the 3×3 identity matrix, $\alpha^2 = m_x^2 + m_y^2 + m_z^2$ and

$$M = \begin{pmatrix} 0 & -m_z & m_y \\ m_z & 0 & -m_x \\ -m_y & m_x & 0 \end{pmatrix} \tag{4.47}$$

Observe that

$$\begin{aligned}
M^2 &= \begin{pmatrix} 0 & -m_z & m_y \\ m_z & 0 & -m_x \\ -m_y & m_x & 0 \end{pmatrix}\begin{pmatrix} 0 & -m_z & m_y \\ m_z & 0 & -m_x \\ -m_y & m_x & 0 \end{pmatrix} \\
&= \begin{pmatrix} -(m_y^2+m_z^2) & m_x m_y & m_x m_z \\ m_x m_y & -(m_x^2+m_z^2) & m_y m_z \\ m_x m_z & m_y m_z & -(m_x^2+m_y^2) \end{pmatrix} \\
&= -\alpha^2 I_3 + \begin{pmatrix} m_x^2 & m_x m_y & m_x m_z \\ m_x m_y & m_y^2 & m_y m_z \\ m_x m_z & m_y m_z & m_z^2 \end{pmatrix} \\
&= -\alpha^2 I_3 + N \qquad (4.48)
\end{aligned}$$

which yields

$$MN = NM = 0. \qquad (4.49)$$

Succesful iteration yields for $i \geq 0$;

$$\begin{aligned}
M^2 &= -\alpha^2 I + N && (4.50) \\
M^3 &= -\alpha^2 IM + NM = -\alpha^2 M && (4.51) \\
M^4 &= -\alpha^2 M^2 && (4.52) \\
M^5 &= \alpha^4 M && (4.53) \\
&\ldots \\
M^{4i+1} &= \alpha^{4i} M && (4.54) \\
M^{4i+2} &= \alpha^{4i} M^2 && (4.55) \\
M^{4i+3} &= -\alpha^{4i+2} M && (4.56) \\
M^{4i+4} &= -\alpha^{4i+2} M^2 && (4.57)
\end{aligned}$$

Substituting these values into the exponential map

$$exp(M) = \sum_{i=0}^{\infty} \frac{M^i}{i!} = I + M + \frac{M^2}{2} + \frac{M^3}{3!} + \ldots \qquad (4.58)$$

gives us the required result, known as Rodrigues' formula:

$$\begin{aligned} exp(M) &= \sum_{i=0}^{\infty} \frac{M^i}{i!} \\ &= I + M + \frac{M^2}{2} + \frac{M^3}{3!} + ... \\ &= I + \frac{1}{\alpha}\left(\alpha - \frac{\alpha^3}{3!} + \frac{\alpha^5}{5!} - ...\right)M + \frac{1}{\alpha^2}\left(\frac{1}{2} - \frac{\alpha^2}{4!} + \frac{\alpha^4}{6!} - ...\right)M^2 \\ &= I + \frac{sin\alpha}{\alpha}M + \frac{(1-cos\alpha)}{\alpha}M^2 \end{aligned} \tag{4.59}$$

4.3 Special Euclidean Group

The group of rigid body motions that preserves Euclidean metric is called special Euclidean group. This is the group of isometries that preserves orientation of the vector space $\mathbb{R}^n$ which contains rotations and translations and it can be represented as

$$SE(n,\mathbb{R}) = \left\{ A \in GL(n+1,\mathbb{R}) \mid A = \begin{pmatrix} R & t \\ 0 & 1 \end{pmatrix}, R \in SO(n,\mathbb{R}),\ t \in \mathbb{R}^n \right\} \tag{4.60}$$

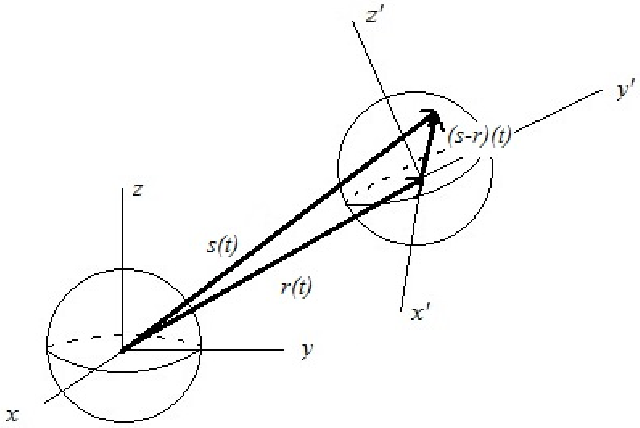

Figure 4.2 Position of a rigid body with respect to time t in three dimension

Briefly, an element in this set can be illustrated with a rotation R and a translation t from an initial position to a final position. To proceed explanation of this group further, firstly we will analyze the meaning of "semi-direct product".

Definition 4.3.1. For a multiplicative group G and a commutative group H with a linear action G on H;

$$G \times H \to H, \qquad (g,h) \mapsto h \tag{4.61}$$

the semi-direct product of G and H, $G \ltimes H$, can be defined as

$$(g_1,h_1)(g_2,h_2) = (g_1 g_2, h_1 + g_1(h_2)) \tag{4.62}$$

$$(g,h)^{-1} = (g^{-1}, -g^{-1}(h)) \tag{4.63}$$

where $g, g_1, g_2 \in G$ and $h, h_1, h_2 \in H$ (Selig, 2005).

Denote rigid body transformations (displacements) as pairs (Ω, v) where $\Omega \in SO(n,\mathbb{R})$ and $v \in \mathbb{R}^n$. In fact, pure rotation can be stated as $(\Omega, 0)$ and pure translation can be stated as (I, v). For $\Omega_1, \Omega_2 \in SO(n,\mathbb{R})$ and $v_1, v_2 \in \mathbb{R}^n$, the composition of two different transformations can be expressed as

$$(\Omega_2, v_2)(\Omega_1, v_1) = (\Omega_2\Omega_1, \Omega_2 v_1 + v_2) \tag{4.64}$$

which reveals that the general rigid body transformations, in other words, the elements of the special Euclidean group is the semi-direct product of the special orthogonal group with $\mathbb{R}^n$. Moreover, we can express it as

$$SE(n,\mathbb{R}) = SO(n,\mathbb{R}) \ltimes \mathbb{R}^n \tag{4.65}$$

Consider $SE(n,\mathbb{R}) \to GL(n+1,\mathbb{R})$ with $(\Omega, v) \mapsto \begin{pmatrix} \Omega & v \\ 0 & 1 \end{pmatrix}$. This map is clearly one-to-one and

$$\begin{pmatrix} \Omega_2 & v_2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \Omega_1 & v_1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \Omega_2\Omega_1 & \Omega_2 v_1 + v_2 \\ 0 & 1 \end{pmatrix} \tag{4.66}$$

As we can see it is similar to the equation of the product of pairs and so it is a homomorphism. This yields that we can use the representation of matrices. Inverse of this matrix can be expressed as;

$$\begin{pmatrix} \Omega & v \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \Omega^T & -\Omega^T v \\ 0 & 1 \end{pmatrix} \tag{4.67}$$

Dimension of the special Euclidean group is

$$dimSE(n,\mathbb{R}) = \begin{pmatrix} n+1 \\ 2 \end{pmatrix} = \frac{n(n+1)}{2} \tag{4.68}$$

or more conveniently, using the information of semi-direct product we have illustrated before

$$\begin{aligned} dim\,SE(n,\mathbb{R}) &= dim\,SO(n,\mathbb{R}) + dim\,(\mathbb{R}^n) \\ &= \begin{pmatrix} n \\ 2 \end{pmatrix} + n \\ &= \frac{n(n+1)}{2} \\ &= \begin{pmatrix} n+1 \\ 2 \end{pmatrix} \end{aligned} \tag{4.69}$$

Specifically, the groups $SE(2,\mathbb{R})$ and $SE(3,\mathbb{R})$ are of great importance, because these are planar displacements and rigid body displacements in three dimensions respectively. The critical subgroups of $SE(3,\mathbb{R})$ are

1. The group of rotations in three dimensions about the origin:
   $SO(3,\mathbb{R}) = \{A \in GL(3,\mathbb{R}) | \, A^T A = I, \; detA = 1\}$
2. The set of planar displacements:
   $SE(3,\mathbb{R}) = \left\{ A \in GL(4,\mathbb{R}) \mid A = \begin{pmatrix} R & t \\ 0 & 1 \end{pmatrix}, \; R \in SO(3,\mathbb{R}), \; t \in \mathbb{R}^n \right\}$
3. The set of orthogonal matrices in two dimensions:
   $SO(2,\mathbb{R}) = \{A \in GL(2,\mathbb{R}) \mid A^T A = I, \; detA = 1\}$

Example 4.3.2. $SE(n,\mathbb{R})$ is a Lie group and its Lie algebra is

$$\mathfrak{se}(n,\mathbb{R}) = \left\{ \begin{pmatrix} \Omega & v \\ 0 & 0 \end{pmatrix} \in \mathfrak{gl}(n+1,\mathbb{R}) \; : \; \Omega \in \mathfrak{so}(n,\mathbb{R}), v \in \mathbb{R}^n \right\} \tag{4.70}$$

where Ω is an $n \times n$ skew-symmetric matrix with real components.

Any two elements in $SE(n,\mathbb{R})$ is a continuous mapping under the matrix multiplication, that is, for $\Omega_1, \Omega_2 \in SO(n,\mathbb{R})$ and $v_1, v_2 \in \mathbb{R}^n$,

$$\begin{pmatrix} \Omega_2 & v_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \Omega_1 & v_1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \Omega_2\Omega_1 & \Omega_2 v_1 + v_2 \\ 0 & 1 \end{pmatrix} \tag{4.71}$$

is clearly continuous and the inverse element

$$\begin{pmatrix} \Omega & v \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} \Omega^T & -\Omega^T v \\ 0 & 1 \end{pmatrix} \tag{4.72}$$

is continuous too, which implies that $SE(n,\mathbb{R})$ is a continuous group. The special Euclidean group of order n is a subgroup of the general linear group of order $n+1$. Since we have proven before that $GL(n,\mathbb{R})$ is a differentiable manifold, so is $SE(n,\mathbb{R})$. Hence we have the necessary information to state that $SE(n,\mathbb{R})$ is a Lie group.

Recall the definition of Lie algebra regarding the matrix Lie groups. If the exponential map of a matrix is in $SE(n,\mathbb{R})$, then the matrix that is producing the exponential map is in $\mathfrak{se}(n,\mathbb{R})$. If $e^{\omega\Upsilon} \in SE(n,\mathbb{R})$, then

$$\frac{d}{d\omega}(e^{\omega\Upsilon})|_{\omega=0} = \Upsilon \tag{4.73}$$

with the $(n+1)^{th}$ row is all constructed by zeros. Now, assume that Υ is in the form

$$\Upsilon = \begin{pmatrix} \Omega & v \\ 0 & 0 \end{pmatrix} \tag{4.74}$$

where $\Omega \in SO(n,\mathbb{R})$ and $v \in \mathbb{R}^n$. We will use induction on i for the matrix Υ. Assume that $i-1 = i-1$ is true for all $i \geq 0$;

$$\Upsilon^{i-1} = \begin{pmatrix} \Omega^{i-1} & \Omega^{i-2}v \\ 0 & 0 \end{pmatrix} \tag{4.75}$$

Then

$$\Upsilon^{i} = \Upsilon^{i-1}\Upsilon = \begin{pmatrix} \Omega^{i-1} & \Omega^{i-2}v \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \Omega & v \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \Omega^{i} & \Omega^{i-1}v \\ 0 & 0 \end{pmatrix} \tag{4.76}$$

Therefore the exponential function becomes;

$$\begin{aligned}
exp(\Upsilon) &= \sum_{i=0}^{\infty}\frac{\Upsilon^i}{i!}\\
&= I_{n+1}+\frac{1}{i!}\sum_{i=1}^{\infty}\begin{pmatrix}\Omega^i & \Omega^{i-1}v\\ 0 & 0\end{pmatrix}\\
&= I_{n+1}+\begin{pmatrix}\displaystyle\sum_{i=1}^{\infty}\frac{\Omega^i}{i!} & \displaystyle\left(\sum_{i=1}^{\infty}\frac{\Omega^{i-1}}{i!}\right)v\\ 0 & 0\end{pmatrix}\\
&= \begin{pmatrix}I_n+\displaystyle\sum_{i=1}^{\infty}\frac{\Omega^i}{i!} & \displaystyle\left(\sum_{i=1}^{\infty}\frac{\Omega^{i-1}}{i!}\right)v\\ 0 & 1\end{pmatrix}\\
&= \begin{pmatrix}\displaystyle\sum_{i=0}^{\infty}\frac{\Omega^i}{i!} & \displaystyle\left(\sum_{i=1}^{\infty}\frac{\Omega^{i-1}}{i!}\right)v\\ 0 & 1\end{pmatrix}\\
&= \begin{pmatrix}e^{\Omega} & Cv\\ 0 & 1\end{pmatrix} \qquad (4.77)
\end{aligned}$$

where I_{n+1} is the $(n+1)\times(n+1)$ identity matrix with

$$C=\sum_{i=1}^{\infty}\frac{\Omega^{i-1}}{i!}. \qquad (4.78)$$

Then $e^{\omega\Omega}$ is of the form

$$exp(\omega\Upsilon)=\begin{pmatrix}e^{\omega\Omega} & P\\ 0 & 1\end{pmatrix}. \qquad (4.79)$$

Since $e^{\omega\Omega}$ is in the Lie algebra of the special orthogonal group we have

$$\Omega^T+\Omega=0. \qquad (4.80)$$

Hence the Lie algebra of the special Euclidean group is

$$\mathfrak{se}(n,\mathbb{R})=\left\{\begin{pmatrix}\Omega & v\\ 0 & 0\end{pmatrix}\in\mathfrak{gl}(n+1,\mathbb{R})\ :\ \Omega\in\mathfrak{so}(n,\mathbb{R}),v\in\mathbb{R}^n\right\} \qquad (4.81)$$

Meanwhile we have shown the exponential map for special Euclidean group. The exponential map $exp : \mathfrak{se}(n,\mathbb{R}) \to SE(n,\mathbb{R})$ of $SE(n,\mathbb{R})$ is

$$e^{\Upsilon} = \begin{pmatrix} e^{\Omega} & Cv \\ 0 & 1 \end{pmatrix} \tag{4.82}$$

where

$$\Upsilon = \begin{pmatrix} \Omega & v \\ 0 & 0 \end{pmatrix} \in \mathfrak{se}(n,\mathbb{R}) \tag{4.83}$$

with

$$C = \sum_{i=1}^{\infty} \frac{\Omega^{i-1}}{i!}. \tag{4.84}$$

CHAPTER FIVE
CONCLUSION

After stating basic definitions for Lie groups, with the help of equation (3.6) from monoparametric subgroups, exponential map is evaluated and stated clearly at definition (3.2.1.). By stating the relation between $\phi_{\mathcal{W}}$ and $\phi_{p\mathcal{W}}$, a general formula for exponential map is constructed. Some of the properties of exponential map such as (3.19), (3.21), (3.22), (3.24) is examined. Then by briefly explaining the meaning of infinitesimal generators, the generators for the special orthogonal group of order three are found and the exponential map is produced related to them. It is shown that the generated matrix belongs to the Lie algebra of the special orthogonal group of order three. The usage of skew-symmetric matrices, which are the profound part for rigid body motions, led to Cayley maps as seen at (3.66). The general linear group, special orthogonal group, special Euclidean group for order n are introduced briefly and then detailed examples are carried out regarding to these groups. Their Lie algebras as well as their exponential maps are found. Specifically, for special orthogonal group, the exponential map is found for order 3 and it has seen that the relation is similar to the Rodrigues' equation. The relation between Cayley transformation and exponential map may be investigated for $SE(2,\mathbb{R})$, $SO(2,\mathbb{R})$ or more in future studies.

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