EXPONENTIAL TYPE OPERATORS AND THEIR APPROXIMATION PROPERTIES

by

KÜBRA ALTINAY

THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES
OF
THE ABANT İZZET BAYSAL UNIVERSITY
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE
DEGREE OF
MASTER OF SCIENCE
IN
THE DEPARTMENT OF MATHEMATICS

AUGUST 2014

Approval of the Graduate School of Natural and Applied Sciences

Prof. Dr. Yaşar DÜRÜST
Director

I certify that this thesis satisfies all the requirements as a thesis for the degree of Master of Science.

Prof. Dr. Zafer ERCAN
Head of Department

This is to certify that we have read this thesis and that in our opinion it is fully adequate, in scope and quality, as a thesis for the degree of Master of Science.

Assoc. Prof. Dr. Harun KARSLI
Supervisor

Examining Committee Members

Prof. Dr. Cenap ÖZEL	(AİBÜ)	
Assoc.Prof. Dr. Harun KARSLI	(AİBÜ)	
Assist.Prof. Dr. Ayşe Çiçek GÖZÜTOK	(AİBÜ)	

ABSTRACT

EXPONENTIAL TYPE OPERATORS AND THEIR APPROXIMATION PROPERTIES

ALTINAY, Kübra

M.Sc., Department of Mathematics

Supervisor: Assoc. Prof. Dr. Harun Karslı

August 2014, 75 pages

This thesis is a survey on some approximation properties of exponential type operators in the Approximation Theory. One of the fundamental problems of analysis is to approximate a given function f in some sense or other by functions having certain properties, and generally, by functions which have 'better' properties than f. It is to be expected that the better behaved functions are to be constructed from the given f by some smoothing operation on f itself.

This thesis consists of four chapters. The first chapter is devoted to the introduction. The second chapter contains concepts, definitions and also some important theorems, which we need further studies and calculations. In the third chapter, we give some informations about the exponential type operators and their connection with the differential equations. In the last chapter, we study the uniform convergence and rate of convergence of some exponential type operators by modulus of continuity and Lipschitz Class functions. Also, we study Korovkin type theorems and Voronovskaya type theorems for these operators.

Keywords: Exponential Type Operators, Modulus Of Continuity, Korovkin Type Theorems, Lipschitz Class Functions, Voronovskaya Type Theorems.

ÖZET

ÜSTEL TİPLİ OPERATÖRLER VE ONLARIN YAKINSAKLIK ÖZELLİKLERİ

ALTINAY, Kübra
Yüksek lisans, Matematik Bölümü
Tez Yöneticisi: Doç. Dr. Harun Karslı

Ağustos 2014, 75 sayfa

Bu tez, Yaklaşım Teorisindeki üstel tipli operatörlerin yaklaşım özellikleri üzerine yapılan bir çalışmadır. Analizin temel problemlerinden biri f gibi bazı kötü özelliklere sahip bir fonksiyona, daha iyi özellikleri olan başka bir fonksiyonla yaklaşmaktır.

Bu tez dört bölümden oluşmaktadır. Birinci kısım giriş bölümüne ayrılmıştır. İkinci bölüm kavramlar, tanımlar ve ayrıca ileriki çalışmalar ve hesaplamalar için gerekli olan bazı teoremler içermektedir. Üçüncü bölümde üstel tipli operatörlerle ilgili bazı bilgiler ve onların diferansiyel denklemlerle olan ilişkileri verildi. Son bölümde ise bazı üstel tipli operatörlerin düzgün yakınsaklık özellikleri çalışıldı ve yakınsaklık oranları süreklilik modülü ve Lipschitz Sınıfından fonksiyonlar yardımıyla incelendi. Ayrıca bu operatörler için Korovkin tipli teoremler ve Voronovskaya tipli teoremler çalışıldı.

Anahtar Kelimeler: Üstel tipli operatörler, Süreklilik Modülü, Korovkin tipli teoremler, Lipschitz sınıfından fonksiyonlar, Voronovskaya tipli teoremler.

To my family, my husband and my supervisor...

ACKNOWLEDGEMENTS

I would like to express sincere appreciation to my supervisor, Assoc. Prof. Dr. Harun Karslı for his motivation, helpful discussions, encouragement, patience and constant guidance during this work. I would also like to thank other examining committee members for their suggestions.

I am grateful to express my appreciation to my family, my husband, relatives and friends for their continuous helps.

TABLE OF CONTENTS

NOTATIONS

$f_n(x) \rightrightarrows f(x)$	$Uniform\ convergence\ of\ \{f_n\}\ to\ f$
$\mathbb{N}$	$\mathbb{N} = \{1, 2, ...\}$ $the\ set\ of\ natural\ numbers$
$C[a,b]$	$The\ set\ of\ continuous\ functions\ on\ closed\ interval\ [a,b]$
$B[a,b]$	$The\ set\ of\ bounded\ functions\ on\ closed\ interval\ [a,b]$
$P[a,b]$	$The\ set\ of\ polynomials\ on\ closed\ interval\ [a,b]$
$C_{\rho,K_f}[0,\infty)$	$The\ weighted\ space\ for\ the\ function\ f$
$\omega(f;\delta)$	$The\ modulus\ of\ continuity\ of\ f$
$\Omega(f;\delta)$	$The\ modulus\ of\ continuity\ of\ f\ in\ weighted\ spaces$
$Lip_M(\alpha)$	$Lipschitz\ class\ functions$
$\|f\|$	$Norm\ of\ the\ function\ f$
$C_0\,(0,1)$	$The\ set\ of\ all\ functions\ f : I = (0,1) \to \mathbb{R},\ continuous\ and\ bounded$
$\varphi_{n,k}\,(x)$	$The\ k-th\ moment$
$(B_n f)\,(x)$	$Bernstein\ polynomials$
$(S_n f)\,(x)$	$Sz\acute{a}sz\ operator$
$(K_n f)\,(x)$	$Baskakov\ operator$
$(W_n f)\,(x)$	$Gauss - Weierstrass\ operator$
$(P_n f)\,(x)$	$Post - Widder\ operator$

CHAPTER 1

INTRODUCTION

One of the fundamental problems of analysis is to approximate a given function f in some sense or other by functions having certain properties, and generally, by functions which have 'better' properties than f. Approximation Theory is that area of analysis which, at its cone, is concerned with the ability to approximate functions by simpler and more easily calculated functions.

This theory takes its origin from at 19th century. At the beginning of the century the functions were viewed by the formulas, such as series, or as a solutions of differential equations. However largely as a consequence of the claims Fourier and result of Dirichlet and the modern concept of a function by its properties were introduced.

The birth of Approximation Theory becomes an unavailable development after a new definitions of the functions. It is in the theory of Fourier series that we find some of the first result of the approximation theory. This include conditions as a functions that ensure the pointwise convergence and uniform convergence of its Fourier Series.

The first question we ask in approximation theory concerns the possibility of approximation. Is the given family of functions from which we plan to approximate dense in the set of functions we wish to approximate? The Weierstrass Approximation Theorems spawned numerous generalization which were applied to other families of functions. They also led to the development of two general methods for determining the density. These are Stone-Weierstrass Theorem generalizing the Weierstrass Theorem to the normed space. Especially in S.W.

Theorem the normed space must be compact. A different and more modern approach to density theorem is due to Functional Analysis.

Suppose that a given function f can be shown that the limit of set of functions $\varphi_n(x)$ at the point which has good properties at x_0. Namely,

$$\lim_{n \to \infty} \varphi_n(x_0) = f(x_0)$$

be as described above. In this case the series of $\varphi_n(x)$ approximates or converges to f.

The second fundamental problem of the approaching theorem is finding the rate of approach to the problem.

$$\lim_{n \to \infty} \varphi_n(x_0) = f(x_0)$$

while $n \to \infty$, then the series $(\alpha_n) = (f(x_0) - \varphi_n(x_0))$ is a zero sequence. If we find an another zero sequence β_n with $\alpha_n = o(\beta_n)$, then one has

$$f(x_0) - \varphi_n(x_0) = o(\beta_n).$$

This Landau representation shows that the $f(x_0) - \varphi_n(x_0)$ is approaches to zero more rapidly than β_n, when $n \to \infty$.

In calculus we consider the real line $\mathbb{R}$ and real-valued functions on $\mathbb{R}$ (or on a subset of $\mathbb{R}$). Obviously, any such function is a mapping of its domain into $\mathbb{R}$. In functional analysis we consider more genaral spaces, such as metric spaces and normed spaces, and mappings of these spaces. In the case of vector spaces and, in particular, normed spaces, a mapping is called an operator.

The start point of this thesis is the following,

$$\frac{\partial}{\partial t}\omega\left(\lambda,t,u\right)=\frac{\lambda}{p\left(t\right)}\omega\left(\lambda,t,u\right)\left(u-t\right) \tag{1.1}$$

where $p\left(t\right)$ is a polynomial of degree ≤ 2 , $\omega>0$, then

$$\frac{\partial\omega}{\partial t}=\frac{\lambda}{p\left(t\right)}\omega\left(u-t\right)$$

where $\omega=\omega\left(\lambda,t,u\right)$. Dividing both of sides by ω, we get

$$\frac{\frac{\partial\omega}{\partial t}}{\omega}=\frac{\lambda}{p\left(t\right)}\left(u-t\right).$$

Integrating both of sides,

$$\int\frac{\frac{\partial\omega}{\partial t}}{\omega}dt=\int\frac{\lambda}{p\left(t\right)}\left(u-t\right)dt$$

then we have

$$\ln\omega=\lambda\int\frac{\left(u-t\right)}{p\left(t\right)}dt+f\left(u\right).$$

If we take $e^{f(u)}=\varphi\left(u\right)$ where φ is an arbitrary function, then we get

$$\omega=\varphi\left(u\right)e^{\lambda\int\frac{(u-t)}{p(t)}dt}.$$

Although this type equation seems to be special type equation, it contains most of equations which we know in general, such as; Heat Equation and Wave Equation. The solutions of the above differential equation are called exponential type operators. The well known exponential type operators are Bernstein

Operator, Szász Operator, Baskakov Operator, Gauss-Weierstrass Operator and Post-Widder Operator. For further reading, we refer the reader to [5], [15], [20], [21], [22], [23] and [24].

CHAPTER 2

PRELIMINARIES

In this chapter, we give some necessary definitions and theorems, which are used in this thesis.

2.1 Some Definitions

Definition 2.1. If A is a subset of the topological space X and if x is a point of X, we say that x is a *limit point* (or *accumulation point*) of A if every neighborhood of x intersects A in some point other than x itself. Said differently, x is a limit point of A if it belongs to the closure of $A - \{x\}$. The point x may lie in A or not; for this definition it does not matter.

Definition 2.2. We say that $f(x)$ approaches the limit L as x approaches a, and we write

$$\lim_{x \to a} f(x) = L$$

if the following condition is satisfied:
For every number $\varepsilon > 0$ there exists a number $\delta > 0$, depending on ε, such that

$$0 < |x - a| < \delta \quad \textit{implies} \quad |f(x) - L| < \varepsilon.$$

Definition 2.3. Suppose $\{f_n\}$, $n = 1, 2, 3, ..,$ is a sequence of functions defined on set E, and suppose that the sequence of numbers $\{f_n(x)\}$ converges for every $x \in E$. We shall say that $\{f_n\}$ converges to f *pointwise* on E, if

$$\lim_{n \to \infty} f_n(x) = f(x) \qquad , \quad (x \in E)$$

holds.

Definition 2.4. We say that a sequence of functions $\{f_n\}$, $n = 1, 2, 3, ..,$ converges *uniformly* on E to a function f if for every $\varepsilon > 0$ there is an integer N such that $n \geq N$ implies

$$|f_n(x) - f(x)| \leq \varepsilon$$

for all $x \in E$.

Definition 2.5. The sequence f_n uniformly converge to f on $A \Leftrightarrow \forall\varepsilon > 0\ \exists n_0$ such that $n \geq n_0$ and $\forall x \in A$, $|f_n(x) - f(x)| < \varepsilon$.

Definition 2.6. ($C[a, b]$) As a set X we take the set of all real- valued functions $x, y, \cdots$ which are functions of an independent real variable t and are defined and continuous on a given closed interval $J = [a, b]$. Choosing the metric defined by

$$d(x, y) = \max_{t \in J} |x(t) - y(t)|$$

where *max* denotes the *maximum*, we obtain a metric space which is denoted by $C[a, b]$.

Definition 2.7. A map $T : X \to Y$ between two vector spaces is called an operator.

Definition 2.8. A linear operator T is an operator such that

- The domain $D(T)$ of T is a vector space and the range $R(T)$ lies in a vector space over the same field,

- for each $x, y \in D(T)$ and scalar α,

$$T(x + y) = Tx + Ty$$

and

$$T(\alpha x) = \alpha T x.$$

(Kreyszig, 1989)

Definition 2.9. (Integral Operator) We can define an integral operator $T : C[0,1] \rightarrow C[0,1]$ by

$$y = Tx, \qquad where \qquad y(t) = \int_0^1 x(t)k(t,\tau)d\tau.$$

Here k is a given function which is called the kernel of T and is assumed to be continuous on the closed square $G = J \times J$ in the $t\tau - plane$, where $J = [0,1]$. This operator is linear.

Definition 2.10. ($L_p(-\infty,\infty)$ Space) L_p is the set of functions which are Lebesgue integrable to the p^{th} power over $\mathbb{R}$, for $1 \leq p < \infty$, and essentially bounded (bounded almost everywhere) on $\mathbb{R}$ if $p = \infty$. For $f \in L_p(-\infty,\infty)$

$$||f||_p = \left\{ \int_{-\infty}^{\infty} |f(t)|^p dt \right\}^{\frac{1}{p}},$$

if $1 \leq p < \infty$ and in case $p = \infty$

$$||f||_\infty = ess \sup_{x \in \mathbb{R}} |f(t)|.$$

(Butzer and Nessel, 1971)

Definition 2.11. ($L_p[a,b]$ Space) $L_p[a,b]$ is the set of functions which are Lebesgue integrable to the p^{th} power of $[a,b]$. For $f \in L_p[a,b]$

$$||f||_p = \left\{ \int_a^b |f(t)|^p dt \right\}^{1/p}.$$

(Butzer and Nessel, 1971)

Definition 2.12. ($L_1[a, b]$ Space) $L_1[a, b]$ is the set of functions which are Lebesgue integrable to the first power of the period $[a, b]$. For $f \in L_1[a, b]$

$$||f||_1 = \int_a^b |f(t)|dt.$$

(Butzer and Nessel, 1971)

Definition 2.13. ($L_p[-\pi, \pi]$ Space) $L_p[-\pi, \pi]$ is the set of functions which are Lebesgue integrable to the p^{th} power of the period $[-\pi, \pi]$. For $f \in L_p[-\pi, \pi]$

$$||f||_p = \left\{ \int_{-\pi}^{\pi} |f(t)|^p dt \right\}^{1/p}.$$

(Butzer and Nessel, 1971)

Definition 2.14. ($X(\mathbb{R})$ Space) $X(\mathbb{R})$ always denotes one of the spaces C or L_p, $1 \leq p < \infty$. For $f, g \in X(\mathbb{R})$ we write $f(x) = g(x)$ (a.e.) if equality holds for all $x \in X(\mathbb{R})$ in case $X(\mathbb{R}) = C$, and almost everywhere in case $X(\mathbb{R}) = L_p$, $1 \leq p < \infty$, i.e., if $||f - g|| = 0$. In this event we also write $f = g$ in $X(\mathbb{R})$.

(Butzer and Nessel, 1971)

Definition 2.15. (Minkowski' s Inequality) Let $f, g \in X(\mathbb{R})$. Then $(f + g) \in X(\mathbb{R})$ and

$$||f + g||_{X(\mathbb{R})} \leq ||f||_{X(\mathbb{R})} + ||g||_{X(\mathbb{R})}.$$

If p is such that $1 \leq p \leq \infty$, the conjugate number q is defined through $(1/p) + (1/q) = 1$ in case $1 < p < \infty$; $q = \infty$ if, $p = 1$, and $q = 1$ if $p = \infty$.

Definition 2.16. (Hölder' s Inequality) Let $f \in L_p$, $1 \leq p \leq \infty$, and $g \in L_q$. Then $fg \in L_1$ and $||fg||_1 \leq ||f||_p ||g||_q$. Namely;

$$\left| \int fg \right| \leq \left(\int |f|^p \right)^{1/p} \left(\int |g|^q \right)^{1/q}.$$

Definition 2.17. (Hölder- Minkowski Inequality) Let $f(x,y)$ be defined and measurable on $\mathbb{R}^2$. If $||f(x,y)||_{X(\mathbb{R})} \in L_1$, then

$$\left|\left|\int_{-\infty}^{\infty} f(x,y)dy\right|\right|_{X(\mathbb{R})} \leq \int_{-\infty}^{\infty} ||f(x,y)||_{X(\mathbb{R})}dy.$$

This is also known as the generalized Minkowski inequality.

Definition 2.18. (Lipschitz condition) A finite function $f(x)$ defined on $[a,b]$ is said to satisfy a Lipschitz condition if there exists a constant K such that for any two points x and y in $[a,b]$,

$$|f(x) - f(y)| \leq K|x-y|$$

holds true.

(Natanson, Volume I, 1964)

Definition 2.19. A finite function $f(x)$ is defined on $[a,b]$, is said to satisfy a Hölder condition if there exists a constant K such that for any two points x and y in $[a,b]$,

$$|f(x) - f(y)| \leq K|x-y|^{\alpha}$$

where $0 < \alpha \leq 1$. Then the functions satisfying Hölder condition define the following set of functions ;

$$Lip_K\alpha = \{f : [a,b] \to \mathbb{R}, |f(x) - f(y)| \leq K|x-y|^{\alpha}\},$$

for all x,$y \in [a,b]$ and $0 < \alpha \leq 1$.

Definition 2.20. (Kernel) A kernel is a function $\phi_n(t,x)$ which $(n = 1,2,3,...)$

defined in the square $(a \leq t \leq b, a < x < b)$ and such that;

$$\lim_{n\to\infty} \int\limits_{\beta}^{\theta} \phi_n(t,x)\,dt = 1$$

provided where, $a \leq \beta < x < \theta \leq b$. It is self-evident that $\phi_n(t,x)$ is assumed summable with respect to t for every fixed x.

And also note that; an integral of the form;

$$f_n(x) = \int\limits_{a}^{b} \phi_n(t,x)\, f(t)\,dt$$

where $\phi_n(t,x)$ is a kernel, is called a *singular integral*.

(Natanson Volume II, 1960)

Definition 2.21. Let M_f is a constant number depend on f. Also,we say that, if f is continuous then f belongs to $C_\rho(\mathbb{R})$ function space.

$$C_\rho(\mathbb{R}) = \{f : f \in B_\rho(\mathbb{R}) \ \textit{and } f \textit{ is continuous}\}.$$

Let f is defined on $\mathbb{R}$ and satisfies $|f(x)| \leq M_f \rho(x)$. Then we say that f belongs to $B_\rho(\mathbb{R})$ function space. Indeed,

$$B_\rho(\mathbb{R}) = \{f : |f(x)| \leq M_f \rho(x)\}.$$

$C_\rho(\mathbb{R})$ and $B_\rho(\mathbb{R})$ are both weighted space and we can see that $C_\rho(\mathbb{R}) \subset B_\rho(\mathbb{R})$. The norm of these space is defined as;

$$\|f\|_\rho = \sup \frac{|f(x)|}{\rho(x)}$$

Let $\rho(x)$ is a monotone increasing, continuous $\rho(x) \geq 1$ and $\lim_{|x|\to\infty} \rho(x) = \infty$ is said to be weighted function. $B_\rho(\mathbb{R})$ and $C_\rho(\mathbb{R})$ are normed space with the norm which is defined above.

Definition 2.22. A function sequence (f_n) converges uniformly a function f in $C_\rho(\mathbb{R})$ if and only if the equation

$$\lim_{n\to\infty} \|f_n - f\|_{C_\rho(\mathbb{R})} = 0$$

or more simply the equation;

$$\lim_{n\to\infty} \sup \frac{|f_n(x) - f(x)|}{\rho(x)} = 0$$

holds. And uniform convergence is denoted by $f_n(x) \rightrightarrows f(x)$.

2.2 Some Theorems

Theorem 2.1. *(P.P.Korovkin 1953) Let* $f \in C[a,b]$ *and for every real number*

$$|f(x)| \leq M_f \tag{2.1}$$

holds. If linear positive operator sequence (L_n) *,*

i. $L_n(1,x) \rightrightarrows 1$

ii. $L_n(t,x) \rightrightarrows x$

iii. $L_n(t^2,x) \rightrightarrows x^2$

for every $x \in [a,b]$*,then for every* $f \in C[a,b]$*,* $L_n(f;x) \rightrightarrows f(x)$ *on* $[a,b]$*.*

Proof. Let $f \in C[a,b]$,then for every positive ϵ , we can find a positive δ such that,

$$|f(t) - f(x)| < \epsilon$$

when $|t-x| \leq \delta$. Using the triangle inequality in equation (2.1),we get

$$|f(t) - f(x)| \leq |f(t)| + |f(x)| \leq 2M_f \tag{2.2}$$

Since $|t-x| > \delta$,then $\frac{|t-x|}{\delta} > 1$ and so

$$\frac{(t-x)^2}{\delta^2} > 1 \tag{2.3}$$

From (2.2) and (2.3),we obtain

$$|f(t) - f(x)| \leq 2M_f \frac{(t-x)^2}{\delta^2}$$

So we get,

$$|t-x| \leq \delta \text{ için } |f(t) - f(x)| < \epsilon$$

$$|t-x|>\delta \text{ için } |f(t)-f(x)| \leq 2M_f \frac{(t-x)^2}{\delta^2} \tag{2.4}$$

Therefore,for every $x, t \in [a,b]$

$$|f(t)-f(x)| < \epsilon + 2M_f \frac{(t-x)^2}{\delta^2}$$

holds. Now,we will show the equation

$$\lim_{n\to\infty} \|L_n(f)-f\|_{C[a,b]} = 0$$

satisfied when the operator sequence (L_n) has the properties $(i), (ii), (iii)$,

$$\begin{aligned} |L_n(f(t);x)-f(x)| &= |L_n(f(t);x)-f(x)+L_n(f(x);x)-L_n(f(x);x)| \\ &= |L_n(f(t);x)-L_n(f(x);x)+L_n(f(x);x)-f(x)| \\ &= |L_n(f(t)-f(x);x)+f(x)(L_n(1;x)-1)| \end{aligned}$$

By the triangle inequality,we get

$$|L_n(f(t);x)-f(x)| \leq |L_n(f(t)-f(x);x)| + |f(x)|\,|L_n(1;x)-1|$$

then from (2.1),

$$|L_n(f(t);x)-f(x)| \leq L_n(|f(t)-f(x)|;x) + M_f\,|L_n(1;x)-1|$$

holds and we can write from equation (2.4),since (L_n) is monotone increasing

$$|L_n(f(t);x)-f(x)| \leq L_n\left(\epsilon + 2\frac{M_f}{\delta^2}(t-x)^2;x\right) + M_f\,|L_n(1;x)-1| \tag{2.5}$$

holds. On the other hand (L_n) is linear,we can write

$$\begin{aligned} L_n\left(\epsilon+2\frac{M_f}{\delta^2}(t-x)^2;x\right) &= L_n(\epsilon;x)+L_n\left(2\frac{M_f}{\delta^2}(t-x)^2;x\right) \\ &= \epsilon L_n(1;x)+2\frac{M_f}{\delta^2}L_n\left(t^2-2xt+x^2;x\right) \\ &= \epsilon L_n(1;x)+2\frac{M_f}{\delta^2}\{L_n\left(t^2;x\right)-x^2-x^2+2x^2 \\ &\quad -2xL_n(1;x)+x^2L_n(1;x)\} \\ &= \epsilon L_n(1;x)+2\frac{M_f}{\delta^2}\{L_n\left(t^2;x\right)-x^2+2x^2-2xL_n(1;x) \\ &\quad +x^2L_n(1;x)-x^2\} \\ &= \epsilon L_n(1;x)+2\frac{M_f}{\delta^2}\{\left(L_n\left(t^2;x\right)-x^2\right)+2x(x-L_n(t;x)) \\ &\quad +x^2(L_n(1;x)-1)\} \end{aligned}$$

If we use last equality an equation(2.5),we get

$$\begin{aligned} |L_n(f(t);x)-f(x)| &\leq \epsilon L_n(1;x)+2\frac{M_f}{\delta^2}\{\left(L_n\left(t^2;x\right)-x^2\right) \\ &\quad +2x(x-L_n(t;x))+x^2(L_n(1;x)-1) \qquad (2.6) \\ &\quad +M_f|(L_n(1;x)-1)|\} \end{aligned}$$

then using $(i),(ii),(iii)$ an equation in (2.6),we obtain

$$\|L_n(f)-f\|=\lim_{n\to\infty}\left\{\max_{a\leq x\leq b}|L_n(f;x)-f(x)|\right\}=0.$$

This proof is completed.

Theorem 2.2. *(Uniform Boundedness Theorem) Let (T_n) be a sequence of bounded linear operators $T_n : X \to Y$ from a Banach space X into a normed space Y such that $(\|T_nx\|)$ is bounded for every $x \in X$,say,*

$$\|T_nx\| \leqq c_x \quad n=1,2\ldots$$

where c_x is a real number. Then the sequence of the norms $\|T_n\|$ is bounded, that is , there is a c is such that

$$\|T_n\| \leqq c \quad n = 1, 2 \dots \ .$$

(Kreyszig, 1989)

Theorem 2.3. *(Modulus of Continuity) Let $f \in C[a,b]$ and $\delta > 0$ be given. Then modulus of continuity is given by;*

$$\omega(f;\delta) = \sup_{|t-x|\leq\delta\ ,\ t,x\in[a,b]} |f(t) - f(x)| \, .$$

and it has the following properties;

i) $\omega(f;\delta) \geq 0$

ii) If $\delta_1 \leq \delta_2$, then $\omega(f;\delta_1) \leq \omega(f;\delta_2)$

iii) Let $m \in \mathbb{N}$, then $\omega(f;m\delta) \leq m\omega(f;\delta)$

iv) Let $\lambda \in \mathbb{R}^+$, then $\omega(f;\lambda\delta) \leq (\lambda+1)\,\omega(f;\delta)$

v) $\lim\limits_{\delta\to 0}\omega(f;\delta) = 0$

vi) $|f(t) - f(x)| \leq \omega(f;|t-x|)$

vii) $|f(t) - f(x)| \leq \left(\frac{|t-x|}{\delta} + 1\right)\omega(f;\delta)$

Proof.

i) Since $|f(t) - f(x)| \geq 0$, then $\omega(f;\delta) = \sup\limits_{|x-y|\leq\delta} |f(x) - f(y)| \geq 0$ for every $x, y \in [a,b]$.

ii) If $\delta_1 \leq \delta_2$ then the zone $|t-x| \leq \delta_2$ is bigger than the zone $|t-x| \leq \delta_1$.

iii) From the definition we can write;

$$\omega(f;m\delta) = \sup_{|t-x|\leq m\delta\ ,\ t,x\in[a,b]} |f(t) - f(x)| \ , \ m \in \mathbb{N}$$

If $|t-x| \leq m\delta$ then

$$x - m\delta \leq t \leq x + m\delta$$

Let $t = x + mh$, then $|h| \leq \delta$ and we can write

$$\begin{aligned}
\omega(f; m\delta) &= \max_{|h|\leq\delta\ ,\ t,x\in[a,b]} |f(x+mh) - f(x)| \\
&= \max_{|h|\leq\delta\ ,\ t,x\in[a,b]} \left| \sum_{k=0}^{m-1} [f(x+(k+1)h) - f(x+kh)] \right|
\end{aligned}$$

then using triangle inequality,

$$\begin{aligned}
\omega(f; m\delta) &\leq \max_{|h|\leq\delta\ ,\ t,x\in[a,b]} \sum_{k=0}^{m-1} |f(x+(k+1)h) - f(x+kh)| \\
&\leq \max_{|h|\leq\delta\ ,\ t,x\in[a,b]} \sum_{k=0}^{m-1} |f(x+(k+1)h) - f(x+kh)| \\
&\leq \sum_{k=0}^{m-1} \max_{|h|\leq\delta\ ,\ t,x\in[a,b]} |f(x+(k+1)h) - f(x+kh)| \\
&= \omega(f;\delta) + ... + \omega(f;\delta)
\end{aligned}$$

Therefore we get;

$$\omega(f; m\delta) \leq m\omega(f;\delta).$$

iv) Let $\lambda \in \mathbb{R}^+$ then $[|\lambda|] \leq \lambda \leq [|\lambda|] + 1$. Then by using (ii) we can write;

$$\omega(f; \lambda\delta) \leq \omega(f; ([|\lambda|] + 1)\delta)$$

Due to $[|\lambda|]$ is positive, we can use (iii) to right side of inequality and we get;

$$\omega(f; ([|\lambda|] + 1)\delta) \leq (\lambda + 1)\omega(f;\delta)$$

In conclusion, we can get

$$\omega (f; \lambda\delta) \leq (\lambda + 1) \omega (f; \delta).$$

v) Let consider the inequality $|t - x| \leq \delta$. $\delta \to 0$ means that $t \to x$. Due to function f is continuous, if $t \to x$ then $|f(t) - f(x)| \to 0$. If the function is continuous on the closed interval $[a, b]$ then it is uniformly continuous, therefore if it coes to zero then its maximum also goes to zero. This completes the proof.

vi) Let $\delta = |t - x|$ in the $\omega (f; \delta)$ then we obtain,

$$\omega (f; |t - x|) = \sup_{t,x \in [a,b]} |f(t) - f(x)|$$

Due to supremum of $|f(t) - f(x)|$ equal to $\omega (f; |t - x|)$, then

$$|f(t) - f(x)| \leq \omega (f; |t - x|).$$

vii) From (vi), we get

$$|f(t) - f(x)| \leq \omega \left(f; \frac{|t - x|}{\delta} \delta \right)$$

Then using (iv), we obtain

$$|f(t) - f(x)| \leq \left(\frac{|t - x|}{\delta} + 1 \right) \omega (f; \delta)$$

This completes the proof.

Theorem 2.4. *(Modulus of Continuity in Weighted Space) Let $f \in C_{\rho,K_f}[0, \infty)$ and $\delta > 0$ be given. Then modulus of continuity in weighted space is given by;*

$$\Omega (f; \delta) = \sup_{x \geq 0, |h| \leq \delta} \frac{|f(x + h) - f(x)|}{(1 + x^2)(1 + h^2)}$$

and it has the following properties;

i) $\Omega\left(f;\delta\right)\geq 0$

ii) $\delta_1>\delta_2\Longrightarrow\Omega\left(f;\delta_1\right)>\Omega\left(f;\delta_2\right)$

iii)$\forall m\in\mathbb{N},\Omega\left(f;m\delta\right)\leq 4m\left(1+\delta^2\right)\,\Omega\left(f;\delta\right)$

iv)$\forall\lambda\in\mathbb{R}^+$, $\Omega\left(f;\lambda\delta\right)\leq 4\left(1+\lambda\right)\,\left(1+\delta^2\right)\Omega\left(f;\delta\right)$

v)$\forall f\in C_{\rho,K_f}[0,\infty)$, $\lim\limits_{\delta\to 0}\Omega\left(f;\delta\right)=0$

vi)$\left|f\left(t\right)-f\left(x\right)\right|\leq\left(1+x^2\right)\left(1+\left(t-x\right)^2\right)\Omega\left(f;\left|t-x\right|\right)$

vii)$\left|f\left(t\right)-f\left(x\right)\right|\leq 4\left(\frac{|t-x|}{\delta}+1\right)\left(1+x^2\right)\left(1+\left(t-x\right)^2\right)\left(1+\delta^2\right)\Omega\left(f;\delta\right).$

CHAPTER 3

EXPONENTIAL TYPE OPERATORS

In this chapter we will show that exponential type operators satisfy the equation (1.1).

An exponential type operator is a positive linear integral operator

$$(S_n f)(x) = \int_{-\infty}^{\infty} W(n,x,t) f(t)\, dt$$

whose kernel $W(n,x,t)$ satisfies the partial differential equation

$$\frac{\partial W}{\partial x} = \frac{n(t-x)}{p(x)} W$$

and the normalization condition

$$\int_{-\infty}^{\infty} W(n,x,t)\, dt = 1.$$

We will some examples about exponential type operators.

Firstly, we will show that the Bernstein Operator

$$B_n(x) = (B_n f)(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k}$$

satisfies equation (1.1).

We consider the kernel

$$W(n,x,k) = \binom{n}{k} x^k (1-x)^{n-k}$$

and

$$p\left(x\right)=x(1-x)\ ,\ x\in\left(0,1\right).$$

Therefore

$$\begin{aligned}
\frac{\partial W\left(n,x,k\right)}{\partial x} &= \binom{n}{k}\left[kx^{k-1}(1-x)^{n-k}-x^{k}(n-k)(1-x)^{n-k-1}\right]\\
&= \binom{n}{k}\left[\frac{k}{x}x^{k}(1-x)^{n-k}-\frac{(n-k)}{(1-x)}(1-x)^{n-k}x^{k}\right]\\
&= \binom{n}{k}\left[x^{k}(1-x)^{n-k}\left(\frac{k}{x}-\frac{(n-k)}{(1-x)}\right)\right]\\
&= \binom{n}{k}\left[x^{k}\left(1-x\right)^{n-k}\frac{(k-nx)}{x(1-x)}\right]\\
&= \binom{n}{k}x^{k}\left(1-x\right)^{n-k}\frac{n(\frac{k}{n}-x)}{x(1-x)}\\
&= \frac{n}{x(1-x)}(\frac{k}{n}-x)\binom{n}{k}x^{k}\left(1-x\right)^{n-k}\\
&= \frac{n}{p(x)}\left(\frac{k}{n}-x\right)W\left(n,x,k\right).
\end{aligned}$$

Then we have

$$\frac{\partial W\left(n,x,k\right)}{\partial x}=\frac{n\left(t-x\right)}{p\left(x\right)}W\left(n,x,k\right).$$

Secondly, we will show that the Szász Operator

$$S_{n}\left(x\right)=\left(S_{n}f\right)\left(x\right)=\sum_{k=0}^{\infty}f\left(\frac{k}{n}\right)e^{-nx}\frac{\left(nx\right)^{k}}{k!}$$

satisfies equation (1.1).

We consider the kernel

$$W\left(n,x,k\right)=e^{-nx}\frac{\left(nx\right)^{k}}{k!}$$

and

$$p\left(x\right)=x\ ,\ x\in\left(0,+\infty\right).$$

Therefore

$$\begin{aligned}
\frac{\partial W(n,x,k)}{\partial x} &= \frac{1}{k!}\left(-n.e^{-nx}(nx)^k + e^{-nk}n^k k x^{k-1}\right) \\
&= \frac{1}{k!}\left(-n.e^{-nx}(nx)^k + e^{-nk}n^k k x^{k-1}\right) \\
&= \frac{1}{k!}\left(-n.e^{-nx}(nx)^k + e^{-nk}\frac{k}{x}(nx)^k\right) \\
&= e^{-nx}\frac{(nx)^k}{k!}(\frac{k-nx}{n}) \\
&= \frac{n}{x}(\frac{k}{n}-x)e^{-nx}\frac{(nx)^k}{k!} \\
&= \frac{n}{p(x)}\left(\frac{k}{n}-x\right)W(n,x,k)
\end{aligned}$$

Then we have

$$\frac{\partial W(n,x,k)}{\partial x} = \frac{n(t-x)}{p(x)}W(n,x,k).$$

Thirdly, we will show that the Baskakov Operator

$$(K_n f)(x) = \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right)\binom{n+k-1}{k}\frac{x^k}{(1+x)^{n+k}}$$

satisfies equation (1.1).

We consider the kernel

$$W(n,x,k) = \binom{n+k-1}{k}\frac{x^k}{(1+x)^{n+k}}$$

and

$$p(x) = x(1+x)\ ,\ x \in (0,+\infty).$$

Therefore

$$\begin{aligned}
\frac{\partial W(n,x,k)}{\partial x} &= \frac{\partial}{\partial x}\left[\binom{n+k-1}{k}\frac{x^k}{(1+x)^{n+k}}\right] \\
&= \binom{n+k-1}{k}\frac{\partial}{\partial x}(\frac{x^k}{(1+x)^{n+k}}) \\
&= \binom{n+k-1}{k}\left[\frac{k.x^{k-1}(1+x)^{n+k} - x^k(n+k)(1+x)^{n+k-1}}{(1+x)^{2(n+k)}}\right] \\
&= \binom{n+k-1}{k}\frac{x^k\left(\frac{k}{x}-\frac{n+k}{1+x}\right)}{(1+x)^{n+k}} \\
&= \frac{n}{x(1+x)}(\frac{k}{n}-x)\binom{n+k-1}{k}\frac{x^k}{(1+x)^{n+k}} \\
&= \frac{n}{p(x)}\left(\frac{k}{n}-x\right)W(n,x,k).
\end{aligned}$$

Then we have

$$\frac{\partial W(n,x,k)}{\partial x} = \frac{n(t-x)}{p(x)}W(n,x,k).$$

Finally, we will show that the Post-Widder Operator

$$P_n(x) = (P_n f)(x) = \frac{1}{(n-1)!}\left(\frac{n}{x}\right)^n \int_0^\infty e^{\frac{-nt}{x}} t^{n-1} f(t)\, dt$$

satisfies equation (1.1).

We consider the kernel

$$W(n,x,t) = \frac{1}{(n-1)!}\left(\frac{n}{x}\right)^n e^{\frac{-nt}{x}} t^{n-1}$$

and

$$p(x) = x^2 \ , \ x \in (0,+\infty).$$

Therefore

$$\begin{aligned}
\frac{\partial W(n,x,t)}{\partial x} &= \frac{\partial}{\partial x}\left[\frac{1}{(n-1)!}\left(\frac{n}{x}\right)^{n} e^{\frac{-nt}{x}} t^{n-1}\right] \\
&= \frac{1}{(n-1)!}t^{n-1}\left[n^{n}\left(-n\right)x^{-n-1}e^{\frac{-nt}{x}} + n^{n}x^{-n}\frac{nt}{x^{2}}e^{\frac{-nt}{x}}\right] \\
&= \frac{1}{(n-1)!}t^{n-1}\left(\frac{n}{x}\right)^{n} e^{\frac{-nt}{x}}\left[\frac{-n}{x}+\frac{nt}{x^{2}}\right] \\
&= \frac{1}{(n-1)!}t^{n-1}\left(\frac{n}{x}\right)^{n} e^{\frac{-nt}{x}} n\left(\frac{t-x}{x^{2}}\right) \\
&= n\left(\frac{t-x}{x^{2}}\right)\frac{1}{(n-1)!}t^{n-1}\left(\frac{n}{x}\right)^{n} e^{\frac{-nt}{x}} \\
&= \frac{n}{p(x)}\left(\frac{k}{n}-x\right)W(n,x,t).
\end{aligned}$$

Then we have

$$\frac{\partial W(n,x,t)}{\partial x} = \frac{n\left(t-x\right)}{p\left(x\right)}W(n,x,t).$$

CHAPTER 4

APPROXIMATION PROPERTIES OF EXPONENTIAL TYPE OPERATORS

4.1 Bernstein Operator and Its Approximation Properties

Definition 4.1. For a function $f(x)$ defined on the closed interval $[0,1]$ the expression.

$$B_n(x) = (B_n f)(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k (1-x)^{n-k} \tag{4.1}$$

is called the Bernstein polynomial of order n of the function $f(x)$. $B_n(x)$ is a polynomial in x of degree $\leq n$.The polynomials $B_n(x)$ were introduced by S. Bernstein [3] to give an especially simple proof of the approximation theorem of Weierstrass. For if $f(x)$ is continuous on $[0,1]$,then

$$\lim_{n\to\infty} B_n(x) = f(x)$$

uniformly in $[0,1]$ here are many other expressions, the so-called "singular integrals", which have, in common with the $B_n(x)$, the peculiarity of approximating the "generating" function $f(x)$ and of reproducing some of its properties. The best-known singular integral is the Dirichlet's integral

$$s_n(x) = \frac{1}{\pi}\int_{-\pi}^{+\pi} f(t)\frac{\sin\left(n+\frac{1}{2}\right)(t-x)}{2\sin\frac{1}{2}(t-x)}dt$$

representing the partial sums $s_n(x)$ of the Fourier series of the function $f(x)$ integrable on $[-\pi, +\pi]$. Another example is the Fejer integral which represents the arithmetic means $\sigma_n = (s_0 + s_1 + ... + s_n) / (n+1)$ of the above $s_n(x)$. In general, a singular integral may be written in the form

$$\Phi_n(x) = \int_a^b f(t) K_n(x,t) \, dt$$

where $K_n(x,t)$ is the "kernel", defined for $a \leq x \leq b$, $a \leq t \leq b$, which has the property that for functions $f(x)$ of a certain class and in a certain sense, $\Phi_n(x)$ converges to $f(x)$ as $n \to \infty$.

The Bernstein polynomial (2.1) is a finite sum of a type corresponding to the integral (2.3). Both (2.1) and (2.3) are special cases of singular Stieltjes integrals. (2.1) may be written in the form of a Stieltjes integral in the variable t,

$$(B_n f)(x) = \int_0^1 f(t) \, d_t K_n(x,t)$$

with the kernel which is constant in any interval $k/n \leq t < (k+1)/k$, $k = 0, 1, ..., n-1$, and has the jump

$$\binom{n}{k} x^k (1-x)^{n-k}$$

at the basic point of interpolation $t = k/n$. In this sense, the theory of the Bernstein polynomials; as well as the theory of Fourier series; is a chapter of the theory of singular integrals. If one would like to compare these theories, the Bernstein polynomial $B_n(x)$ would correspond rather to the Fejer's mean $\sigma_n(x)$ than to the partial sum of the Fourier series $s_n(x)$.

Theorem 4.1. *Bernstein operator is linear and positive operator.*

Proof. First we will show B_n is linear.

$$\begin{aligned}(B_n(\alpha f+\beta g))(x) &= \sum_{k=0}^{n}(\alpha f+\beta g)\left(\frac{k}{n}\right)x^k(1-x)^{n-k} \\ &= \sum_{k=0}^{n}(\alpha f)\left(\frac{k}{n}\right)x^k(1-x)^{n-k}+\sum_{k=0}^{n}(\beta g)\left(\frac{k}{n}\right)x^k(1-x)^{n-k} \\ &= \alpha(B_nf)(x)+\beta(B_ng)(x).\end{aligned}$$

Since $x\in[0,1]$, and if $f(x)\geq 0$, then

$$(B_nf)(x)=\sum_{k=0}^{n}f\left(\frac{k}{n}\right)x^k(1-x)^{n-k}\geq 0.$$

So B_n is positive.

Theorem 4.2. *Let $f:[0,1]\to\mathbb{R}$, $f\in B[0,1]$ and $p_{n,k}(x)$ is the Bernstein basis. Then, $B_n:B[0,1]\to P[0,1]$ and*

$$\|B_nf\|_{P[0,1]}\leq\|f\|_{B[0,1]},$$

holds true for every $n\in\mathbb{N}$.

Proof . Note that $P[0,1]\subset C[0,1]$. The norm of the space $P[0,1]$ can be considered as the maximum norm. We need to evaluate $\|B_nf\|_{B[0,1]}$.

$$\begin{aligned}|(B_nf)(x)| &= \left|\sum_{k=0}^{n}f\left(\frac{k}{n}\right)p_{n,k}(x)\right| \\ &= \left|\sum_{k=0}^{n}f\left(\frac{k}{n}\right)\binom{n}{k}x^k(1-x)^{n-k}\right| \\ &\leq \sum_{k=0}^{n}\left|f\left(\frac{k}{n}\right)\right|\left|\binom{n}{k}x^k(1-x)^{n-k}\right| \\ &\leq \sum_{k=0}^{n}\max\left|f\left(\frac{k}{n}\right)\right|\left|\binom{n}{k}x^k(1-x)^{n-k}\right|\end{aligned}$$

$$
\begin{aligned}
&\leq \max\left|f\left(\frac{k}{n}\right)\right|\sum_{k=0}^{n}\binom{n}{k}x^{k}\left(1-x\right)^{n-k} \\
&= \max\left|f\left(\frac{k}{n}\right)\right|
\end{aligned}
$$

By taking into account that $f \in B\left[0,1\right]$ since $\left\|f\right\|_{C[0,1]} = \max\limits_{x\in[0,1]}\left|f\left(x\right)\right|$, then

$$\left|\left(B_n f\right)\left(x\right)\right| \leq \left\|f\right\|_{B[0,1]}$$

for every $x \in [0,1]$; hence we obtain

$$\left\|B_n f\right\|_{P[0,1]} \leq \left\|f\right\|_{B[0,1]}$$

and so $B_n : B\left[0,1\right] \to P\left[0,1\right]$.

Theorem 4.3. *Bernstein operator (4.1) satisfies the following equations.*

i)$B_n\left(1;x\right) \rightrightarrows 1$

ii)$B_n\left(t;x\right) \rightrightarrows x$

iii)$B_n\left(t^2;x\right) \rightrightarrows x^2$

Proof. We will show that (i),(ii),(iii) hold true for all $x \in [0,1]$ and $n \in \mathbb{N}$. Firstly,

$$B_n\left(1;x\right) = \sum_{k=0}^{n}\binom{n}{k}x^{k}\left(1-x\right)^{n-k} = \left(x+\left(1-x\right)\right)^{n} = 1.$$

Secondly,

$$\begin{aligned} B_n(t;x) &= \sum_{k=1}^{n} \frac{k}{n} \binom{n}{k} x^k (1-x)^{n-k} \\ &= \sum_{k=1}^{n} \frac{k}{n} \frac{n!}{k!\,(n-k)!} x^k (1-x)^{n-k} \quad , \quad k \to k+1 \\ &= \sum_{k=0}^{n-1} \frac{(n-1)!}{k!\,(n-1-k)!} x^{k+1} (1-x)^{n-1-k} \end{aligned}$$

$$\begin{aligned} &= x \sum_{k=0}^{n-1} \binom{n-1}{k} x^k (1-x)^{n-1-k} \\ &= x\,(x + (1-x))^{n-1} = x. \end{aligned}$$

Thirdly,

$$\begin{aligned}
B_n\left(t^2;x\right) &= \sum_{k=1}^{n}\frac{k^2}{n^2}\binom{n}{k}x^k\left(1-x\right)^{n-k}\\
&= \sum_{k=1}^{n}\frac{k-1+1}{n}\frac{(n-1)!}{(k-1)!\,(n-k)!}x^k\left(1-x\right)^{n-k}\\
&= \sum_{k=2}^{n}\frac{k-1}{n}\frac{(n-1)!}{(k-1)!\,(n-k)!}x^k\left(1-x\right)^{n-k}\\
&\quad+\sum_{k=1}^{n}\frac{1}{n}\frac{(n-1)!}{(k-1)!\,(n-k)!}x^k\left(1-x\right)^{n-k}\\
&= \sum_{k=0}^{n-2}\frac{1}{n}\frac{(n-1)!}{(k)!\,(n-2-k)!}x^{k+2}\left(1-x\right)^{n-2-k}\\
&\quad+\sum_{k=0}^{n-1}\frac{1}{n}\frac{(n-1)!}{(k)!\,(n-1-k)!}x^{k+1}\left(1-x\right)^{n-1-k}\\
&= \frac{n-1}{n}x^2\sum_{k=0}^{n-2}\frac{(n-2)!}{(k)!\,(n-2-k)!}x^k\left(1-x\right)^{n-2-k}\\
&\quad+\frac{1}{n}x\sum_{k=0}^{n-1}\frac{(n-1)!}{(k)!\,(n-1-k)!}x^k\left(1-x\right)^{n-1-k}\\
&= \frac{n-1}{n}x^2\sum_{k=0}^{n-2}\binom{n-2}{k}x^k\left(1-x\right)^{n-2-k}\\
&\quad+\frac{1}{n}x\sum_{k=0}^{n-1}\binom{n-1}{k}x^k\left(1-x\right)^{n-1-k}\\
&= \frac{n-1}{n}x^2\left(x+(1-x)\right)^{n-2}+\frac{1}{n}x\left(x+(1-x)\right)^{n-1}\\
&= \frac{n-1}{n}x^2+\frac{1}{n}x = x^2+\frac{x\left(1-x\right)}{n}.
\end{aligned}$$

This completes the proof.

Theorem 4.4. *(Bernstein 1912) For a function $f(x)$ bounded on $[0,1]$, the relation*

$$\lim_{n\to\infty}\left(B_nf\right)(x) = f(x)$$

holds at each point of continuity x of f; and the relation holds uniformly on $[0,1]$ *if $f(x)$ is continuous on this interval.*

(G.G. Lorentz)

Proof . We shall compute the value of

$$T=\sum_{k=0}^{n}(k-nx)^2\,p_{n,k(x)}=\sum_{k=0}^{n}\left\{k(k-1)-(2nx-1)k+n^2x^2\right\}p_{n,k}(x).$$

Clearly

$$\sum_{k=0}^{n}p_{n,k}(x)=1.$$

Moreover,we have

$$\sum_{k=0}^{n}kp_{n,k}(x)=nx\sum_{\mu=0}^{n-1}\binom{n-1}{\mu}x^{\mu}(1-x)^{n-1-\mu}=nx$$

$$\sum_{k=0}^{n}k(k-1)\,p_{n,k}(x)=n(n-1)\,x^2\sum_{\mu=0}^{n-2}\binom{n-2}{\mu}x^{\mu}(1-x)^{n-2-\mu}=n(n-1)\,x^2$$

and, we get

$$T=n^2x^2-(2nx-1)\,nx+n(n-1)\,x^2=nx(1-x).$$

Since $x(1-x)\leq\frac{1}{4}$ on $[0,1]$, we obtain the inequality

$$\begin{aligned}\sum_{\left|\frac{k}{n}-x\right|\geq\delta}p_{n,k}(x) &\leq \frac{1}{\delta^2}\sum_{\left|\frac{k}{n}-x\right|\geq\delta}\left(\frac{k}{n}-x\right)^2p_{n,k}(x)\\ &\leq \frac{1}{n^2\delta^2}T\\ &= \frac{x(1-x)}{n\delta^2}\\ &\leq \frac{1}{4n\delta^2}.\end{aligned}\tag{4.2}$$

If now the function f is bounded, say $|f(u)| \le M$ in $0 \le u \le 1$ and x a point of continuity, for a given $\epsilon > 0$ we can find a $\delta > 0$ such that $\left|x - x^{'}\right| < \delta$ implies $\left|f(x) - f\left(x^{'}\right)\right| < \epsilon$. We have

$$\begin{aligned}|f(x) - (B_n f)(x)| &= \left|\sum_{k=0}^{n}\left\{f(x) - f\binom{k}{n}\right\} p_{n,k}(x)\right| \\ &\le \sum_{\left|\frac{k}{n}-x\right|<\delta}\left|f(x) - f\binom{k}{n}\right| p_{n,k}(x) + \sum_{\left|\frac{k}{n}-x\right|\ge\delta}\left|f(x) - f\binom{k}{n}\right| p_{n,k}(x).\end{aligned}$$

The first sum is $\le \epsilon \sum p_{n,k}(x) = \epsilon$, the second is, by (4.2), $\le 2M\frac{1}{4n\delta^2}$.Therefore

$$|f(x) - (B_n f)(x)| \le \epsilon + M\frac{1}{2n\delta^2} \tag{4.3}$$

and if n is sufficiently large,

$$|f(x) - (B_n f)(x)| < 2\epsilon.$$

Finally, if $f(x)$ is continuous in the whole interval $[0,1]$ then (4.3)holds with a δ independent of x, so that

$$(B_n f)(x) \to f(x).$$

uniformly.

This completes the proof.

Theorem 4.5. *(T. Popoviciu) If f is a continuous and $\omega(\delta)$ is the modulus of continuity of $f(x)$,then*

$$|f(x) - (B_n f)(x)| < \frac{5}{4}\omega\left(\frac{1}{\sqrt{n}}\right) \tag{4.4}$$

(G.G. Lorentz)

Proof. For arbitrary x_1,x_2 in $[0,1]$ and a $\delta > 0$ we denote by $\lambda = \lambda(x_1, x_2; \delta)$

the integer $\left[|x_1 - x_2|\,\delta^{-1}\right]$; the difference $f(x_1) - f(x_2)$ then a sum of $\lambda + 1$ differences of $f(x)$ on intervals of length $< \delta$.

Therefore

$$|f(x_1) - f(x_2)| \le (\lambda + 1)\,\omega(\delta)$$

and it follows that

$$\begin{aligned}
|f(x) - (B_n f)(x)| &\le \sum_{k=0}^{n} \left| f(x) - f\left(\frac{k}{n}\right) \right| p_{n,k}(x) \\
&\le \omega(\delta) \sum_{k=0}^{n} \left\{ 1 + \lambda\left(x, \frac{k}{n}; \delta\right) \right\} p_{n,k}(x) \\
&= \omega(\delta) \left\{ 1 + \sum_{\lambda \ge 1} \lambda\left(x, \frac{k}{n}; \delta\right) p_{n,k}(x) \right\} \\
&\le \omega(\delta) \left\{ 1 + \frac{1}{\delta} \sum_{\lambda \ge 1} \left| x - \frac{k}{n} \right| p_{n,k}(x) \right\} \\
&\le \omega(\delta) \left\{ 1 + \frac{1}{\delta^2} \sum_{k=0}^{n} \left(\frac{k}{n} - x \right)^2 p_{n,k}(x) \right\} \\
&\le \omega(\delta) \left\{ 1 + \frac{1}{4n\delta^2} \right\}
\end{aligned}$$

by (4.2) .We obtain (4.4) by choosing $\delta = \frac{1}{\sqrt{n}}$.

The smallest value of the constant C for which

$$|f(x) - (B_n f)(x)| \le C\omega\left(\frac{1}{\sqrt{n}}\right) \tag{4.5}$$

is true for each function f and each n , is not known ; in any case we have $C \le \frac{5}{4}$. Moreover, $C \ge 1$, as is seen from the following example. Let $\delta_n = o\left(\frac{1}{\sqrt{n}}\right)$ and suppose that $f_n(x)$ is the function which is equal to zero at x_0, $0 < x_0 < 1$, equal to 1 in $[0, x_0 - \delta_n]$ and $[x_0 + \delta_n, 1]$ and linear in the rest of $[0, 1]$. For large n, we have $\omega\left(\frac{1}{\sqrt{n}}\right) = 1$ for the function f_n ; also,

$$|f_n(x_0) - (B_n f)(x_0)| = (B_n f)(x_0) \geq \sum_{\left|\frac{k}{n}-x_0\right|\geq\delta_n} p_{n,k(x_0)} = 1 - \epsilon_n \qquad \epsilon_n \to 0.$$

Therefore (4.5) cannot be true $C < 1$.

The function $\omega(\delta)$ cannot therefore be replaced in (4.5) by any other function decreasing to zero more rapidly. This may be shown also by the example of one single continuous function. The function $f(x) = |x - x_0|^\alpha$ with fixed $0 < x_0 < 1$, $0 < \alpha \leq 1$, belongs to the class $Lip\alpha$ .We have $\omega(\delta) = \delta^\alpha$ with $\delta = \frac{1}{\sqrt{n}}$

$$\begin{aligned} |f(x_0) - (B_n f)(x_0)| &= (B_n f)(x_0) = \sum_{k=0}^{n} \left|\frac{k}{n} - x\right|^\alpha p_{n,k(x_0)} \\ &\geq n^{-\frac{1}{2}\alpha} \sum_{\left|\frac{k}{n}-x_0\right|\geq n^{-\frac{1}{2}}} p_{n,k(x_0)} \cong Cn^{-\frac{1}{2}\alpha}. \end{aligned}$$

for a certain constant $C > 0$.

Theorem 4.6. *(Voronovskaya) Let $f(x)$ be bounded in $[0,1]$ and suppose that the second derivative $f''(x)$ exists at a certain x of $[0,1]$, then*

$$\lim_{n\to\infty} n\left[B_n(f;x) - f(x)\right] = \frac{1}{2}x(1-x)f''(x). \tag{4.6}$$

(G.G. Lorentz)

Proof. In particular, if $f''(x) \neq 0$, the difference $f(x) - (B_n f)(x)$ is exactly of order n^{-1}. In order to prove (4.6), we write

$$f\left(\frac{k}{n}\right) = f(x) + \left(\frac{k}{n} - x\right)f'(x) + \left(\frac{k}{n} - x\right)^2\left[\frac{1}{2}f''(x) + \mu\left(\frac{k}{n} - x\right)\right]$$

where $\mu(h)$ is bounded, $|\mu(h)| \leq H$ for all h and converges to zero with h.Multiplying by $p_{n,k}(x)$ and summing,

$$\begin{aligned}(B_n f)(x) &= \sum_{k=0}^{n} f\left(\frac{k}{n}\right) p_{n,k}(x) \\ &= f(x) + \frac{1}{2} f''(x)\, n^{-2} T_{n,2}(x) + \sum_{k=0}^{n} \left(\frac{k}{n} - x\right)^2 \mu\left(\frac{k}{n} - x\right) p_{n,k}(x).\end{aligned}$$

The last term on the right can easily be estimated. Let $\epsilon > 0$ be arbitrary and $\delta > 0$ such that $|h| < \delta$ implies $|\mu(h)| < \epsilon$

$$\begin{aligned}& \epsilon \sum_{\left|\frac{k}{n} - x\right| \le \delta} \left(\frac{k}{n} - x\right)^2 p_{n,k}(x) + H \sum_{\left|\frac{k}{n} - x\right| > \delta} p_{n,k}(x) \\ \le\ & \epsilon n^{-2} T_{n,2} + AHn^{-2}.\end{aligned}$$

Since $T_{n,2} = nx(1-x)$, this is $< 2\epsilon n^{-1}$ for all sufficiently large n . Thus

$$(B_n f)(x) = f(x) + \frac{x(1-x)}{2n} f''(x) + \frac{\epsilon_n}{n}, \quad \epsilon_n \to 0$$

for $n \to \infty$, which is equivalent to (4.6) .

In a similar way, using the polynomails $T_{n,s}(x)$ we obtain

$$(B_n f)(x) = f(x) + \sum_{s=1}^{2k} \frac{1}{s!} n^{-s} f^{(s)}(x)\, T_{n,s}(x) + \epsilon_n n^{-k}, \quad \epsilon_n \to 0$$

for $n \to \infty$, if the derivative $f^{(2k)}(x)$ exists at x.

$$n^{-2k} T_{n,2k}(x) = \frac{(2k)!}{2^k k!} (x(1-x))^k n^{-k} + O\left(n^{-k-1}\right)$$

and we obtain the formula

$$\lim_{n\to\infty} n^k \left[(B_n f)(x) - f(x) - \sum_{s=1}^{2k-1} \frac{1}{s!} n^{-s} T_{n,s}(x)\, f^{(s)}(x) \right] = \left(\frac{1}{2} x(1-x)\right)^k \frac{1}{k!} f^{2k}(x).$$

which describes the asymtotic behaviour of the difference $B_n - f$ for large n.

Theorem 4.7. *If* $f \in Lip_M(\alpha)$ *where* $0 < \alpha \leq 1$, *then*

$$\|B_n(f;x) - f(x)\|_{C[0,1]} \leq O\left(\frac{1}{4n}\right)^{\frac{\alpha}{2}}.$$

Proof. We can write that

$$|B_n(f;x) - f(x)| = \left|\sum_{k=0}^{n} f(\frac{k}{n})\binom{n}{k}x^k(1-x)^{n-k} - f(x)\right|$$

and we know that

$$\sum_{k=0}^{n} P_{k,n}(x) = 1$$

then,

$$\begin{aligned} |B_n(f;x) - f(x)| &= \left|\sum_{k=0}^{n} f(\frac{k}{n})\binom{n}{k}x^k(1-x)^{n-k} - f(x)\sum_{k=0}^{n} P_{n,k}(x)\right| \\ &= \sum_{k=0}^{n}\left|f(\frac{k}{n}) - f(x)\right| P_{n,k}(x). \end{aligned}$$

By the Lipschitz Condition, then

$$|B_n(f;x) - f(x)| \leq \sum_{k=0}^{n} M\left|\frac{k}{n} - x\right|^{\alpha} P_{n,k}(x).$$

Now, we will use the Hölder Inequality

$$|B_n(f;x) - f(x)| = \sum_{k=0}^{n} M\left|\frac{k}{n} - x\right|^{\alpha} P_{n,k}^{\frac{\alpha}{2}}(x)\, P_{n,k}^{\frac{2-\alpha}{2}}(x)$$

thus,

$$\begin{aligned}
|B_n(f;x)-f(x)| &\leq M\left(\sum_{k=0}^{n}\left|\frac{k}{n}-x\right|^2 P_{k,n}(x)\right)^{\frac{\alpha}{2}}\left(\sum_{k=0}^{n}P_{k,n}(x)\right)^{\frac{2-\alpha}{2}} \\
&= M\left(\sum_{k=0}^{n}\left(\left(\frac{k}{n}\right)^2 P_{k,n}(x)-2x\frac{k}{n}P_{k,n}(x)+x^2P_{k,n}(x)\right)\right)^{\frac{\alpha}{2}} \\
&= M(B_n\left(t^2;x\right)-2xB_n(t;x)+x^2B_n(1;x))^{\frac{\alpha}{2}} \\
&= M\left(x^2+\frac{x(1-x)}{n}-2x.x+x^2\right)^{\frac{\alpha}{2}} \\
&= M\left(\frac{x(1-x)}{n}\right)^{\frac{\alpha}{2}}
\end{aligned}$$

where $p=\frac{2}{\alpha}$,$q=\frac{2}{2-\alpha}$ and $\frac{1}{p}+\frac{1}{q}=1$,we get,

$$|B_n(f;x)-f(x)|\leq M\left(\frac{x(1-x)}{n}\right)^{\frac{\alpha}{2}}.$$

Taking its maximum,

$$\max_{x\in[0,1]}|B_n(f;x)-f(x)|\leq M\left(\frac{1}{4n}\right)^{\frac{\alpha}{2}}$$

Thus

$$\|B_n(f;x)-f(x)\|_{C[0,1]}\leq O\left(\frac{1}{4n}\right)^{\frac{\alpha}{2}}.$$

4.2 Szász Operator and Its Approximation Properties

Definition 4.2. For a function $f(x)$ defined on $C[0,\infty)$ and $x \in [0,\infty)$ the expression

$$S_n(x) = (S_n f)(x) = \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right) e^{-nx} \frac{(nx)^k}{k!}$$

is called the Szász Operator.

Theorem 4.8. *Szász operator is linear and positive operator.*

Proof. First we will show S_n is linear.

$$\begin{aligned}(S_n(\alpha f + \beta g))(x) &= \sum_{k=0}^{\infty} (\alpha f + \beta g)\left(\frac{k}{n}\right) e^{-nx} \frac{(nx)^k}{k!} \\ &= \sum_{k=0}^{\infty} (\alpha f)\left(\frac{k}{n}\right) e^{-nx} \frac{(nx)^k}{k!} + \sum_{k=0}^{\infty} (\beta g)\left(\frac{k}{n}\right) e^{-nx} \frac{(nx)^k}{k!} \\ &= \alpha (S_n f)(x) + \beta (S_n g)(x).\end{aligned}$$

Since $x \in [0, A]$, and if $f(x) \geq 0$, then

$$(S_n f)(x) = \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right) e^{-nx} \frac{(nx)^k}{k!} \geq 0.$$

So S_n is positive.

Theorem 4.9. *Let $f : [0, A] \to \mathbb{R}$, $f \in C[0, A]$. Then, S_n define an operator $C[0, A] \to C[0, A]$ and for every $n \in \mathbb{N}$*

$$\|S_n f\|_{C[0,A]} \leq \|f\|_{C[0,A]},$$

holds true.

Proof . We need to evaluate $\|S_n f\|_{C[0,A]}$. By taking into account that $f \in$

$C[0,A]$,

$$\begin{aligned} |(S_n f)(x)| &= \left|\sum_{k=0}^{\infty} f\left(\frac{k}{n}\right) e^{-nx}\frac{(nx)^k}{k!}\right| \\ &\le \sum_{k=0}^{\infty}\left|f\left(\frac{k}{n}\right)\right|\left|e^{-nx}\frac{(nx)^k}{k!}\right| \\ &\le \sum_{k=0}^{\infty}\max\left|f\left(\frac{k}{n}\right)\right|\left|e^{-nx}\frac{(nx)^k}{k!}\right| \\ &\le \max\left|f\left(\frac{k}{n}\right)\right|\sum_{k=0}^{\infty}\left|e^{-nx}\frac{(nx)^k}{k!}\right| \\ &= \max\left|f\left(\frac{k}{n}\right)\right| \end{aligned}$$

since $\|f\|_{C[0,A]} = \max_{x\in[0,A]} |f(x)|$, then

$$|(S_n f)(x)| \le \|f\|_{C[0,A]}$$

for every $x \in [0,A]$; hence we obtain

$$\|S_n f\|_{C[0,A]} \le \|f\|_{C[0,A]}$$

and so $S_n : C[0,A] \to C[0,A]$.

Theorem 4.10. *For $A \in \mathbb{R}^+$, $(S_n f)(x)$ is continuous on $[0,A]$ and $S_n f \rightrightarrows f$ at the positive semi-axes where f is bounded. That is,if $f \in C[0,A]$ then*

$$(S_n f) \rightrightarrows (f), \quad x \in [0,A].$$

Proof. We will show that;

i) $S_n(1;x) \rightrightarrows 1$

ii) $S_n(t;x) \rightrightarrows x$

iii) $S_n(t^2;x) \rightrightarrows x^2$ hold true for all $x \in [0,\infty)$ and $n \in \mathbb{N}$..

Firstly,

$$\begin{aligned} S_n(1;x) &= \sum_{k=0}^{\infty} e^{-nx} \frac{(nx)^k}{k!} \\ &= e^{-nx} \sum_{k=0}^{\infty} \frac{(nx)^k}{k!} \\ &= e^{-nx} e^{nx} \\ &= 1. \end{aligned}$$

Secondly,

$$\begin{aligned} S_n(t;x) &= e^{-nx} \sum_{k=0}^{\infty} \frac{k}{n} \frac{(nx)^k}{k!} \\ &= e^{-nx} \sum_{k=0}^{\infty} \frac{k}{n} \frac{n^k x^k}{k!} \\ &= e^{-nx} \sum_{k=1}^{\infty} \frac{k}{n} \frac{n^k x^k}{k!} \\ &= e^{-nx} \sum_{k=1}^{\infty} \frac{k}{n} \frac{n^{k-1} x^{k-1} x}{(k-1)!} \end{aligned}$$

where $(k \to k+1)$.

$$\begin{aligned} &= x e^{-nx} \sum_{k=0}^{\infty} \frac{n^k x^k}{k!} \\ &= x e^{-nx} e^{nx} \\ &= x. \end{aligned}$$

Thirdly,

$$\begin{aligned}
S_n\left(t^2;x\right) &= e^{-nx}\sum_{k=0}^{\infty}\frac{k^2}{n^2}\frac{(nx)^k}{k!}\\
&= e^{-nx}\sum_{k=0}^{\infty}\frac{k^2}{n^2}\frac{n^k x^k}{k!}\\
&= e^{-nx}\sum_{k=1}^{\infty}\frac{k^2}{n^2}\frac{n^k x^k}{k!}
\end{aligned}$$

$$\begin{aligned}
&= e^{-nx}\sum_{k=1}^{\infty}(\frac{k-1}{n}+\frac{1}{n})\frac{n^{k-1}x^{k-1}x}{(k-1)!}\\
&= e^{-nx}\left(\sum_{k=1}^{\infty}\frac{k-1}{n}\frac{n^{k-1}x^{k-1}x}{(k-1)!}+\sum_{k=1}^{\infty}\frac{1}{n}\frac{n^{k-1}x^{k-1}x}{(k-1)!}\right)\\
&= e^{-nx}\left(\sum_{k=2}^{\infty}\frac{k-1}{n}\frac{n^{k-1}x^{k-1}x}{(k-1)!}+\sum_{k=1}^{\infty}\frac{1}{n}\frac{n^{k-1}x^{k-1}x}{(k-1)!}\right)\\
&= e^{-nx}\left(\sum_{k=2}^{\infty}\frac{n^{k-2}x^{k-2}x^2}{(k-2)!}+\sum_{k=1}^{\infty}\frac{n^{k-1}x^{k-1}x}{(k-1)!}\right)
\end{aligned}$$

where $k \to k+2$ and $k \to k+1$.

$$\begin{aligned}
&= e^{-nx}(x^2\sum_{k=0}^{\infty}\frac{n^k x^k}{k!}+\frac{x}{n}\sum_{k=0}^{\infty}\frac{n^k x^k}{k!})\\
&= e^{-nx}x^2e^{nx}+\frac{x}{n}e^{-nx}e^{nx}\\
&= x^2+\frac{x}{n}
\end{aligned}$$

thus we get,

$$\begin{aligned}
S_n(t^2;x) &= x^2+\frac{x}{n}\\
S_n(t^2;x) &= x^2 \quad (n\to\infty)\,.
\end{aligned}$$

Therefore $\forall f \in C\left[0,A\right]$ in $\left[0,A\right].$

$$S_n\left(f;x\right) \rightrightarrows f\left(x\right) \ \ \left(n \to \infty\right).$$

Theorem 4.11. *If* $f \in C\left[0,A\right],$

$$\left\|S_n\left(f;x\right) - f\left(x\right)\right\|_{C[0,A]} \leq \left(1+\sqrt{A}\right)\omega\left(f;\frac{1}{\sqrt{n}}\right).$$

Proof. Firstly,we can write that

$$\begin{aligned}
\left|S_n\left(f;x\right) - f\left(x\right)\right| &= \left|e^{-nx}\sum_{k=0}^{\infty} f\left(\frac{k}{n}\right)\frac{(nx)^k}{k!} - f\left(x\right)\right| \\
&= \left|\sum_{k=0}^{\infty} f\left(\frac{k}{n}\right)P_{n,k}(x) - \sum_{k=0}^{\infty} f\left(x\right)P_{n,k}(x)\right| \\
&\leq \sum_{k=0}^{\infty}\left|f\left(\frac{k}{n}\right) - f\left(x\right)\right| P_{k,n}(x).
\end{aligned}$$

By modulus of continuity,

$$\left|S_n\left(f;x\right) - f\left(x\right)\right| \leq \sum_{k=0}^{\infty}\left(1+\frac{\left|\frac{k}{n}-x\right|}{\delta_n}\right)\omega\left(f;\delta_n\right)P_{n,k}(x)$$

then,

$$
\begin{aligned}
|S_n(f;x)-f(x)| &= \omega(f;\delta_n)\left[\sum_{k=0}^{\infty}P_{n,k}(x)+\frac{1}{\delta_n}\sum_{k=0}^{\infty}\left|\frac{k}{n}-x\right|P_{n,k}(x)\right]\\
&= \omega(f;\delta_n)\left[\sum_{k=0}^{\infty}P_{n,k}(x)+\frac{1}{\delta_n}\sum_{k=0}^{\infty}\left|\frac{k}{n}-x\right|P_{n,k}^{\frac{1}{2}}(x).P_{n,k}^{\frac{1}{2}}(x)\right]\\
&\le \omega(f;\delta_n)\left[1+\frac{1}{\delta_n}\left(\sum_{k=0}^{\infty}\left|\frac{k}{n}-x\right|^2P_{n,k}(x)\right)^{\frac{1}{2}}.\left(\sum_{k=0}^{\infty}P_{n,k}(x)\right)^{\frac{1}{2}}\right]\\
&= \omega(f;\delta_n)\left[1+\frac{1}{\delta_n}\left(\sum_{k=0}^{\infty}(\frac{k}{n})^2P_{n,k}(x)-2\frac{k}{n}xP_{n,k}(x)+x^2P_{n,k}(x)\right)^{\frac{1}{2}}\right]\\
&= \omega(f;\delta_n)\left[1+\frac{1}{\delta_n}\left((S_n(t^2;x)-2xS_n(t;x)+x^2S_n(1;x)\right)^{\frac{1}{2}}\right]\\
&= \omega(f;\delta_n)\left[1+\frac{1}{\delta_n}\left(\frac{x}{n}\right)^{\frac{1}{2}}\right]
\end{aligned}
$$

we get,

$$|S_n(f;x)-f(x)| = \omega(f;\delta_n)\left[1+\frac{1}{\delta_n}\left(\frac{x}{n}\right)^{\frac{1}{2}}\right].$$

Choosing $\delta_n=\frac{1}{\sqrt{n}}$

$$\|S_n(f;x)-f(x)\|_{C[0,A]} \le \left(1+\sqrt{A}\right)\omega\left(f;\frac{1}{\sqrt{n}}\right).$$

Theorem 4.12. *(Voronovskaya) Let $f(x)$ be bounded in $[0,A]$ and suppose that the second derivative $f''(x)$ exists at a certain point x of $(0,A)$, then*

$$\lim_{n\to\infty} n\,[S_n(f;x)-f(x)] = \frac{1}{2}xf''(x). \tag{4.7}$$

Proof. In particular, if $f''(x)\neq 0$, the difference $f(x)-S_n(x)$ is exactly of order n^{-1}. In order to prove (4.7), we write

$$f\left(\frac{k}{n}\right) = f(x) + \left(\frac{k}{n} - x\right) f'(x) + \left(\frac{k}{n} - x\right)^2 \left[\frac{1}{2} f''(x) + \mu\left(\frac{k}{n} - x\right)\right]$$

where $\mu(h)$ is bounded, $|\mu(h)| \leq H$ for all h and converges to zero with h.Multiplying by $e^{-nx}\frac{(nx)^k}{k!}$by and summing,

$$\begin{aligned} S_n(x) &= \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right) e^{-nx} \frac{(nx)^k}{k!} \\ &= f(x) + \frac{1}{2} f''(x) T_{n,2}(x) + e^{-nx} \sum_{k=0}^{\infty} \left(\frac{k}{n} - x\right)^2 \mu\left(\frac{k}{n} - x\right) \frac{(nx)^k}{k!}. \end{aligned}$$

The last term on the right can easily be estimated. Let $\epsilon > 0$ be arbitrary and $\delta > 0$ such that $|h| < \delta$ implies $|\mu(h)| < \epsilon$

$$\begin{aligned} I &= e^{-nx} \sum_{\left|\frac{k}{n} - x\right| \leq \delta} \left(\frac{k}{n} - x\right)^2 \mu\left(\frac{k}{n} - x\right) \frac{(nx)^k}{k!} + e^{-nx} \sum_{\left|\frac{k}{n} - x\right| > \delta} \left(\frac{k}{n} - x\right)^2 \mu\left(\frac{k}{n} - x\right) \frac{(nx)^k}{k!} \\ &\leq \epsilon T_{n,2} + H e^{-nx} \sum_{\left|\frac{k}{n} - x\right| > \delta} \left(\frac{k}{n} - x\right)^2 \frac{(nx)^k}{k!}. \end{aligned}$$

Since $T_{n,2} = \frac{x}{n}$. Thus

$$\leq \epsilon \frac{x}{n} + H e^{-nx} \sum_{\left|\frac{k}{n} - x\right| > \delta} \left(\frac{k}{n} - x\right)^2 \frac{(nx)^k}{k!}.$$

In a similar way, we can show the second expression by J ,

$$J = \sum_{\left|\frac{k}{n} - x\right| > \delta} \left(\frac{k}{n} - x\right)^2 \frac{(nx)^k}{k!}.$$

And we know that

$$\left|\frac{k}{n} - x\right| > \delta \Longrightarrow \frac{\left(\frac{k}{n} - x\right)^2}{\delta^2} > 1.$$

Then

$$J \leq \sum_{\left|\frac{k}{n}-x\right|>\delta} \frac{\left(\frac{k}{n}-x\right)^2}{\delta^2} \left(\frac{k}{n}-x\right)^2 \frac{(nx)^k}{k!}$$

and we obtain

$$J \leq \frac{1}{\delta^2} \sum_{\left|\frac{k}{n}-x\right|>\delta} \left(\frac{k}{n}-x\right)^4 \frac{(nx)^k}{k!}.$$

We can expand the last expression from $k = 0$ to ∞ ,

$$\begin{aligned} I &\leq \epsilon\frac{x}{n} + \frac{H}{\delta^2} e^{-nx} \sum_{k=0}^{\infty} \left(\frac{k}{n}-x\right)^4 \frac{(nx)^k}{k!} \\ &\leq \epsilon\frac{x}{n} + \frac{H}{\delta^2} T_{n,4} \end{aligned}$$

and using by $T_{n,4} = \frac{x}{n^3} + \frac{3x^2}{n^2}$.

$$\begin{aligned} I &\leq \epsilon\frac{x}{n} + \frac{H}{\delta^2}\left(\frac{x}{n^3} + \frac{3x^2}{n^2}\right) \\ &= \epsilon\frac{x}{n} + \frac{H}{\delta^2}\frac{1}{n^2}\left(\frac{x}{n} + 3x^2\right). \end{aligned}$$

Taking its maximum in $x \in [0, A]$,

$$\begin{aligned} I &\leq \epsilon\frac{A}{n} + \frac{1}{n^2}\left[\frac{H}{\delta^2}\left(\frac{A}{n} + 3A^2\right)\right] \\ &\leq \epsilon\frac{A}{n} + \frac{1}{n^2}\left[\frac{H}{\delta^2}\left(A + 3A^2\right)\right] \\ &= \frac{1}{n}\left\{\epsilon A + \frac{1}{n}B\right\} \\ &\leq \frac{C}{n}\left\{\epsilon + \frac{1}{n}\right\} \end{aligned}$$

where $B = \frac{H}{\delta^2}\left(A + 3A^2\right)$ and $C = \max\{A, B\}$.

Choosing $\epsilon_n = \max\left\{\epsilon, \frac{1}{n}\right\}$ and $I = \frac{1}{n}O\left(\epsilon_n\right)$,then we obtain

$$\begin{aligned} S_n(f;x) &= f(x) + \frac{1}{2}\frac{x}{n}f''(x) + \frac{1}{n}O\left(\epsilon_n\right) \\ S_n(f;x) - f(x) &= \frac{1}{2}\frac{x}{n}f''(x) + \frac{1}{n}O\left(\epsilon_n\right). \end{aligned}$$

Thus,

$$\lim_{n\to\infty} n\left[S_n(f;x) - f(x)\right] = \frac{1}{2}xf''(x)$$

where $n \to \infty$ and $\epsilon_n \to 0$.

Theorem 4.13. *If $f \in Lip_M\left(\alpha\right)$ where $0 < \alpha \leq 1$, then*

$$\left\|S_n\left(f;x\right) - f\left(x\right)\right\|_{C[0,A]} = O\left(\left(\frac{A}{n}\right)^{\frac{1}{2}}\right).$$

Proof. Firstly,we can write that

$$\left|S_n\left(f;x\right) - f\left(x\right)\right| = \left|\sum_{k=0}^{\infty} f\left(\frac{k}{n}\right) e^{-nx}\frac{(nx)^k}{k!} - f\left(x\right)\right|$$

and we know that

$$\sum_{k=0}^{\infty} e^{-nx}\frac{(nx)^k}{k!} = 1$$

then,

$$\begin{aligned} \left|S_n\left(f;x\right) - f\left(x\right)\right| &= \left|\sum_{k=0}^{\infty} f\left(\frac{k}{n}\right) e^{-nx}\frac{(nx)^k}{k!} - f\left(x\right)\sum_{k=0}^{\infty} e^{-nx}\frac{(nx)^k}{k!}\right| \\ &= \sum_{k=0}^{\infty}\left|f(\frac{k}{n}) - f\left(x\right)\right| e^{-nx}\frac{(nx)^k}{k!}. \end{aligned}$$

By the Lipschitz Condition,then

$$\left|S_n\left(f;x\right) - f\left(x\right)\right| \leq \sum_{k=0}^{\infty} M\left|\frac{k}{n} - x\right|^{\alpha} e^{-nx}\frac{(nx)^k}{k!}.$$

Now,we will use the Hölder Inequality

$$|S_n(f;x)-f(x)| = \sum_{k=0}^{\infty} M\left|\frac{k}{n}-x\right|^{\alpha}\left(e^{-nx}\frac{(nx)^k}{k!}\right)^{\frac{\alpha}{2}}\left(e^{-nx}\frac{(nx)^k}{k!}\right)^{\frac{2-\alpha}{2}}$$

thus,

$$\begin{aligned}
|B_n(f;x)-f(x)| &\leq M\left(\sum_{k=0}^{\infty}\left|\frac{k}{n}-x\right|^2 e^{-nx}\frac{(nx)^k}{k!}\right)^{\frac{\alpha}{2}}\left(\sum_{k=0}^{\infty} e^{-nx}\frac{(nx)^k}{k!}\right)^{\frac{2-\alpha}{2}} \\
&= M\left(\sum_{k=0}^{\infty}\left(\left(\frac{k}{n}\right)^2 e^{-nx}\frac{(nx)^k}{k!} - 2x\frac{k}{n}e^{-nx}\frac{(nx)^k}{k!} + x^2 e^{-nx}\frac{(nx)^k}{k!}\right)\right)^{\frac{\alpha}{2}} \\
&= M(S_n\left(t^2;x\right) - 2xS_n(t;x) + x^2S_n(1;x))^{\frac{\alpha}{2}} \\
&= M\left(x^2+\frac{x}{n}-2x^2+x^2\right)^{\frac{\alpha}{2}} \\
&= M\left(\frac{x}{n}\right)^{\frac{\alpha}{2}}
\end{aligned}$$

where $p=\frac{2}{\alpha}$,$q=\frac{2}{2-\alpha}$ and $\frac{1}{p}+\frac{1}{q}=1$,we get,

$$|S_n(f;x)-f(x)| \leq M\left(\frac{x}{n}\right)^{\frac{\alpha}{2}}$$

Taking its maximum,

$$\max_{x\in[0,A]}|S_n(f;x)-f(x)| \leq M\left(\frac{A}{n}\right)^{\frac{\alpha}{2}}.$$

Thus

$$\|S_n(f;x)-f(x)\|_{C[0,A]} = O\left(\left(\frac{A}{n}\right)^{\frac{1}{2}}\right).$$

Theorem 4.14. *Suppose that $L_n : C_{\rho_1}(\mathbb{R}) \to B_{\rho_2}(\mathbb{R})$ is linear positive operator sequence, $C_{\rho_1}(\mathbb{R}) \to B_{\rho_2}(\mathbb{R})$ is uniformly bounded and $\lim\limits_{x\to\infty}\frac{\rho_1(x)}{\rho_2(x)} = 0$ .If*

$\lim_{n\to\infty} |L_n(f;x) - f(x)| = 0$,

$$\lim_{n\to\infty} \|L_n(f) - f\|_{\rho_2} = 0 \tag{4.8}$$

where for $|x| \le s, \forall f \in C_{\rho_1}(\mathbb{R})$.

Theorem 4.15. *Let* $\rho_2(x) \ge 1,\ x \in [0,\infty)$ *continuous and monotone increasing function,*

$$\lim_{x\to\infty} \frac{1+x^2}{\rho_2(x)} = 0. \tag{4.9}$$

Then $\forall f \in C_{1+x^2}(\mathbb{R}^+)$

$$\lim_{n\to\infty} \|S_n(f) - f\|_{\rho_2} = 0.$$

Proof. In order to prove this theorem ,we will show (4.8)holds true. Firstly, we will investigate that $S_n(f;x)$ is linear positive operator sequence from $C_{\rho_1}(\mathbb{R}^+)$ to $B_{\rho_1}(\mathbb{R}^+)$.

$$\|S_n(\rho_1)\|_{\rho_2} = \sup_{x\in\mathbb{R}^+} \frac{|S_n(1+t^2;x)|}{\rho_2(x)}$$

then

$$\begin{aligned} \|S_n(\rho_1)\|_{\rho_2} &= \sup_{x\in\mathbb{R}^+} \frac{\left|1+x^2+\frac{x}{n}\right|}{\rho_2(x)} \\ &= \sup_{x\in\mathbb{R}^+} \left| \frac{1+x^2}{\rho_2(x)} + \frac{x}{\rho_2(x)}\frac{1}{n} \right|. \end{aligned} \tag{4.10}$$

On the other hand,$1+x^2$ and $\rho_2(x)$ are continuous functions and $\rho_2(x) \ge 1$, so $\frac{1+x^2}{\rho_2(x)}$ is continuous. Every continuous functions are bounded in closed interval,then there exists a constant number such that M where $0 \le x \le N$ for $\frac{1+x^2}{\rho_2(x)} \le M$ .In additon ,from (4.9) and definition of limit, $\frac{1+x^2}{\rho_2(x)} < \epsilon$ is satisfied for $N < x$.

Thus,we can write

$$\frac{1+x^2}{\rho_2(x)} < M + \epsilon. \tag{4.11}$$

Also

$$0 < \frac{x}{\rho_2(x)} < \frac{1+x^2}{\rho_2(x)} < M + \epsilon$$

and then,

$$\frac{x}{\rho_2(x)} < M + \epsilon. \tag{4.12}$$

We get,

$$\|S_n(\rho_1)\|_{\rho_2} \leq 2(M+\epsilon)$$

by using eq.(4.11) and eq.(4.12)in eq.(4.10).If $2(M+\epsilon) = M_1$, we obtain that $\|S_n(\rho_1)\|_{\rho_2} \leq M_1$. Thus $C_{\rho_1}(\mathbb{R}^+) \to B_{\rho_1}(\mathbb{R}^+)$.

Now, we will show that $S_n : C_{\rho_1}(\mathbb{R}^+) \to B_{\rho_1}(\mathbb{R}^+)$ is uniformly bounded. There exists a constant K for $\|S_n\|_{C_{\rho_1} \to B_{\rho_1}} \leq K$.

$$\begin{aligned} \|S_n\|_{C_{\rho_1} \to B_{\rho_1}} &= \|S_n(\rho_1)\|_{\rho_1} \\ &= \sup_{x\in\mathbb{R}^+} \frac{|S_n(1+t^2;x)|}{1+x^2} \\ &= \sup_{x\in\mathbb{R}^+} \frac{\left|1+x^2+\frac{x}{n}\right|}{1+x^2} \\ &= \sup_{x\in\mathbb{R}^+} \left\{1+\frac{x}{1+x^2}\frac{1}{n}\right\}. \end{aligned} \tag{4.13}$$

On the other hand, if $x \leq 1+x^2$, then $\frac{x}{1+x^2} \leq 1$ and $\frac{1}{n} \leq 1$.We can use these inequalities in eq.(4.13) ,then we obtain $\|S_n\|_{C_{\rho_1} \to B_{\rho_1}} \leq 2$. Thus, $K = 2$ and uniformly bounded is satisfied. Therefore, $f \in C_{1+x^2}[0,s] \subset C[0,s]$ for $|x| \leq s$ and since from Korovkin theorem

$$\lim_{n\to\infty} \max_{0\leq x\leq s} |S_n(f;x) - f(x)| \geq |S_n(f;x) - f(x)| \geq 0$$

holds.Then we obtain

$$\lim_{n\to\infty} |S_n(f;x) - f(x)| = 0.$$

If we consider its norm,then we get

$$\lim_{n\to\infty} \left\| S_n\left(f;x\right) - f\left(x\right) \right\|_{\rho_2} = 0.$$

This proof is completed.

Theorem 4.16. *If* $f \in C_{\rho,K_f}[0,\infty)$,

$$\left\| S_n\left(f\right) - f \right\|_{\rho^2} \leq 40\Omega\left(f;\frac{1}{\sqrt{n}}\right).$$

Proof. Firstly, we can write that

$$S_n(f;x) = e^{-nx}\sum_{k=0}^{\infty} f\left(\frac{k}{n}\right)\frac{(nx)^k}{k!}$$

and we know this form

$$S_n(1;x) = e^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!} = 1.$$

Using by linearity,

$$\begin{aligned} \left|S_n(f;x) - f\left(x\right)\right| &= \left| e^{-nx}\sum_{k=0}^{\infty} f\left(\frac{k}{n}\right)\frac{(nx)^k}{k!} - f\left(x\right)\sum_{k=0}^{\infty} e^{-nx}\frac{(nx)^k}{k!}\right| \\ &\leq e^{-nx}\sum_{k=0}^{\infty}\left| f\left(\frac{k}{n}\right) - f\left(x\right)\right|\frac{(nx)^k}{k!} \end{aligned}$$

and using by triangle inequality, $e^{-nx} \geq 0$, $\frac{(nx)^k}{k!} \geq 0$;we obtain

$$\left|S_n(f;x) - f\left(x\right)\right| \leq e^{-nx}\sum_{k=0}^{\infty}\left| f\left(\frac{k}{n}\right) - f\left(x\right)\right|\frac{(nx)^k}{k!}. \tag{4.14}$$

Since property (vii) of modulus of continuity, choosing $t = \frac{k}{n}$,

$$\left| f\left(\frac{k}{n}\right) - f(x) \right| \leq 4\left(\frac{\left|\frac{k}{n} - x\right|}{\delta} + 1\right)\left(1 + x^2\right)\left(1 + \left(\frac{k}{n} - x\right)^2\right)\left(1 + \delta^2\right)\Omega(f;\delta).$$

Using this inequality in(4.14), we get

$$\begin{aligned} |S_n(f;x) - f(x)| &\leq 4\left(1 + x^2\right)\left(1 + \delta^2\right)\Omega(f;\delta) \\ &\quad \sum_{k=0}^{\infty}\left(\frac{\left|\frac{k}{n} - x\right|}{\delta} + 1\right)\left(1 + \left(\frac{k}{n} - x\right)^2\right)\frac{(nx)^k}{k!}e^{-nx} \\ &= 4\left(1 + x^2\right)\left(1 + \delta^2\right)\Omega(f;\delta)\left\{\sum_{k=0}^{\infty}\frac{(nx)^k}{k!}e^{-nx}\right. \\ &\quad + \sum_{k=0}^{\infty}\left(\frac{k}{n} - x\right)^2\frac{(nx)^k}{k!}e^{-nx} + \frac{1}{\delta}\sum_{k=0}^{\infty}\left|\frac{k}{n} - x\right|\frac{(nx)^k}{k!}e^{-nx} \\ &\quad \left. + \frac{1}{\delta}\sum_{k=0}^{\infty}\left|\frac{k}{n} - x\right|\left(\frac{k}{n} - x\right)^2\frac{(nx)^k}{k!}e^{-nx}\right\}. \end{aligned} \tag{4.15}$$

Let we denote these equations by A and B,

$$A = \sum_{k=0}^{\infty}\left|\frac{k}{n} - x\right|\frac{(nx)^k}{k!}e^{-nx}$$

and

$$B = \sum_{k=0}^{\infty}\left|\frac{k}{n} - x\right|\left(\frac{k}{n} - x\right)^2\frac{(nx)^k}{k!}e^{-nx}.$$

We can write

$$A = \sum_{k=0}^{\infty}\left|\frac{k}{n} - x\right|\left(\frac{(nx)^k}{k!}e^{-nx}\right)^{\frac{1}{2}}\left(\frac{(nx)^k}{k!}e^{-nx}\right)^{\frac{1}{2}}.$$

By Cauchy-Schwarz inequality ,

$$\begin{aligned} A &\leq \left(\sum_{k=0}^{\infty}\left(\frac{k}{n}-x\right)^{2}\frac{(nx)^{k}}{k!}e^{-nx}\right)^{\frac{1}{2}}\left(\sum_{k=0}^{\infty}\frac{(nx)^{k}}{k!}e^{-nx}\right)^{\frac{1}{2}} \\ &= \left(\sum_{k=0}^{\infty}\left(\frac{k}{n}-x\right)^{2}\frac{(nx)^{k}}{k!}e^{-nx}\right)^{\frac{1}{2}}\left(S_{n}\left(1;x\right)\right)^{\frac{1}{2}} \end{aligned}$$

then,we get

$$A \leq \left(\sum_{k=0}^{\infty}\left(\frac{k}{n}-x\right)^{2}\frac{(nx)^{k}}{k!}e^{-nx}\right)^{\frac{1}{2}}.$$

On the other hand,

$$\begin{aligned} \sum_{k=0}^{\infty}\left(\frac{k}{n}-x\right)^{2}\frac{(nx)^{k}}{k!}e^{-nx} &= \sum_{k=0}^{\infty}\left[\frac{k^{2}}{n^{2}}-2x\frac{k}{n}+x^{2}\right]\frac{(nx)^{k}}{k!}e^{-nx} \\ &= \sum_{k=0}^{\infty}\frac{k^{2}}{n^{2}}\frac{(nx)^{k}}{k!}e^{-nx}-2x\sum_{k=0}^{\infty}\frac{k}{n}\frac{(nx)^{k}}{k!}e^{-nx} \\ &\quad +x^{2}\sum_{k=0}^{\infty}\frac{(nx)^{k}}{k!}e^{-nx} \\ &= S_{n}\left(t^{2};x\right)-2xS_{n}\left(t;x\right)+x^{2}S_{n}\left(1;x\right). \end{aligned}$$

One has,

$$\begin{aligned} \sum_{k=0}^{\infty}\left(\frac{k}{n}-x\right)^{2}\frac{(nx)^{k}}{k!}e^{-nx} &= x^{2}+\frac{x}{n}-2xx+x^{2} \\ &= \frac{x}{n}. \end{aligned} \tag{4.16}$$

Thus

$$A \leq \frac{\sqrt{x}}{\sqrt{n}}. \tag{4.17}$$

In similar way,we can write that

$$B = \sum_{k=0}^{\infty}\left|\frac{k}{n}-x\right|\left(\frac{(nx)^{k}}{k!}e^{-nx}\right)^{\frac{1}{2}}\left(\frac{k}{n}-x\right)^{2}\left(\frac{(nx)^{k}}{k!}e^{-nx}\right)^{\frac{1}{2}}.$$

By Cauchy-Schwarz inequality,we obtain

$$B \leq \left(\sum_{k=0}^{\infty}\left(\frac{k}{n}-x\right)^{2}\frac{(nx)^{k}}{k!}e^{-nx}\right)^{\frac{1}{2}}\left(\sum_{k=0}^{\infty}\left(\frac{k}{n}-x\right)^{4}\frac{(nx)^{k}}{k!}e^{-nx}\right)^{\frac{1}{2}}.$$

Also, we know that

$$\sum_{k=0}^{\infty}\left(\frac{k}{n}-x\right)^{4}\frac{(nx)^{k}}{k!}e^{-nx}=\frac{3x^{2}}{n^{2}}+\frac{x}{n^{3}}.$$

Therefore

$$B \leq \frac{\sqrt{x}}{\sqrt{n}}\left(\frac{3x^{2}}{n^{2}}+\frac{x}{n^{3}}\right)^{\frac{1}{2}}. \tag{4.18}$$

Using eq.(4.16) ,eq.(4.17) and eq.(4.18)in eq.(4.15) ,we get

$$|S_n(f;x)-f(x)| = 4\left(1+x^{2}\right)\left(1+\delta^{2}\right)\Omega(f;\delta)\left\{1+\frac{x}{n}+\frac{1}{\delta}\frac{\sqrt{x}}{\sqrt{n}}+\frac{1}{\delta}\frac{\sqrt{x}}{\sqrt{n}}\left(\frac{3x^{2}}{n^{2}}+\frac{x}{n^{3}}\right)^{\frac{1}{2}}\right\}.$$

Dividing both of sides by $(1+x^2)^2$,we obtain

$$\begin{aligned}
\frac{|S_n(f;x)-f(x)|}{(1+x^{2})^{2}} &\leq 4\left(1+\delta^{2}\right)\Omega(f;\delta)\left\{\frac{1}{(1+x^{2})}+\frac{x}{(1+x^{2})}\frac{1}{n}\right.\\
&\quad\left.+\frac{1}{\delta}\frac{\sqrt{x}}{\sqrt{n}}\frac{1}{(1+x^{2})}+\frac{1}{\delta}\frac{\sqrt{x}}{(1+x^{2})}\frac{1}{\sqrt{n}}\left(3x^{2}+\frac{x}{n}\right)^{\frac{1}{2}}\right\}\\
&= 4\left(1+\delta^{2}\right)\Omega(f;\delta)\left\{\frac{1}{(1+x^{2})}+\frac{x}{(1+x^{2})}\frac{1}{n}\right.\\
&\quad\left.+\frac{1}{\delta}\frac{\sqrt{x}}{\sqrt{n}}\frac{1}{(1+x^{2})}+\frac{1}{\delta}\frac{\sqrt{x}}{(1+x^{2})}\frac{1}{\sqrt{n}}\left(3x^{2}+\frac{x}{n}\right)^{\frac{1}{2}}\right\}\\
&\leq 4\left(1+\delta^{2}\right)\Omega(f;\delta)\left\{\frac{1}{(1+x^{2})}+\frac{x}{(1+x^{2})}\right.\\
&\quad\left.+\frac{1}{\delta}\frac{\sqrt{x}}{\sqrt{n}}\frac{1}{(1+x^{2})}+\frac{1}{\delta}\frac{\sqrt{x}}{(1+x^{2})}\frac{1}{\sqrt{n}}\left(3x^{2}+x\right)^{\frac{1}{2}}\right\}. \quad (4.19)
\end{aligned}$$

In additon, these inequalities

$$\begin{aligned}1 &\leq 1+x^2 \Longrightarrow \frac{1}{1+x^2} \leq 1\\ x &\leq 1+x^2 \Longrightarrow \frac{x}{1+x^2} \leq 1\\ \sqrt{x} &< x \leq 1+x^2 \Longrightarrow \frac{\sqrt{x}}{1+x^2} \leq 1\end{aligned}$$

are satisfied. And, the inequality holds true

$$\frac{\sqrt{x}\left(3x^2+x\right)^{\frac{1}{2}}}{1+x^2} \leq 2.$$

Then,we get

$$\begin{aligned}\frac{\sqrt{x}\left(3x^2+x\right)^{\frac{1}{2}}}{1+x^2} &= \frac{\sqrt{x\left(3x^2+x\right)}}{1+x^2} = \frac{\sqrt{x\left(3x^2+x\right)}}{\sqrt{\left(1+x^2\right)^2}} = \sqrt{\frac{3x^3}{\left(1+x^2\right)^2}+\frac{x^2}{\left(1+x^2\right)^2}}\\ \frac{x^3}{\left(1+x^2\right)^2} &= \frac{x^2}{\left(1+x^2\right)}.\frac{x}{\left(1+x^2\right)} < 1.1 \Longrightarrow \frac{x^3}{\left(1+x^2\right)^2} < 1\end{aligned}$$

and

$$x^2 < 1+x^2 < \left(1+x^2\right)^2 \Longrightarrow \frac{x^2}{\left(1+x^2\right)^2} < 1.$$

Since

$$\frac{\sqrt{x}\left(3x^2+x\right)^{\frac{1}{2}}}{1+x^2} \leq \sqrt{3.1+1} = 2$$

then we obtain,

$$\frac{\sqrt{x}\left(3x^2+x\right)^{\frac{1}{2}}}{1+x^2} \leq 2. \tag{4.20}$$

Taking its supremum in eq.(4.19)and using by eq.(4.20),we get

$$\left\|S_n\left(f\right)-f\right\|_{\rho^2} \leq 4\left(1+\delta^2\right)\Omega\left(f;\delta\right)\left\{1+1+\frac{1}{\delta}\frac{1}{\sqrt{n}}+\frac{1}{\delta}\frac{1}{\sqrt{n}}2\right\}.$$

Choosing $\delta = \frac{1}{\sqrt{n}}$,

$$\begin{aligned}
\left\| S_n(f) - f \right\|_{\rho^2} &\leq 4\left(1 + \frac{1}{n}\right)\Omega\left(f; \frac{1}{\sqrt{n}}\right)\{1 + 1 + 1 + 2\} \\
&\leq 40\Omega\left(f; \frac{1}{\sqrt{n}}\right).
\end{aligned}$$

This proof is completed.

4.3 Baskakov Operator and Its Approximation Properties

Definition 4.3. For a function $f(x)$ defined on $C[0,\infty)$ and $x \in [0,\infty)$ the expression

$$(K_n f)(x) = \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right)\binom{n+k-1}{k}\frac{x^k}{(1+x)^{n+k}}$$

is called the Baskakov Operator.

Theorem 4.17. *Baskakov operator is linear and positive operator.*

Proof. First we will show K_n is linear.

$$\begin{aligned}(K_n(\alpha f+\beta g))(x) &= \sum_{k=0}^{\infty}(\alpha f+\beta g)\left(\frac{k}{n}\right)\binom{n+k-1}{k}\frac{x^k}{(1+x)^{n+k}} \\ &= \sum_{k=0}^{\infty}(\alpha f)\left(\frac{k}{n}\right)P_{n,k}(x)+\sum_{k=0}^{\infty}(\beta g)\left(\frac{k}{n}\right)P_{n,k}(x) \\ &= \alpha(K_n f)(x)+\beta(K_n g)(x).\end{aligned}$$

Since $x \in [0,A]$, and if $f(x) \geq 0$, then

$$(K_n f)(x) = \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right)\binom{n+k-1}{k}\frac{x^k}{(1+x)^{n+k}} \geq 0.$$

So K_n is positive.

Theorem 4.18. *Let $f : [0,\infty) \to \mathbb{R}$, $f \in C[0,\infty)$. Then, $K_n : C[0,\infty) \to C[0,\infty)$ and for every $n \in \mathbb{N}$*

$$\|K_n f\|_{C[0,\infty)} \leq \|f\|_{C[0,\infty)},$$

holds true.

Proof . We need to evaluate $\|K_n f\|_{C[0,\infty)}$. By taking into account that

$f \in C[0,\infty)$

$$
\begin{aligned}
|(S_n f)(x)| &= \left|\sum_{k=0}^{\infty} f\left(\frac{k}{n}\right)\binom{n+k-1}{k}\frac{x^k}{(1+x)^{n+k}}\right| \\
&\leq \sum_{k=0}^{\infty}\left|f\left(\frac{k}{n}\right)\right|\left|\binom{n+k-1}{k}\frac{x^k}{(1+x)^{n+k}}\right| \\
&\leq \sum_{k=0}^{\infty}\sup\left|f\left(\frac{k}{n}\right)\right|\left|\binom{n+k-1}{k}\frac{x^k}{(1+x)^{n+k}}\right| \\
&\leq \sup\left|f\left(\frac{k}{n}\right)\right|\sum_{k=0}^{\infty}\left|\binom{n+k-1}{k}\frac{x^k}{(1+x)^{n+k}}\right| \\
&= \sup\left|f\left(\frac{k}{n}\right)\right|
\end{aligned}
$$

since $\|f\|_{C[0,\infty)} = \sup\limits_{x\in[0,\infty)} |f(x)|$, then

$$|(K_n f)(x)| \leq \|f\|_{C[0,\infty)}$$

for every $x \in [0,\infty)$; hence we obtain

$$\|K_n f\|_{C[0,\infty)} \leq \|f\|_{C[0,\infty)}$$

and so $K_n : C[0,\infty) \to C[0,\infty)$.

Theorem 4.19. *For $A \in \mathbb{R}^+$, $(K_n f)(x)$ is continuous on $[0,A]$ and $K_n f \rightrightarrows f$ at the positive semi-axes where f is bounded. That is,if $f \in C[0,A]$ then*

$$(K_n f) \rightrightarrows (f), \quad x \in [0,A].$$

Proof. Let $K_n : C[0,\infty) \to C[0,\infty)$. Then

i)$K_n(1;x) \rightrightarrows 1$

ii)$K_n(t;x) \rightrightarrows x$

iii)$K_n(t^2;x) \rightrightarrows x^2$ for all $x \in [0,\infty)$ and $n \in \mathbb{N}$.

Firstly,the Taylor Series at $t = 0$

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} t^k$$

$$\begin{aligned}
f(t) &= (1+x-xt)^{-n} \\
f'(t) &= xn(1+x-xt)^{-(n+1)} \Longrightarrow f'(0) = nx(1+x)^{-(n+1)} \\
f''(t) &= x^2 n(n+1)(1+x-xt)^{-(n+2)} \Longrightarrow f''(0) = n(n+1)x^2(1+x)^{-(n+2)} \\
f^{(k)}(t) &= x^k n(n+1)...(n+k-1)(1+x-xt)^{-(n+k)} \\
&\Longrightarrow f^{(k)}(0) = n(n+1)\dots(n+k-1)x^k(1+x)^{-(n+k)} \\
&\Longrightarrow (1+x-xt)^{-n} = \sum_{k=0}^{\infty} \frac{(n-1)!n(n+1)...(n+k-1)x^k(1+x)^{-(n+k)}}{(n-1)!k!} \\
&= \sum_{k=0}^{\infty} \binom{n+k-1}{k} \left(\frac{x}{1+x}\right)^k \frac{1}{(1+x)^n} t^k.
\end{aligned}$$

For $t = 1$, and hence

$$1 = \sum_{k=0}^{\infty} \binom{n+k-1}{k} \left(\frac{x}{1+x}\right)^k \frac{1}{(1+x)^n}$$

then

$$K_n(1;x) = 1.$$

Secondly,

$$\begin{aligned}
K_n(t;x) &= \sum_{k=0}^{\infty} \frac{k}{n} \binom{n+k-1}{k} \left(\frac{x}{1+x}\right)^k \frac{1}{(1+x)^n} \\
&= \sum_{k=1}^{\infty} \frac{k}{n} \frac{(n+k-1)!}{k!(n-1)!} \left(\frac{x}{1+x}\right)^k \frac{1}{(1+x)^n}
\end{aligned}$$

where $k \to k+1$

$$
\begin{aligned}
&= \sum_{k=0}^{\infty} \frac{1}{n} \frac{(n+k)!}{k!\,(n-1)!} \left(\frac{x}{1+x}\right)^{k+1} \frac{1}{(1+x)^n} \\
&= \sum_{k=0}^{\infty} \binom{n+k}{k} \left(\frac{x}{1+x}\right)^{k+1} \frac{1}{(1+x)^n} \\
&= x \sum_{k=0}^{\infty} \binom{n+k}{k} \left(\frac{x}{1+x}\right)^{k} \frac{1}{(1+x)^{n+1}} \\
&= x.\,(1+x-x)^{-(n+1)} \\
&= x.
\end{aligned}
$$

Thirdly,

$$
\begin{aligned}
K_n\left(t^2; x\right) &= \sum_{k=1}^{\infty} \frac{k^2}{n^2} \binom{n+k-1}{k} \left(\frac{x}{1+x}\right)^{k} \frac{1}{(1+x)^n} \\
&= \sum_{k=1}^{\infty} \frac{k^2}{n^2} \frac{(n+k-1)!}{k!\,(n-1)!} \left(\frac{x}{1+x}\right)^{k} \frac{1}{(1+x)^n}
\end{aligned}
$$

where $k \to k+2$ and $k \to k+1$

$$
\begin{aligned}
&= \sum_{k=2}^{\infty} \frac{1}{n^2} \frac{(n+k-1)!}{(k-2)!\,(n-1)!} \left(\frac{x}{1+x}\right)^{k} \frac{1}{(1+x)^n} \\
&\quad + \sum_{k=1}^{\infty} \frac{1}{n^2} \frac{(n+k-1)!}{(k-1)!\,(n-1)!} \left(\frac{x}{1+x}\right)^{k} \frac{1}{(1+x)^n} \\
&= \sum_{k=0}^{\infty} \frac{1}{n.n} \frac{(n+k+1)!}{k!\,(n-1)!} \left(\frac{x}{1+x}\right)^{k+2} \frac{1}{(1+x)^n} \\
&\quad + \sum_{k=0}^{\infty} \frac{1}{n.n} \frac{(n+k)!}{k!\,(n-1)!} \left(\frac{x}{1+x}\right)^{k+1} \frac{1}{(1+x)^n} \\
&= \sum_{k=0}^{\infty} \frac{1}{n.n} \frac{(n+k+1)!}{k!n!} \left(\frac{x}{1+x}\right)^{k+2} \frac{1}{(1+x)^n} \\
&\quad + \sum_{k=0}^{\infty} \frac{1}{n.n} \frac{(n+k)!}{k!n!} \left(\frac{x}{1+x}\right)^{k+1} \frac{1}{(1+x)^n}
\end{aligned}
$$

$$\begin{aligned}
&= \sum_{k=0}^{\infty}\frac{1}{n}\frac{(n+k+1)!}{k!\,(n+1)!}\left(\frac{x}{1+x}\right)^{k+2}\frac{1}{(1+x)^{n}} \\
&\quad +\sum_{k=0}^{\infty}\frac{1}{n}\frac{(n+k)!}{k!n!}\left(\frac{x}{1+x}\right)^{k+1}\frac{1}{(1+x)^{n}} \\
&= \sum_{k=0}^{\infty}\frac{n+1}{n}\binom{n+k+1}{k}\left(\frac{x}{1+x}\right)^{k+2}\frac{1}{(1+x)^{n}} \\
&\quad +\sum_{k=0}^{\infty}\frac{1}{n}\binom{n+k}{k}\left(\frac{x}{1+x}\right)^{k+1}\frac{1}{(1+x)^{n}} \\
&= \frac{n+1}{n}x^{2}\sum_{k=0}^{\infty}\binom{n+k+1}{k}\left(\frac{x}{1+x}\right)^{k}\frac{1}{(1+x)^{n+2}} \\
&\quad +\frac{1}{n}.x\sum_{k=0}^{\infty}\binom{n+k}{k}\left(\frac{x}{1+x}\right)^{k}\frac{1}{(1+x)^{n+1}}
\end{aligned}$$

thus we get,

$$\begin{aligned}
K_n(t^2;x) &= \frac{n+1}{n}x^2+\frac{1}{n}x \\
K_n(t^2;x) &= x^2 \quad (n\to\infty)\,.
\end{aligned}$$

Therefore $\forall f\in C\,[0,A]$ in $[0,A]\,.$

$$K_n\,(f;x)\rightrightarrows f\,(x)\ \ (n\to\infty)\,.$$

Theorem 4.20. *If $f\in C\,[0,A]$,*

$$\left\|K_n\,(f;x)-f\,(x)\right\|_{C[0,A]}\le\left(1+\sqrt{A\,(A+1)}\right)\omega\left(f;\frac{1}{\sqrt{n}}\right).$$

Proof. Firstly,we can write that

$$\begin{aligned}
|K_n(f;x)-f(x)| &= \left|\sum_{k=0}^{\infty} f\left(\frac{k}{n}\right)\binom{n+k-1}{k}\left(\frac{x}{1+x}\right)^k \frac{1}{(1+x)^n} - f(x)\right| \\
&= \left|\sum_{k=0}^{\infty} f\left(\frac{k}{n}\right)P_{n,k}(x) - \sum_{k=0}^{\infty} f(x)\,P_{n,k}(x)\right| \\
&\leq \sum_{k=0}^{\infty}\left|f\left(\frac{k}{n}\right)-f(x)\right|P_{n,k}(x).
\end{aligned}$$

By modulus of continuity,

$$|K_n(f;x)-f(x)| \leq \sum_{k=0}^{\infty}\left(1+\frac{\left|\frac{k}{n}-x\right|}{\delta_n}\right)\omega(f;\delta_n)\,P_{n,k}(x)$$

then,

$$\begin{aligned}
|K_n(f;x)-f(x)| &= \omega(f;\delta_n)\left[\sum_{k=0}^{\infty}P_{n,k}(x)+\frac{1}{\delta_n}\sum_{k=0}^{\infty}\left|\frac{k}{n}-x\right|P_{n,k}(x)\right] \\
&= \omega(f;\delta_n)\left[\sum_{k=0}^{\infty}P_{n,k}(x)+\frac{1}{\delta_n}\sum_{k=0}^{\infty}\left|\frac{k}{n}-x\right|P_{n,k}^{\frac{1}{2}}(x).P_{n,k}^{\frac{1}{2}}(x)\right] \\
&\leq \omega(f;\delta_n)\left[1+\frac{1}{\delta_n}\left(\sum_{k=0}^{\infty}\left|\frac{k}{n}-x\right|^2P_{n,k}(x)\right)^{\frac{1}{2}}.\left(\sum_{k=0}^{\infty}P_{n,k}(x)\right)^{\frac{1}{2}}\right] \\
&= \omega(f;\delta_n)\left[1+\frac{1}{\delta_n}\left(\sum_{k=0}^{\infty}(\frac{k}{n})^2P_{n,k}(x)-2\frac{k}{n}xP_{n,k}(x)+x^2P_{n,k}(x)\right)^{\frac{1}{2}}\right] \\
&= \omega(f;\delta_n)\left[1+\frac{1}{\delta_n}\left((K_n(t^2;x)-2xK_n(t;x)+x^2K_n(1;x)\right)^{\frac{1}{2}}\right] \\
&= \omega(f;\delta_n)\left[1+\frac{1}{\delta_n}\left(\frac{x+x^2}{n}\right)^{\frac{1}{2}}\right]
\end{aligned}$$

we get,

$$|K_n(f;x)-f(x)| = \omega(f;\delta_n)\left[1+\frac{1}{\delta_n}\left(\frac{x+x^2}{n}\right)^{\frac{1}{2}}\right].$$

Choosing $\delta_n = \frac{1}{\sqrt{n}}$

$$\|K_n(f;x) - f(x)\|_{C[0,A]} \leq \left(1 + \sqrt{A(A+1)}\right)\omega\left(f;\frac{1}{\sqrt{n}}\right).$$

Theorem 4.21. *If* $f \in Lip_M(\alpha)$ *where* $0 < \alpha \leq 1$*, then*

$$\|K_n(f;x) - f(x)\|_{C[0,A]} = O\left(\left(\frac{A^2 + A}{n}\right)^{\frac{1}{2}}\right).$$

Proof. Firstly,we can write that

$$|K_n(f;x) - f(x)| = \left|\sum_{k=0}^{\infty} f\left(\frac{k}{n}\right) P_{n,k}(x) - f(x)\right|$$

and we know that

$$\sum_{k=0}^{\infty} \binom{n+k-1}{k} \left(\frac{x}{1+x}\right)^k \frac{1}{(1+x)^n} = 1$$

then,

$$\begin{aligned} |K_n(f;x) - f(x)| &= \left|\sum_{k=0}^{\infty} f\left(\frac{k}{n}\right) P_{n,k}(x) - f(x)\sum_{k=0}^{\infty} P_{n,k}(x)\right| \\ &= \sum_{k=0}^{\infty} \left|f\left(\frac{k}{n}\right) - f(x)\right| P_{n,k}(x). \end{aligned}$$

By the Lipschitz Condition,then

$$|K_n(f;x) - f(x)| \leq \sum_{k=0}^{\infty} M \left|\frac{k}{n} - x\right|^{\alpha} P_{n,k}(x).$$

Now,we will use the Hölder Inequality

$$|S_n(f;x) - f(x)| = \sum_{k=0}^{\infty} M \left|\frac{k}{n} - x\right|^{\alpha} (P_{n,k}(x))^{\frac{\alpha}{2}} (P_{n,k}(x))^{\frac{2-\alpha}{2}}$$

thus,

$$\begin{aligned}
\left|B_n(f;x)-f(x)\right| &\leq M\left(\sum_{k=0}^{\infty}\left|\frac{k}{n}-x\right|^2 P_{n,k}(x)\right)^{\frac{\alpha}{2}}\left(\sum_{k=0}^{\infty}P_{n,k}(x)\right)^{\frac{2-\alpha}{2}} \\
&= M\left(\sum_{k=0}^{\infty}\left(\left(\frac{k}{n}\right)^2 P_{n,k}(x)-2x\frac{k}{n}P_{n,k}(x)+x^2P_{n,k}(x)\right)\right)^{\frac{\alpha}{2}} \\
&= M(K_n\left(t^2;x\right)-2xK_n(t;x)+x^2K_n(1;x))^{\frac{\alpha}{2}} \\
&= M\left(\frac{n+1}{n}x^2+\frac{x}{n}-2x^2+x^2\right)^{\frac{\alpha}{2}} \\
&= M\left(\frac{x^2}{n}+\frac{x}{n}\right)^{\frac{\alpha}{2}}
\end{aligned}$$

where $p=\frac{2}{\alpha}$,$q=\frac{2}{2-\alpha}$ and $\frac{1}{p}+\frac{1}{q}=1$,we get,

$$\left|K_n(f;x)-f(x)\right| \leq M\left(\frac{x^2+x}{n}\right)^{\frac{\alpha}{2}}.$$

Taking its maximum,

$$\max_{x\in[0,A]}\left|K_n(f;x)-f(x)\right| \leq M\left(\frac{A^2+A}{n}\right)^{\frac{\alpha}{2}}.$$

Thus

$$\left\|S_n(f;x)-f(x)\right\|_{C[0,A]}=O\left(\left(\frac{A^2+A}{n}\right)^{\frac{1}{2}}\right).$$

4.4 Gauss-Weierstrass Operator and Its Approximation Properties

Definition 4.4. For a function $f(x)$ defined on $C(-\infty,+\infty)$ and $x \in (-\infty,+\infty)$ the expression

$$W_n(x) = (W_n f)(x) = \frac{n}{\sqrt{\pi}} \int_{-\infty}^{\infty} f(t)\, e^{-n^2(t-x)^2} dt$$

is called the Gauss-Weierstrass Operator.

Theorem 4.22. *Gauss-Weierstrass operator is linear and positive operator.*

Proof. First we will show W_n is linear.

$$\begin{aligned}
(W_n(\alpha f + \beta g))(x) &= \frac{n}{\sqrt{\pi}} \int_{-\infty}^{\infty} (\alpha f + \beta g)(x)\, e^{-n^2(t-x)^2} dt \\
&= \frac{n}{\sqrt{\pi}} \int_{-\infty}^{\infty} (\alpha f)(x)\, e^{-n^2(t-x)^2} dt + \int_{-\infty}^{\infty} (\beta g)(x)\, e^{-n^2(t-x)^2} dt \\
&= \alpha (W_n f)(x) + \beta (W_n g)(x).
\end{aligned}$$

Since $x \in (-\infty,\infty)$, and if $f(x) \geq 0$, then

$$(W_n f)(x) = \frac{n}{\sqrt{\pi}} \int_{-\infty}^{\infty} f(t)\, e^{-n^2(t-x)^2} dt \geq 0.$$

So W_n is positive.

Some Properties

1. We see that $\omega_n(z) > 0$ $\forall n \in \mathbb{N}$ and $\forall z \in \mathbb{R}$ where $\omega_n(z)$ of kernel W_n.

2.$\omega_n(z)$ is an even function as a function of z.

3. Recall that $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$ (Gauss Integral)

4.Now, we will investigate the limit form of $\omega_n(z)$ as $n \to \infty$.

$$\lim_{n\to+\infty} \omega_n(z) = \lim_{n\to+\infty} \frac{n}{\sqrt{\pi}} e^{-n^2 z^2} \text{ whenever } z \neq 0.$$

i) $\lim\limits_{n\to+\infty} \frac{n}{\sqrt{\pi}} e^{-n^2 z^2} = 0.$

ii) $\lim\limits_{n\to+\infty} \frac{n}{\sqrt{\pi}}.1 = +\infty$ whenever $z = 0.$

iii) $\lim\limits_{n\to+\infty} \omega_n(z) = 0$, $\lim\limits_{n\to+\infty} \omega_n(0) = +\infty$ $(z \neq 0)$

iv) $\int\limits_{-\infty}^{\infty} \omega_n(z)\, dz = 1.$

5. We set any $\delta > 0$ and $z \geq \delta$. Then $\omega_n(z)$ takes its supremum on $[\delta, +\infty)$ and the point δ. That is,

$$\omega_n(\delta) = \frac{n}{\sqrt{\pi}} e^{-n^2 \delta^2}$$

is the supremum on $(-\infty, -\delta] \cup [\delta, +\infty)$.

Hence

$$\lim_{n\to+\infty} \omega_n(\delta) = 0$$

In other words

$$\lim_{n\to+\infty} \left(\sup_{|z|\geq\delta} \omega_n(z) \right) = 0$$

Another representation of (5) is given by

$$\begin{aligned}\int_{\delta}^{+\infty} \omega_n(x)\,dx &= \frac{n}{\sqrt{\pi}}\int_{\delta}^{+\infty} e^{-n^2x^2}dx \\ &= \frac{n}{\sqrt{\pi}}\int_{\delta}^{+\infty} e^{-t^2}\frac{dt}{n} \\ &= \frac{1}{\sqrt{\pi}}\int_{n\delta}^{+\infty} e^{-t^2}dt\end{aligned}$$

So we have

$$\lim_{n\to+\infty}\int_{\delta}^{+\infty} \omega_n(x)\,dx = 0.$$

Theorem 4.23. *For a function $f(x) \in C_0$, the relation*

$$\lim_{n\to\infty} W_n(x) = f(x)$$

holds at each point of continuity x of f.

Proof.

$$\begin{aligned}|W_n(f;x) - f(x)| &= \left|\frac{n}{\sqrt{\pi}}\int_{-\infty}^{+\infty} f(t)\,W_n(t-x)\,dt - f(x)\,1\right| \\ &= \left|\frac{n}{\sqrt{\pi}}\int_{-\infty}^{+\infty} f(t)\,W_n(t-x)\,dt - f(x)\,\frac{n}{\sqrt{\pi}}\int_{-\infty}^{+\infty} W_n(t-x)\,dt\right| \\ &= \left|\frac{n}{\sqrt{\pi}}\int_{-\infty}^{+\infty} [f(t) - f(x)]\,W_n(t-x)\,dt\right| \\ &\le \frac{n}{\sqrt{\pi}}\int_{-\infty}^{+\infty} |f(t) - f(x)|\,W_n(t-x)\,dt\end{aligned}$$

then,

$$
\begin{aligned}
|W_n(f;x) - f(x)| &= \frac{n}{\sqrt{\pi}} \left(\int_{-\infty}^{x-\delta} |f(t) - f(x)| \, W_n(t-x) \, dt \right. \\
&\quad + \int_{x-\delta}^{x+\delta} |f(t) - f(x)| \, W_n(t-x) \, dt \\
&\quad \left. + \int_{x+\delta}^{+\infty} |f(t) - f(x)| \, W_n(t-x) \, dt \right).
\end{aligned}
$$

Divide into three parts;

$$I_1 = \int_{-\infty}^{x-\delta} |f(t) - f(x)| \, W_n(t-x) \, dt$$

$$I_2 = \int_{x-\delta}^{x+\delta} |f(t) - f(x)| \, W_n(t-x) \, dt$$

$$I_3 = \int_{x+\delta}^{+\infty} |f(t) - f(x)| \, W_n(t-x) \, dt$$

then we can write,

$$
\begin{aligned}
I_2 &= \frac{n}{\sqrt{\pi}} \int_{x-\delta}^{x+\delta} |f(t) - f(x)| \, W_n(t-x) \, dt \\
&< \epsilon \frac{n}{\sqrt{\pi}} \int_{x-\delta}^{x+\delta} W_n(t-x) \, dt \\
&< \epsilon \frac{n}{\sqrt{\pi}} \int_{-\infty}^{+\infty} W_n(t-x) \, dt \\
&< \epsilon
\end{aligned}
$$

and

$$
\begin{aligned}
I_1 &= \int_{-\infty}^{x-\delta} |f(t) - f(x)| \, W_n(t-x) \, dt \\
&\leq W_n(-\delta) \int_{-\infty}^{x-\delta} |f(t) - f(x)| \, dt \\
&\leq W_n(-\delta) \, 2M.
\end{aligned}
$$

Note that

$$
\int_{-\infty}^{x-\delta} |f(t) - f(x)| \, dt \leq \int_{-\infty}^{+\infty} |f(t) - f(x)| \, dt \leq \int_{-\infty}^{+\infty} |f(t)| \, dt + \int_{-\infty}^{+\infty} |f(x)| \, dx \leq 2M.
$$

In conclusion

$$
\begin{aligned}
|W_n(f; x) - f(x)| &\leq 2M.W_n(-\delta) + \epsilon + 2M.W_n(\delta) \\
&< \widetilde{\epsilon} \left(\forall \widetilde{\epsilon} > 0 \right).
\end{aligned}
$$

4.5 Post-Widder Operator and Its Approximation Properties

Definition 4.5. For a function $f(x) : C_0(0,+\infty) \to C_0(0,+\infty)$ and $x \in (0,+\infty)$ the expression

$$P_n(x) = (P_n f)(x) = \int_0^\infty \frac{1}{(n-1)!}\left(\frac{n}{x}\right)^n e^{-\frac{nt}{x}} t^{n-1} f(t)\,dt$$

is called the Post-Widder Operator.

Theorem 4.24. *Post-Widder operator is linear and positive operator.*

Proof. First we will show P_n is linear.

$$\begin{aligned}
(P_n(\alpha f + \beta g))(x) &= \frac{n}{\sqrt{\pi}}\int_0^\infty (\alpha f + \beta g)(t)\frac{1}{(n-1)!}\left(\frac{n}{x}\right)^n e^{-\frac{nt}{x}} t^{n-1} dt \\
&= \frac{n}{\sqrt{\pi}}\int_0^\infty (\alpha f)(t)\, p_n(x,t)\,dt + \int_0^\infty (\beta g)(t)\, p_n(x,t)\,dt \\
&= \alpha(P_n f)(x) + \beta(P_n g)(x).
\end{aligned}$$

Since $x \in (0,\infty)$, and if $f(x) \geq 0$, then

$$(P_n f)(x) = \int_0^\infty f(t)\frac{1}{(n-1)!}\left(\frac{n}{x}\right)^n e^{-\frac{nt}{x}} t^{n-1} dt \geq 0.$$

So P_n is positive.

Theorem 4.25. *Let $P_n : C_0(0,+\infty) \to C_0(0,+\infty)$. Then Post-Widder operator satisfies the following equations and $\varphi_x(t) := t - x$ for $t \in [0,\infty)$.*

i) $P_n(\varphi_x(t); x) = 0$

ii) $P_n(\varphi_x^2(t); x) = \frac{x^2}{n}$

iii) $P_n\left(\varphi_x^3\left(t\right);x\right)=\frac{2x^3}{n^2}$

iv) $P_n\left(\varphi_x^4\left(t\right);x\right)=\frac{3(n+2)}{n^3}x^4$ for all $x\in(0,+\infty)$ and $n\in\mathbb{N}$.

Theorem 4.26. *If f is a continuous and $\omega\left(\delta\right)$ is the modulus of continuity of $f\left(x\right)$,then*

$$\left|P_n\left(f;x\right)-f\left(x\right)\right|<3\omega\left(f;\frac{x}{\sqrt{n}}\right).$$

Proof. For arbitrary t,$x\in(0,+\infty)$ and $\delta>0$,

Firstly,we can write that

$$\begin{aligned}\left|P_n\left(f;x\right)-f\left(x\right)\right| &= \left|\int_0^\infty f\left(t\right)p_n\left(x,t\right)dt-f\left(x\right)\right| \\ &= \left|\int_0^\infty f\left(t\right)p_n\left(x,t\right)dt-\int_0^\infty f\left(x\right)P_{n,k}(x)dt\right| \\ &\leq \int_0^\infty \left|f\left(t\right)-f\left(x\right)\right|p_n\left(x,t\right)dt.\end{aligned}$$

Using by modulus of continuity,

$$\left|P_n\left(f;x\right)-f\left(x\right)\right|\leq\int_0^\infty\left(1+\frac{\left|\frac{k}{n}-x\right|}{\delta_n}\right)\omega\left(f;\delta_n\right)p_n\left(x,t\right)dt$$

then,

$$
\begin{aligned}
|P_n(f;x) - f(x)| &= \omega(f;\delta_n)\left[\int_0^\infty p_n(x,t)\,dt + \frac{1}{\delta_n}\int_0^\infty \left|\frac{k}{n} - x\right| p_n(x,t)\,dt\right] \\
&= \omega(f;\delta_n)\left[\int_0^\infty p_n(x,t)\,dt + \frac{1}{\delta_n}\int_0^\infty \left|\frac{k}{n} - x\right| p_n^{\frac{1}{2}}(x,t) p_n^{\frac{1}{2}}(x,t)dt\right] \\
&\leq \omega(f;\delta_n)\left[1 + \frac{1}{\delta_n}\left(\int_0^\infty \left|\frac{k}{n} - x\right|^2 p_n(x,t)dt\right)^{\frac{1}{2}}\left(\int_0^\infty p_n(x,t)dt\right)^{\frac{1}{2}}\right] \\
&= \omega(f;\delta_n)\left[1 + \frac{1}{\delta_n}\left(\int_0^\infty \left|\frac{k}{n} - x\right|^2 p_n(x,t)dt\right)^{\frac{1}{2}}\right] \\
&= \omega(f;\delta_n)\left[1 + \frac{1}{\delta_n}\left(\frac{x^2}{n}\right)^{\frac{1}{2}}\right] \\
&= \omega(f;\delta_n)\left[1 + \frac{1}{\delta_n}\left(\frac{x^2}{n}\right)^{\frac{1}{2}}\right]
\end{aligned}
$$

we get,

$$|P_n(f;x) - f(x)| = \omega(f;\delta_n)\left[1 + \frac{1}{\delta_n}\left(\frac{x^2}{n}\right)^{\frac{1}{2}}\right].$$

Choosing $\delta_n = \frac{2x}{\sqrt{n}}$

$$|P_n(f;x) - f(x)| < 3\omega\left(f;\frac{x}{\sqrt{n}}\right).$$

Theorem 4.27. *(Voronovskaya) Let $f(x)$ be bounded in $(0,\infty)$ and suppose that the second derivative $f''(x)$ exists at a certain x of $(0,\infty)$,then*

$$\lim_{n\to\infty} n\left[P_n(f;x) - f(x)\right] = \frac{x^2}{2} f''(x).$$

Proof. In particular, if $f''(x) \neq 0$, the difference $f(x) - P_n(x)$ is exactly of order n^{-1}. In order to prove this theorem, we write

$$f\left(\frac{k}{n}\right)=f\left(x\right)+\left(\frac{k}{n}-x\right)f^{'}\left(x\right)+\left(\frac{k}{n}-x\right)^{2}\left[\frac{1}{2}f^{''}\left(x\right)+\mu\left(\frac{k}{n}-x\right)\right]$$

where $\mu\left(h\right)$ is bounded, $\left|\mu\left(h\right)\right|\leq H$ for all h and converges to zero with h.Multiplying by $p_n\left(x,t\right)$ and summing,

$$\begin{aligned}P_n\left(x;t\right) &= \int_0^\infty f\left(t\right)p_n\left(x,t\right)dt\\ &= f\left(x\right)+\frac{1}{2}f^{''}\left(x\right)P_n\left(\varphi_x^2\left(t\right);x\right)+\sum_{k=0}^{n}\left(\frac{k}{n}-x\right)^{2}\mu\left(\frac{k}{n}-x\right)p_n\left(x;t\right).\end{aligned}$$

The last term on the right can easily be estimated. Let $\epsilon>0$ be arbitrary and $\delta>0$ such that $\left|h\right|<\delta$ implies $\left|\mu\left(h\right)\right|<\epsilon$

$$\begin{aligned}&\epsilon\sum_{\left|\frac{k}{n}-x\right|\leq\delta}\left(\frac{k}{n}-x\right)^{2}p_n\left(x;t\right)+H\sum_{\left|\frac{k}{n}-x\right|>\delta}p_n\left(x;t\right)\\ \leq\ &\epsilon P_n\left(\varphi_x^2\left(t\right);x\right)+AH.\end{aligned}$$

Since $P_n\left(\varphi_x^2\left(t\right);x\right)=\frac{x^2}{n}$. Thus

$$P_n\left(x\right)=f\left(x\right)+\frac{x^2}{2n}f^{''}\left(x\right)+\frac{\epsilon_n}{n},\quad \epsilon_n\rightarrow 0.$$

for $n\rightarrow\infty$.

Then we obtain

$$\begin{aligned}P_n(f;x) &= f(x)+\frac{x^2}{2n}f''(x)\\ P_n(f;x)-f(x) &= \frac{x^2}{2n}f''(x).\end{aligned}$$

Thus,

$$\lim_{n\to\infty} n\left[P_n(f;x) - f(x)\right] = \frac{x^2}{2} f''(x).$$

where $n \to \infty$ and $\epsilon_n \to 0$.

REFERENCES

[1] ALTOMORE, F. and CAMPITI, M., Korovkin-type Approximation Theory ant its Applications, Walter de Gruyter Berlin, New York, 141-180 (1994).

[2] ANDREWS, G. E., ASKEY, R. and ROY, R., Special Function, Cambridge University Pres, 481-485 (1999).

[3] BERNSTEIN, S., Demenstration du theoreme de Weierstrass, fondee sur le calcul des probabilities, Commun. Soc. Math. Kharkow (2), 13 (1912-13), 1-2.

[4] BLEIMAN, G., BUTZER, P. L. and HAHN, L., A Bernstein-type operator approximating continuous functions on the semi axis, Proc. Netherl. Acad. Sci. A83, Indag. Math., 42;255-262, (1980).

[5] BRYC, W. and Ismail, M. E. H., Approximation Operators, Exponential, $q-$Exponential, and Free Exponential Families, arXiv:math. ST/0512224 v1, December (2005).

[6] BUTZER, P. L. and NESSEL, R. J., Fourier Analysis and Approximation, V. 1, Acedemic Press, New York and London, (1971).

[7] DOGRU, O., Weighted Approximation Properties of Szász-type Operators, Intern. Math. Journal, 2 (9);889-895, (2002).

[8] GADJIEV, A. D., On Korovkin type theorems, Math. Zametki, 20, 781-786 (in Russian), Math. Notes (in English), 20, No.5-6;996-998 (1976).

[9] KOROVKIN, P. P., Linear Operators and Approximation Theory, Hindustan Publishing Corp. (India), Delhi, 20-110 (1960).

[10] KOROVKIN, P. P., On Convergence of Linear Positive Operators in The Space of Continuous Functions (in Russian), Dokl. Akad. Nauk SSSR, 90;961-964 (1953).

[11] KREYSZIG, E., Introductory Functional Analysis with Applications, John WILEY and SONS, New York, Santa Barbara, London, Sydney and Toronto, (1978).

[12] LORENTZ, G.G., Bernstein Polynomials, University of Toronto Press, Toronto, (1953).

[13] MAMEDOV, R. G., An Estimate in Terms of the Modulus of Continuity for Approximation of Functions by Positive Linear Operators (Azerbaijani), Izv. Akad. Nauk Azerb. SSR. Ser. Fiz-Mat. Teh. Nauk, 4;3-11 (1961).

[14] MAMEDOV, R. G., On the Order of the Approximation of Functions by Linear Operators, Dokl. AN SSSR, (in Russian), 18(4);674-676 (1959).

[15] MAY, C. P., Saturation and Inverse Theorems For Combinations of a Class of Exponential-Type Operators, Can. J. Math., Vol. XXVIII, No. 6,1224-1250, (1976).

[16] MIHESAN, V., On the Gamma and Beta Approximation Operators, General Mathematics Vol. 6, Sibiu-Romania, 1;06-3;06 (1998).

[17] NATANSON, I. P., Theory of Functions of a Real Variable, Volume I, Translated from the Russian by Leo F. Boron, Frederick Ungar Pub. Co. New York, (1955).

[18] NATANSON, I. P., Theory of Functions of a Real Variable, Volume II, Translated from the Russian by Leo F. Boron, Frederick Ungar Pub. Co. New York, (1960).

[19] POPOVICIU, T., Sur l'approximation des functions convexes d'ordre superieur, Mathematica (cluj), 10; 49-54, (1935).G.G. Lorentz, Bernstein Polynomials, University of Toronto Toronto,1953.

[20] POP, O. T., The Generalization of Voronovskaja's Theorem for Exponential Operators, Creative Math. & Inf. 16;54-62, (2007).

[21] RATHORE, R. K. S. and AGRAWAL, P. N., Inverse and Saturation Theorems for Derivatives of Exponential Type Operators, Indian J. pure appl. Math., 13(3);476-490, April (1982).

[22] SONG, Z., LIU, X. and GUO, S., Pointwise Estimate for Exponential Type Operators, Southeast Asian Bulletin of Mathematics, 24;463-473, (2000).

[23] TYLIBA, A. and WACHNICKI, E., On Some Class of Exponential Type Operators, Annales Societatis Mathematicae Polonae , XLV (1), 57-70, (2005).

[24] ZENG, X. M. and CHENG, F., Sharp Estimates on Approximation of Exponential Type Operators, International Journal of Applied Mathematical Sciences, ISSN 0973-0176 Vol.No.1, 23-30, (2005).