

T.R.
GEBZE INSTITUTE of TECHNOLOGY
GRADUATE SCHOOL of ENGINEERING and SCIENCES

STABILITY of BIMODAL SYSTEMS in \mathbb{R}^3

İŞİL ÖNER
A THESIS SUBMITTED for THE DEGREE of
DOCTOR of PHILOSOPHY
DEPARTMENT of MATHEMATICS

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PROF. DR. VASFİ ELDEM

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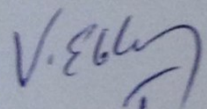
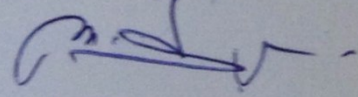
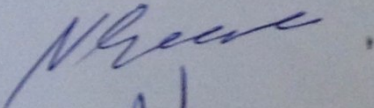
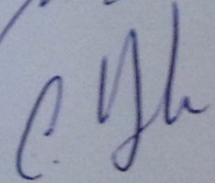
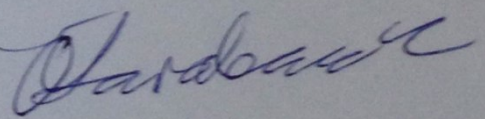


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SUMMARY

In this dissertation, structure and stability of a class of bimodal systems in \mathbb{R}^3 and in \mathbb{R}^2 are investigated. The approach taken employs the structure induced by the eigenvectors of subsystem matrices. In this framework, vector field is allowed to be discontinuous on the switching plane. Furthermore, the trajectories of bimodal system are classified as follows. First, the trajectories starting in one of the modes are classified as trajectories which change mode within a finite time interval, and trajectories which never change mode. Then, this classification is further used for a second classification i) trajectories which change mode only finite number of times as $t \rightarrow \infty$. ii) trajectories which change mode infinite number of times as $t \rightarrow \infty$. It is shown that the trajectories in class i) decay to the origin if and only if all the real eigenvalues of both modes are negative (stable). Then, it is proven that the trajectories in class ii) converge to a certain subset of trajectories which are called trajectories starting from fixed directions. Furthermore, the trajectories in class ii) decay to the origin if and only if the trajectories starting from fixed directions decay to the origin.

Keywords: Bimodal systems, Asymptotic stability, Fix directions, Piecewise linear systems, Discontinuous vector field, Coupling condition.

ÖZET

Bu tezde \mathbb{R}^2 ve \mathbb{R}^3 'te, iki modlu sistemlerin yapısı ve kararlılık problemi araştırılmıştır. Kullanılan yaklaşım alt sistem özvektörlerinin oluşturduğu yapıyı kullanmaktadır. Bu yapıda vektör alanı anahtarlama düzleminde süreksiz olabilmektedir. Ayrıca iki modlu sistemlerin çözümleri aşağıdaki şekilde sınıflandırılmıştır. İlk olarak modların birinden başlayan çözümler, sonlu bir zaman aralığında mod değiştiren çözümler ve hiç mod değiştirmeyen çözümler olarak sınıflandırılmıştır. Sonra, bu sınıflandırma ikinci bir sınıflandırma için aşağıda anlatıldığı gibi kullanılmıştır: i) $t \rightarrow \infty$ iken sonlu kere mod değiştiren çözümler, ii) $t \rightarrow \infty$ iken sonsuz kere mod değiştiren çözümler. i)'deki çözümlerin azalarak sifıra gitmesi için gerekli ve yeterli koşul her iki moda ait reel özdeğerlerin negatif (kararlı) olmasıdır. Sonra ii)'deki çözümlerin sabit doğrultu olarak isimlendirilen başlangıç noktalarından başlayan çözümlere yakınsadığı ispat edilmiştir. Ayrıca, ii) deki çözümlerin azalarak orijine gitmesi için gerekli ve yeterli koşulun sabit doğrultudan başlayan çözümlerin azalarak orijine gitmesi olduğu gösterilmiştir.

Anahtar Kelimeler: İki modlu sistemler, Asimtotik kararlılık, Sabit doğrultular, Parçalı doğrusal sistemler, Süreksiz vektör alanı, Bağlama sabiti.

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LIST of ABBREVIATIONS and ACRONYMS

<u>Abbreviations</u>	<u>Explanations</u>
<u>and Acronyms</u>	
γ^*	: Convergence Rate
\mathbb{R}^n	: n-Dimensional Real Vector Space
$\ker \mathbf{M}$: Null Space of \mathbf{M}
\mathcal{H}	: Plane (in \mathbb{R}^3) or Line (in \mathbb{R}^2) where $\mathbf{c}^T \mathbf{x} = 0$
\mathcal{H}^+	: Open half spaces where $\mathbf{c}^T \mathbf{x} > 0$
\mathcal{H}^-	: Open half spaces where $\mathbf{c}^T \mathbf{x} < 0$
\mathcal{L}_i	: Line passing through to origin
\mathcal{P}_i^+	: Open half plane where $\mathbf{c}^T \mathbf{A}_i \mathbf{x} > 0$
\mathcal{P}_i^-	: Open half plane where $\mathbf{c}^T \mathbf{A}_i \mathbf{x} < 0$
\mathcal{L}_i^+	: Open Half Line where $\mathbf{c}^T \mathbf{A}_i^2 \mathbf{x} > 0$
\mathcal{L}_i^-	: Open Half Line where $\mathbf{c}^T \mathbf{A}_i^2 \mathbf{x} < 0$
\mathcal{S}_i	: Set of initial conditions x_0 where smooth continuation is possible only <i>i</i> th mode
BPLS	: Bimodal Piecewise Linear System
GAS	: Global Asymptotically Stable
PLS	: Piecewise Linear System

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1. INTRODUCTION

Switched systems consist of a finite number of subsystems and a piecewise constant switching signal, which coordinates the transition of trajectories from one subsystem to the other. Since such systems are used for modelling in a rich variety of problems in engineering and physics, they have attracted significant attention during the last two decades. Stability and stabilizability are the basic problems for switched systems. The stability of switched systems with arbitrary switching is investigated by many authors in the literature. A quick summary of these works is given in what follows.

One of the approaches used for the solution of stability problems with arbitrary switching is based on the existence of a common quadratic Lyapunov function for individual subsystems. Along this line, the works of [Mori et al., 1997], [Shorten and Narendra, 1999], [Shorten et al., 2004], [Sun, 2010] can be cited. If a switched system has more than two subsystems, then finding a common quadratic Lyapunov function becomes a difficult task. In such cases, another approach, which is based on non-traditional Lyapunov functions (which are piecewise continuous and piecewise differentiable and formed by concatenating Lyapunov-like functions), is employed. For the details of this approach the works of [Branicky, 1998], [Michel and Hu, 1999], [Liberzon and Morse, 1999] and the survey paper by [Decarlo et al., 2000] may be referred to. The approaches which are based on Lyapunov functions usually yield sufficient conditions for stability. An alternative approach, which is based on the problem of absolute stability, has also become popular in the literature. Since this problem is closely related to the stability of switched systems with arbitrary switching, it has attracted the attention of many researchers. For instance, [Molchanov and Pyatnitskiy, 1989] show that, for a class of linear differential inclusions, the necessary and sufficient conditions for asymptotic stability of the zero solution can be established by using piecewise quadratic Lyapunov functions. The basic idea for solving absolute stability problems is to find the most unstable solution and this yields the necessary and sufficient conditions for stability with arbitrary switching. These conditions obtained only for low order systems. [Pyatnitskiy and Rapoport, 1996], [Rapoport, 1996] formulated the problem of finding the most unstable solution using

a variational approach and they obtained a necessary and sufficient condition for the stability of second order linear switched systems. Since this condition involves the solution of a nonlinear equation with several unknowns, it is not easy to use in practice. Later, the works of [Margaliot and Langholz, 2003], [Holcman and Margaliot, 2003], [Margaliot and Gitizadeh, 2004] provided an easily verifiable necessary and sufficient condition for the stability of second order linear switched systems. Closely related results for the same problem are given by [Boscain, 2002], [Balde et al., 2009]. A more general problem of absolute stability is to determine the Lyapunov exponent of a finite dimensional linear inclusion, which is proposed by [Barabanov, 1988]. In this setup [Barabanov, 1993] provides the necessary and sufficient condition for the stability of third order linear differential inclusions. Along this line, [Margaliot, 2006] gives a simple and efficient numerical scheme for the calculation of the necessary and sufficient condition for the stability of third order linear switched systems with arbitrary switching. In a later work [Barabanov, 2008] investigates the asymptotic behavior of extreme solutions (basically most unstable solutions) and the structure of extreme norms. More detailed overviews of the research on switched systems are given in the books by [Liberzon, 2003], [Sun, 2005], [Sun and Ge, 2011] and also in the survey papers by [Shorten et al., 2007], [Lin and Antsaklis, 2009].

Piecewise linear systems (PLS) constitute a subclass of switched systems where the subsystems are linear and time-invariant and switching is autonomous (state dependent). Bimodal piecewise linear systems (BPLS) comprise a subclass of PLS, where there are only two subsystems. Stability with arbitrary switching is a strong demand for switched systems and conditions for stability are very strict. For instance, subsystems must have Hurwitzian system matrices. On the other hand, stability under autonomous switching is less restrictive and as a result, switched systems exhibit a rich dynamic behavior. This is well-known in the literature. For instance, both subsystems may have Hurwitzian system matrices, but BPLS may be unstable as shown in [Carmona et al., 2006]. Conversely, both subsystems may have non-Hurwitzian system matrices, but BPLS may be globally asymptotically stable (GAS)[Eldem and Şahan, 2009].

PLS are investigated extensively by many authors in the context of stability, stabilizability and controllability. Necessary and sufficient conditions for global

asymptotic stability (GAS) of PLS in \mathbb{R}^2 are given by [Xu and Antsaklis, 2000],[Iwatani and Hara, 2006]. Stabilization of bimodal systems with continuous vector fields is considered by [Camlibel et al., 2008]. The necessary and sufficient conditions for GAS of BPLS (with continuous vector fields) in \mathbb{R}^2 is given by [Camlibel et al., 2003]. The same problem (with discontinuous vector fields) is investigated by [Iwatani and Hara, 2006] and the importance of well-posedness is demonstrated by an example (Example 13 in [Iwatani and Hara, 2006]). Stability of BPLS have also been investigated via Lyapunov methods by [Mori et al., 1997], [Shorten and Narendra, 1999], [Shorten et al., 2004], [Sun, 2010]. Stability of BPLS in \mathbb{R}^3 have also attracted considerable attention in literature. [Carmona et al., 2005], [Carmona et al., 2006] have considered the stability of BPLS in \mathbb{R}^3 with continuous vector fields. In [Carmona et al., 2005], BPLS is transformed to the surface of the unit sphere in \mathbb{R}^3 centered at the origin. In this framework, the authors searched for periodic solutions which is equivalent to the search for invariant cones of the original bimodal system.

In this dissertation, the GAS of certain classes of BPLS in \mathbb{R}^n where $n = 2, 3$ are investigated under autonomous switching. The motivation behind this work is best explained by the following statement given in [Sun, 2010].

“Despite the achievements made so far, it is safe to say that the stability issues are still largely unexploited. In particular, the mechanism of autonomous switching is yet beyond our fully understanding, and verifiable necessary and sufficient stability conditions are still missing.”

Thus, the main focus of this dissertation is to study the structure and the mechanism of autonomous switching and its effects on the GAS of BPLS in \mathbb{R}^2 and \mathbb{R}^3 given as follows.

$$\Sigma_0 : \dot{x} = \begin{cases} A_1 x & \text{if } c^T x \geq 0 \\ A_2 x & \text{if } c^T x \leq 0 \end{cases} \quad (1.1)$$

where x, c are vectors and A_1 and A_2 are real matrices with appropriate dimensions. It is assumed that

- the pairs (c^T, A_1) and (c^T, A_2) are observable (see Remark 4.6 for a discussion on this assumption),

- (c^T, A_2) is in observable canonical form (this assumption does not cause any loss of generality since there is no structural assumption on A_1 , except that the pair (c^T, A_1) is observable),
- Σ_0 is well-posed (this assumption is explained in detail at the beginning of the next section).

In this framework, the vector field is not necessarily continuous on $\ker c^T$. Furthermore, this setting provides convenient tools to classify the trajectories of Σ_0 as explained in the following paragraph.

The dissertation is organized as follows. In Chapter 2, the issue of well-posedness is settled first and the assumptions used throughout the dissertation are stated. Then, in Chapter 3, BPLS in \mathbb{R}^2 are investigated. At the end of the Chapter 3, it is shown that the conjecture given in [Eldem and Şahan, 2014] is valid for the class of BPLS in \mathbb{R}^2 being considered. In the first section of Chapter 4 BPLS in \mathbb{R}^3 , where both modes have only real eigenvalues (with all possible multiplicities), are considered. It is basically shown that such BPLS are GAS if and only if all the eigenvalues of both modes are negative. In Section 4.2, BPLS, where one mode has only real eigenvalues and the other mode has a conjugate pair of complex eigenvalues, are investigated. In this set up, the trajectories of both subsystems are classified as i) the trajectories which change mode in finite time, ii) the trajectories which never change mode. Then, the dynamic behavior of the trajectories after they change mode are determined. This leads to another classification as i) the trajectories which change mode only finite number of times as $t \rightarrow \infty$, and ii) the trajectories which change mode infinite number of times as $t \rightarrow \infty$. Then, it is shown that the trajectories which change mode infinite number of times as $t \rightarrow \infty$, converge to the trajectories starting from certain directions on $\ker c^T$ (called fixed directions in the dissertation, see Definition 4.1). Furthermore, using this classification, the conditions under which a trajectory decays to the origin, are determined. In Section 4.3 possible geometric structures which reveal how the subsystems are coupled on the switching plane are considered. Then, it is shown that the bimodal system is GAS if and only if all the real eigenvalues of both modes are negative and the trajectories starting from fixed directions decay to origin as $t \rightarrow \infty$. Furthermore, it is proven that the trajectories starting from fixed directions decay to origin as $t \rightarrow \infty$ if and only if the constant γ^*

(called the convergence rate in the paper, see Definition 4.7) is strictly less than one. The outline of an algorithm which calculates the fixed directions and corresponding convergence rate γ^* is given in Remark 4.4. Finally, at the end of Section 4, the examples where the coupling constant is changed (without changing the eigenvalues of subsystems) are provided. These examples show that this change can make a BPLS in \mathbb{R}^3 unstable or GAS. Section 5 is on conclusions.

Since GAS under autonomous switching is investigated in this work, the most relevant papers to the results presented here are [Carmona et al., 2005], [Carmona et al., 2006], [Iwatani and Hara, 2006], [Eldem and Şahan, 2014]. A detailed comparison of the results given in these papers to the results given here takes place in Remark 4.6 in the dissertation.

The contribution of the dissertation can be summarized as follows.

- This dissertation provides, for the first time, a complete set of verifiable necessary and sufficient conditions for the GAS of a class of BPLS in \mathbb{R}^3 with discontinuous vector fields.
- It is shown that the discontinuity of the vector field on $\ker c^T$ plays a crucial role in GAS of BPLS. To this end, a constant, which is defined (see Definition 4.9) and referred to as the coupling constant in the dissertation, is used. It is demonstrated that the coupling constant can be changed (without changing the eigenvalues of subsystems) and this change can make BPLS unstable or GAS (see Remark 4.5). This unexpected dynamic behavior of BPLS in \mathbb{R}^3 is first pointed out and demonstrated in this dissertation. Furthermore, this behavior can not be observed in BPLS where the vector field is continuous on $\ker c^T$.
- The structure, induced by the assumption of well-posedness and by the switching plane, naturally yield a classification of the trajectories of Σ_0 , as explained in the previous paragraph. This structure sheds light into the mechanism of autonomous switching and provides a strong clue for the GAS of BPLS in general. The same classification is previously used by [Eldem and Şahan, 2014] (see Remark 4.6).
- Finally, it is shown that the following conjecture given by [Eldem and Şahan, 2014] holds for the class of bimodal systems considered in this dissertation.

Conjecture 1.1: Consider bimodal system given by the equation (1.1) where both modes are observable and systems is well posed. Then, the following hold.

- *If all the trajectories change mode only a finite number of times as $t \rightarrow \infty$, then bimodal system is GAS if and only if all the real eigenvalues of both modes are negative.*
- *If there are trajectories which change mode an infinite number of times as $t \rightarrow \infty$, then the bimodal system is GAS if and only if all real eigenvalues of both modes are negative, and all the trajectories starting from the fixed directions are stable.*

2. PRELIMINARY RESULTS AND DEFINITIONS

One of the main issues in PLS is well-posedness i.e., the existence and uniqueness of the solutions. This problem is addressed in detail by [Imura and van der Schaft, 2000] for BPLS in \mathbb{R}^n . Later, it is shown by [Eldem and Şahan, 2014] that well-posedness conditions given by [Imura and van der Schaft, 2000], induce a joint structure for subsystem matrices of BPLS in \mathbb{R}^3 . In this chapter the issue of well-posedness of Σ_0 given by equation (1.1) is resolved first. Then, the assumptions used at this stage are given. This is followed by definitions related to well-posedness. The necessary and sufficient conditions for well-posedness of bimodal systems in \mathbb{R}^3 and \mathbb{R}^2 are presented in the relevant sections of this chapter. These sections also include a detailed explanation of the structure and the geometry induced by well-posedness conditions in \mathbb{R}^3 and \mathbb{R}^2 . The assumptions used at this stage are given below.

Assumption 1: The pairs (c^T, A_1) and (c^T, A_2) are observable and only (c^T, A_2) is in observable canonical form.

Note that the second part of the above assumption does not cause any loss of generality as A_1 is completely free (except that (c^T, A_1) is observable). In view of the above assumption, the components of Σ_0 can be written as follows.

$$A_1 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 0 & k_1 \\ 1 & 0 & k_2 \\ 0 & 1 & k_3 \end{bmatrix} \text{ and } c = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad (2.1)$$

Commonly used solution structures for PLS are solutions in the sense of Filippov and Carathéodory. In this dissertation solutions in the sense of Carathéodory is considered as defined below. This definition is taken from [Camlibel, 2008].

Definition 2.1: An absolutely continuous function $x(t) : \mathbb{R} \rightarrow \mathbb{R}^n$ is said to be a solution of Σ_0 for the initial condition x_0 in the sense of

- *Carathéodory if $x(t_0) = x_0$ and $x(t)$ satisfies (1.1) for almost all $t \in \mathbb{R}$.*
- *forward Carathéodory for the initial state x_0 , if it is a solution in the sense of Carathéodory, and for each $t_0 \geq 0$, there exists $\epsilon > 0$ such that either $\dot{x} = A_1x$ and*

$c^T x \geq 0$ or $\dot{x} = A_2 x$ and $c^T x \leq 0$, hold for all $t \in [t_0, t_0 + \epsilon]$.

Well-posedness of Σ_0 is defined as follows.

Definition 2.2: The system Σ_0 is said to be well-posed if for every initial state $x(t_0) = x_0 \in \mathbb{R}^n$ and $t_0 \geq 0$, there exists a unique forward Carathéodory solution $x(t)$ of Σ_0 .

An alternative definition of well-posedness of Σ_0 can be given via the approach used in [Imura and van der Schaft, 2000] as follows.

Definition 2.3: Let \mathcal{S}_i ($i = 1, 2$) denote the set of initial conditions in \mathbb{R}^3 such that for every $x_0 \in \mathcal{S}_i$ there exists a unique forward Carathéodory solution $x(t)$ of Σ_0 and $\dot{x} = A_i x$ for all $t \in [t_0, t_0 + \epsilon]$. In this case we say that the solution of Σ_0 smoothly continues in \mathcal{S}_i .

Definition 2.4: The system Σ_0 is said to be well-posed if for every initial state $x(t_0) = x_0 \in \mathbb{R}^n$ smooth continuation is possible only in one of the modes and for $x_0 \in \mathcal{S}_1 \cap \mathcal{S}_2$ the solutions are the same for both modes.

The geometry induced by a bimodal system in \mathbb{R}^3 and well-posedness are investigated in detail in [Eldem and Şahan, 2014]. The following explanation is adopted from [Eldem and Şahan, 2014] for the sake of completeness.

2.1. Geometry of Initial Conditions in \mathbb{R}^3 and Well-Posedness

Note that the plane $\mathcal{H} := \{x \mid c^T x = 0\}$ divides \mathbb{R}^3 into two open half-spaces, \mathcal{H}^+ and \mathcal{H}^- . Then, for any initial condition $x \in \mathcal{H}^+$ (\mathcal{H}^-) where $c^T x > 0$ ($c^T x < 0$) only mode 1 (mode 2) is active, i.e., $\dot{x}(t) = A_1 x(t)$ ($\dot{x}(t) = A_2 x(t)$). In view of the theory of differential equations, for any initial condition $x \in \mathcal{H}^+$ (\mathcal{H}^-) where $c^T x > 0$ ($c^T x < 0$), there exists $\epsilon > 0$ and a local solution $x(t)$ such that $c^T x(t) > 0$ ($c^T x(t) < 0$) for all $t \in [0, \epsilon]$. More specifically, $x(t)$ smoothly continues into \mathcal{H}^+ (\mathcal{H}^-). However, for trajectories starting from \mathcal{H} , uniqueness of solutions can not be claimed, because both modes are allowed to be active on \mathcal{H} . Since both subsystems are observable, it follows that $\dim(\ker c^T \cap \ker(c^T A_i)) = 1$. Thus, $\mathcal{L}_i := \ker c^T \cap$

$\ker(c^T A_i)$ is a line passing through the origin which divides \mathcal{H} into two open half planes \mathcal{P}_i^+ and \mathcal{P}_i^- . On one side of this line $c^T A_i x > 0$ (\mathcal{P}_i^+) and on the other side $c^T A_i x < 0$ (\mathcal{P}_i^-). Similarly, the origin $\ker c^T \cap \ker(c^T A_i) \cap \ker(c^T A_i^2)$ divides \mathcal{L}_i into two open half lines \mathcal{L}_i^+ where $c^T A_i^2 x > 0$ and \mathcal{L}_i^- where $c^T A_i^2 x < 0$.

Lemma 2.1: [Lemma 1 in [Eldem and Şahan, 2014]] Let Σ_0 be as given by equation (2.1). Then, Σ_0 is well-posed if and only if $\ker c^T \cap \ker(c^T A_1) = \ker c^T \cap \ker(c^T A_2)$ (equivalently $a_{31} = 0$) and $a_{32}, a_{21} > 0$.

In view of the above result it follows that $\mathcal{P}_1^+ = \mathcal{P}_2^+$, $\mathcal{P}_1^- = \mathcal{P}_2^-$, $\mathcal{L}_1^+ = \mathcal{L}_2^+$, and $\mathcal{L}_1^- = \mathcal{L}_2^-$. Consequently, $\mathcal{S}_1 = \mathcal{H}^+ \cup \mathcal{P}_1^+ \cup \mathcal{L}_1^+$ and $\mathcal{S}_2 = \mathcal{H}^- \cup \mathcal{P}_2^- \cup \mathcal{L}_2^-$. This implies that $\mathcal{S}_1 \cup \mathcal{S}_2 = \mathbb{R}^3$ and $\mathcal{S}_1 \cap \mathcal{S}_2 = \{0\}$ and this guarantees well-posedness as defined by [Imura and van der Schaft, 2000].

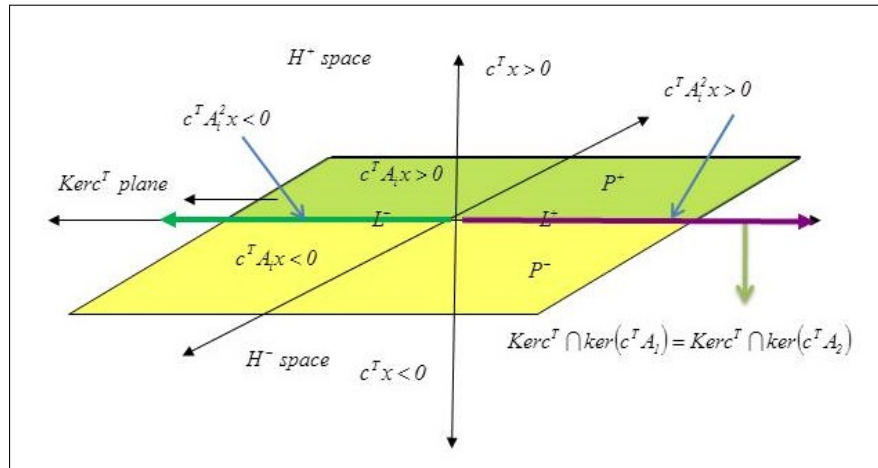


Figure 2.1: Well-posed bimodal system in \mathbb{R}^3

In view of this observation the following definition is in order.

Definition 2.5: \mathcal{P}^+ (\mathcal{P}^-) denotes the half-plane in \mathcal{H} , where the set of initial conditions x_0 satisfy $c^T A_i x_0 > 0$ ($c^T A_i x_0 < 0$) for both modes. \mathcal{L}^+ (\mathcal{L}^-) denotes the half-line in \mathcal{H} , where the set of initial conditions x_0 satisfy $c^T A_i x_0 = 0$ and $c^T A_i^2 x_0 > 0$ ($c^T A_i^2 x_0 < 0$) for both modes.

Assumption 2: $\ker c^T \cap \ker(c^T A_1) = \ker c^T \cap \ker(c^T A_2)$ (equivalently $a_{31} = 0$) and $a_{21} > 0$, $a_{32} > 0$, i.e. Σ_0 is well posed.

Remark 2.1: Note that the assumption on observability could be relaxed. In this case, due to the well-posedness assumption, it is necessary that both modes share the same unobservable subspace and $A_1x = A_2x$ on this subspace (see Theorem 4.2 in [Imura and van der Schaft, 2000]). Furthermore, since the trajectories which dwell in the unobservable subspace do not change mode, detectability of both modes is both necessary and sufficient for these trajectories to decay to the origin. Thus, in our set up, the unobservable case is not interesting, because it does not add anything to the main result.

2.2. Geometry of Initial Conditions in \mathbb{R}^2 and Well-Posedness

This section provides well-posedness conditions for bimodal planar systems. It should be remarked here that planar systems are investigated extensively in the context of stability, stabilizability and controllability, in the literature. Along this line, the works of [Xu and Antsaklis, 2000],[Boscain, 2002],[Iwatani and Hara, 2006],[Arapostathis and Broucke, 2007], [Balde et al., 2009], [Liu et.al, 2012], [Polderman and Langerak, 2012] can be referred to. This section is included in the dissertation both for the sake of completeness and for motivating the approach used in the dissertation.

$\mathcal{H} := \{x \mid c^T x = 0\}$ is a line which divides \mathbb{R}^2 into two open half planes, \mathcal{P}_i^+ ($c^T x > 0$) and \mathcal{P}_i^- ($c^T x < 0$). If the pair (c^T, A_i) is observable, then the point $\ker c^T \cap \ker c^T A_1 = \{0\}$ divides \mathcal{H} into two half lines \mathcal{L}_i^+ ($c^T A_i x > 0$) and \mathcal{L}_i^- ($c^T A_i x < 0$). Then, for any initial condition $x \in \mathcal{P}_i^+$ (\mathcal{P}_i^-) where $c^T x > 0$ ($c^T x < 0$) only mode 1 (mode 2) is active, i.e., $\dot{x}(t) = A_1 x(t)$ ($\dot{x}(t) = A_2 x(t)$).

Lemma 2.2: Suppose that (c^T, A_2) is in observable canonical form and A_1, A_2 and c are as given below.

$$A_1 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, A_2 = \begin{bmatrix} 0 & -\mu_1 \mu_2 \\ 1 & \mu_1 + \mu_2 \end{bmatrix}, c = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \quad (2.2)$$

Then, the bimodal system given by the equations above is well-posed if and only if $a_{21} > 0$.

Proof 2.2: Suppose that $a_{21} > 0$ and $x \in \ker c^T$. Then $x = [a_1 \ 0]^T$ for some real number a_1 . If $c^T A_2 x > 0 (< 0)$, then $a_1 > 0 (a_1 < 0)$. Since $c^T A_1 x = a_1 a_{21}$, it follows that $c^T A_1 x > 0 (< 0)$. Thus, we have, $\mathcal{L}^+ = \mathcal{L}_1^+ = \mathcal{L}_2^+$ and $\mathcal{L}^- = \mathcal{L}_1^- = \mathcal{L}_2^-$. More precisely, for any $x \in \mathcal{H}$, $c^T A_2 x > 0 (< 0)$ implies that $c^T A_1 x > 0 (< 0)$. This is equivalent to the fact that $\mathcal{S}_1 = \mathcal{P}_1^+ \cup \mathcal{L}_1^+$ and $\mathcal{S}_2 = \mathcal{P}_2^- \cup \mathcal{L}_2^-$. Consequently, it follows that $\mathcal{S}_1 \cup \mathcal{S}_2 = \mathbb{R}^2$ and $\mathcal{S}_1 \cap \mathcal{S}_2 = \{0\}$. Hence, from each initial condition in \mathbb{R}^2 smooth continuation is possible only in one of the modes and in view of Lemma 2.1 in [Imura and van der Schaft, 2000], bimodal system is well-posed. Conversely, suppose that the system is well posed and $x = [a_1 \ 0]^T$. Then, if $c^T A_2 x = a_1 > 0 (< 0)$, then necessarily we must have $c^T A_1 x = a_1 a_{21} > 0 (< 0)$, which implies that $a_{21} > 0$, this concludes the proof. \square

In view of the preceding Lemma, Assumption 2 is changed as follows in \mathbb{R}^2 : Bimodal system is well posed or equivalently $a_{21} > 0$. A well-posed bimodal system is depicted in Figure 2.2.

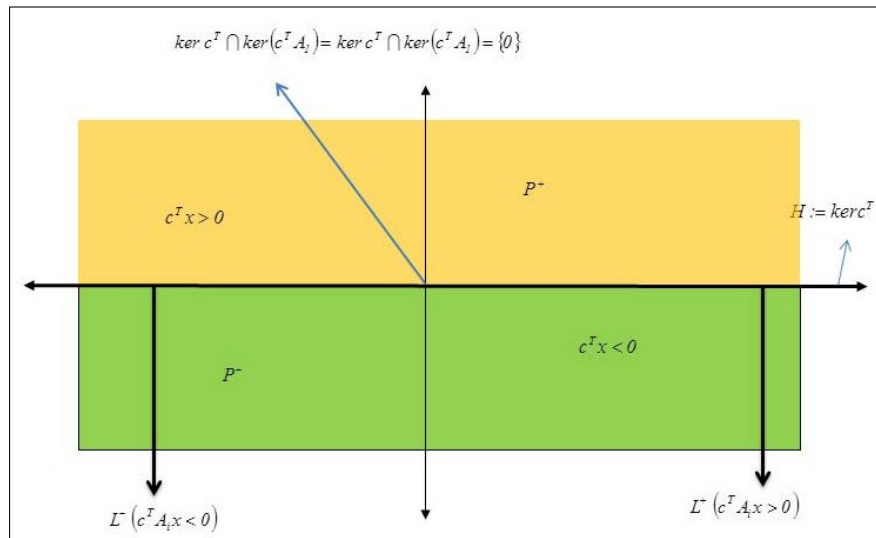


Figure 2.2: Well-posed bimodal system in \mathbb{R}^2

3. STABILITY RESULTS FOR BIMODAL SYSTEMS IN \mathbb{R}^2

In this chapter, the necessary and sufficient conditions for GAS of all cases of well-posed BPLS in \mathbb{R}^2 are presented. Towards this end, it is first shown that by an appropriate choice of eigenvectors the trajectories of BPLS in \mathbb{R}^2 can be classified as i) the trajectories which change mode in a finite time interval (transitive trajectories) and ii) the trajectories which never change mode (non-transitive trajectories). It is also shown that if a mode has only real eigenvalues, then the initial conditions of the trajectories, which start from this mode and never change mode, constitute a convex cone in \mathbb{R}^2 . Continuing along the same lines, it is further shown that if a mode has complex eigenvalues, then all the trajectories which start and smoothly continue in this mode change mode in a finite time interval. These results are then summarized as follows.

- If both modes have only real eigenvalues, then all the trajectories change mode at most once as $t \rightarrow \infty$.
- If only one of the modes have complex eigenvalues, then all the trajectories change mode at most twice as $t \rightarrow \infty$.
- If both modes have complex eigenvalues, then all the trajectories change mode infinite number of times as $t \rightarrow \infty$.

The main result of this chapter is based on the above summary. It should be noted here that GAS of planar BPLS are well known in the literature. This chapter is included in the dissertation for the sake of completeness and for motivating the approach used in the dissertation.

3.1. Both Modes Have Real Eigenvalues

Consider the system given by equation (2.2). In this section it is assumed that all the eigenvalues are real and the eigenvalues of A_1 are $\{\lambda_i\}$, for $i = 1, 2$ and the eigenvalues of A_2 are $\{\mu_i\}$, for $i = 1, 2$ where λ_i and μ_i are real numbers such that

$\lambda_1 \leq \lambda_2$ and $\mu_1 \leq \mu_2$.

Lemma 3.1: Let $\{r_1, r_2\}$ and $\{s_1, s_2\}$ denote the eigenvectors or generalized eigenvectors of A_1 and A_2 , respectively. These eigenvectors (or generalized eigenvectors) can be uniquely chosen such that

$$\text{If } \lambda_1 < \lambda_2 \text{ then } c^T r_i = 1 \text{ for } i = 1, 2, \quad (3.1)$$

$$\text{If } \lambda_1 = \lambda_2 \text{ then } c^T r_1 = 1, c^T r_2 = 0, \quad (3.2)$$

$$\text{If } \mu_1 < \mu_2 \text{ then } c^T s_i = -1 \text{ for } i = 1, 2, \quad (3.3)$$

$$\text{If } \mu_1 = \mu_2 \text{ then } c^T s_1 = -1, c^T s_2 = 0. \quad (3.4)$$

Proof 3.1: Without loss of any generality, suppose that 2nd mode is in observable canonical form. Then, using straightforward calculations, it can be easily shown that if $\lambda_1 < \lambda_2$ or $\mu_1 < \mu_2$ then r_1, r_2 and s_1, s_2 can be calculated as

$$r_1 := \begin{bmatrix} \frac{\lambda_1 - a_{22}}{a_{21}} \\ 1 \end{bmatrix}, r_2 := \begin{bmatrix} \frac{\lambda_2 - a_{22}}{a_{21}} \\ 1 \end{bmatrix}, s_1 := \begin{bmatrix} \mu_2 \\ -1 \end{bmatrix}, s_2 := \begin{bmatrix} \mu_1 \\ -1 \end{bmatrix}. \quad (3.5)$$

On the other hand, if $\lambda_1 = \lambda_2$ or $\mu_1 = \mu_2$ then r_2 and s_2 are as follows.

$$r_2 = \begin{bmatrix} \frac{1}{a_{21}} & 0 \end{bmatrix}^T, s_2 = \begin{bmatrix} -1 & 0 \end{bmatrix}^T. \quad (3.6)$$

The last equation is a direct consequence of observability, which implies that A_1 and A_2 are cyclic matrices. In order to give a general framework, let $z_i(t)$, $i = 1, 2$ denote the trajectories which smoothly continue in the i^{th} mode. Then, the behavior such trajectories in the i^{th} mode can be written as

$$z_1(t) = \alpha_1 r_1 e^{\lambda_1 t} + \alpha_2 p_2 e^{\lambda_2 t} \quad (3.7)$$

$$z_2(t) = \beta_1 s_1 e^{\mu_1 t} + \beta_2 q_2 e^{\mu_2 t} \quad (3.8)$$

where α_1, α_2 , and β_1, β_2 are real constants and $\{r_1, p_2\}, \{s_1, q_2\}$ are vectors (or vector functions) in \mathbb{R}^2 defined as

$$p_2 := r_2, \text{ if } \lambda_1 < \lambda_2 \quad (3.9)$$

$$p_2 := r_1 t + r_2 \text{ and } r_2 = \begin{bmatrix} \frac{1}{a_{21}} & 0 \end{bmatrix}^T, \text{ if } \lambda_1 = \lambda_2, \quad (3.10)$$

and

$$q_2 := s_2, \text{ if } \mu_1 < \mu_2, \quad (3.11)$$

$$q_2 := s_1 t + s_2, \text{ and } s_2 = \begin{bmatrix} -1 & 0 \end{bmatrix}^T, \text{ if } \mu_1 = \mu_2. \quad (3.12)$$

□

Lemma 3.2: The trajectories starting from \mathcal{S}_1 change mode if $\alpha_2 < 0$. If $\alpha_2 \geq 0$, then the trajectories starting from \mathcal{S}_1 do not change mode.

Proof 3.2: For a trajectory starting from \mathcal{S}_1 it follows that $c^T z_1(0) = \alpha_1 + \alpha_2 c^T p_2 \geq 0$. Suppose that $\alpha_2 < 0$ and $\lambda_1 < \lambda_2$. Then, $c^T z_1(0) = \alpha_1 + \alpha_2 c^T p_2 = \alpha_1 + \alpha_2 \geq 0$ which implies that $\alpha_1 \geq -\alpha_2$. Hence,

$$z_1(t) = e^{\lambda_1 t} [\alpha_1 r_1 + \alpha_2 e^{((\lambda_2 - \lambda_1)t)} r_2] \quad (3.13)$$

$$c^T z_1(t) = e^{\lambda_1 t} [\alpha_1 + \alpha_2 e^{((\lambda_2 - \lambda_1)t)}]. \quad (3.14)$$

In view of the above equations we get $\alpha_1 > -\alpha_2$, because in the case of equality smooth continuation to \mathcal{S}_1 is not possible. Furthermore, since $\alpha_2 < 0$ and $\lambda_2 - \lambda_1 > 0$ it is clear that, there exists $\tau_1 > 0$ such that

$$c^T z_1(\tau_1) = e^{\lambda_1 \tau_1} [\alpha_1 + \alpha_2 e^{((\lambda_2 - \lambda_1)\tau_1)}] = 0 \quad (3.15)$$

$$\implies e^{((\lambda_2 - \lambda_1)\tau_1)} = -\frac{\alpha_1}{\alpha_2}. \quad (3.16)$$

Calculating the derivative of $c^T z_1(t)$ at $t = \tau_1$ yields

$$c^T A_1 z_1(\tau_1) = e^{\lambda_1 \tau_1} \alpha_2 (\lambda_2 - \lambda_1) e^{((\lambda_2 - \lambda_1)\tau_1)} < 0 \quad (3.17)$$

as $\alpha_2 < 0$ and $(\lambda_2 - \lambda_1) > 0$. Thus, the trajectory changes mode. Suppose that $\alpha_2 < 0$ and $\lambda_1 = \lambda_2$. Then, $c^T z_1(0) = \alpha_1 + \alpha_2 c^T p_2 = \alpha_1 \geq 0$. If $\alpha_1 = 0$, then

$c^T z_1(t) = [\alpha_2 t e^{((\lambda_2 - \lambda_1)t)}] < 0$ for any $t > 0$ and therefore smooth continuation into \mathcal{S}_1 is not possible. Thus, necessarily $\alpha_1 > 0$. Similar to the previous case, there exists $\tau_1 > 0$ such that

$$c^T z_1(\tau_1) = e^{\lambda_1 \tau_1} [\alpha_1 + \alpha_2 \tau_1 e^{((\lambda_2 - \lambda_1)\tau_1)}] = 0, \quad (3.18)$$

$$\implies e^{((\lambda_2 - \lambda_1)\tau_1)} = -\frac{\alpha_1}{\alpha_2 \tau_1}. \quad (3.19)$$

Calculating the derivative of $c^T z_1(t)$ at $t = \tau_1$ yields the following inequality

$$c^T A_1 z_1(\tau_1) = e^{\lambda_1 \tau_1} [\alpha_1 + \alpha_2 \tau_1 e^{((\lambda_2 - \lambda_1)\tau_1)} + \alpha_2 (\lambda_2 - \lambda_1) e^{((\lambda_2 - \lambda_1)\tau_1)}] \quad (3.20)$$

$$= e^{\lambda_1 \tau_1} [\alpha_2 (\lambda_2 - \lambda_1) e^{((\lambda_2 - \lambda_1)\tau_1)}] < 0 \quad (3.21)$$

as $\alpha_2 < 0$ and $(\lambda_2 - \lambda_1) > 0$. Hence, the trajectory changes mode.

Continuing along the similar lines let us assume that $\alpha_2 \geq 0$. If the equality holds, then necessarily $\alpha_1 > 0$. Otherwise, the trajectory starts from the origin and stays there. Therefore, let us consider the case $\alpha_2 > 0$. If $\alpha_1 \geq 0$, then

$$c^T z_1(t) = e^{\lambda_1 t} [\alpha_1 + (c^T p_2) \alpha_2 e^{((\lambda_2 - \lambda_1)t)}] > 0 \quad (3.22)$$

for all $t > 0$. Consequently, the trajectory does not change mode. On the other hand, if $\alpha_1 < 0$ and $\lambda_2 > \lambda_1$, then necessarily

$$c^T z_1(0) = \alpha_1 + \alpha_2 \geq 0 \implies c^T z_1(t) = e^{\lambda_1 t} [\alpha_1 + \alpha_2 e^{((\lambda_2 - \lambda_1)t)}] > 0. \quad (3.23)$$

Hence, the trajectory does not change mode. If $\lambda_2 = \lambda_1$, then since $c^T z_1(0) = [\alpha_1 + (c^T p_2) \alpha_2] = \alpha_1 \geq 0$, it follows that

$$c^T z_1(t) = e^{\lambda_1 t} [\alpha_1 + (c^T p_2) \alpha_2 e^{((\lambda_2 - \lambda_1)t)}] > 0, \quad (3.24)$$

and consequently the trajectory does not change mode. □

Definition 3.1: Let $z_i(t)$ be a trajectory which starts from \mathcal{S}_i . If there exists a finite $\tau_i > 0$ such that $c^T z_i(\tau_i) = 0$ and $z_i(t)$ changes mode at $t = \tau_i$, then $z_i(t)$ is called a transitive trajectory. Otherwise, it is called a non-transitive trajectory.

Definition 3.2: In view of the above Lemma, the closed convex cone bounded by \mathcal{L}^+ and r_1 (which represents the set of initial conditions of all non-transitive trajectories) will be called the invariant cone of \mathcal{S}_1 and denoted as \mathcal{C}^+ . Similarly, the closed convex cone bounded by \mathcal{L}^- and s_1 will be called invariant cone of \mathcal{S}_2 and denoted as \mathcal{C}^- .

3.2. At Least One of The Modes Has Complex Eigenvalues

Let us consider the system given by equation (2.2) where the eigenvalues of A_1 are $\sigma_1 \pm jw_1$ and the eigenvalues of A_2 are $\{\mu_1, \mu_2\}$. Here, $w_1 > 0$ and σ_1, μ_1, μ_2 are real number such that $\mu_1 \leq \mu_2$. Eigenvectors or generalized eigenvectors of the second mode are given in equation (3.1). Moreover, the corresponding eigenvectors of A_1 can be calculated as follows.

$$A_1 \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} x_1 & y_1 \end{bmatrix} \begin{bmatrix} \sigma_1 & w_1 \\ -w_1 & \sigma_1 \end{bmatrix}. \quad (3.25)$$

Solving for $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$, we get

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} \frac{\sigma_1 - a_{22}}{a_{21}} & \frac{w_1}{a_{21}} \\ 1 & 0 \end{bmatrix}. \quad (3.26)$$

Consequently, the trajectories that start from \mathcal{S}_1 can be expressed as

$$z_1(t) = K e^{\sigma_1 t} [x_1 \sin(w_1 t + \theta_1) + y_1 \cos(w_1 t + \theta_1)]. \quad (3.27)$$

where $K > 0$.

Remark 3.1: Suppose that both modes have complex eigenvalues and the eigenvalues of A_2 are $\sigma_2 \pm jw_2$, where $\sigma_2, w_2 > 0$ are real numbers. Then, it can be shown that

$$A_2 = \begin{bmatrix} 0 & -(\sigma_2^2 + w_2^2) \\ 1 & 2\sigma_2 \end{bmatrix}. \quad (3.28)$$

Moreover, the corresponding eigenvectors of A_2 can be expressed as

$$A_2[x_2 \ y_2] = [x_2 \ y_2] \begin{bmatrix} \sigma_2 & \omega_2 \\ -\omega_2 & \sigma_2 \end{bmatrix}, \quad (3.29)$$

where

$$[x_2 \ y_2] = \begin{bmatrix} \sigma_2 & -\omega_2 \\ -1 & 0 \end{bmatrix}. \quad (3.30)$$

It can be easily shown that all trajectories that start from \mathcal{S}_2 also change mode.

Lemma 3.3: All trajectories which start from \mathcal{S}_1 (\mathcal{S}_2) are transitive if A_1 (A_2) has complex eigenvalues.

Proof 3.3: The trajectories that start from \mathcal{S}_1 can be expressed as

$$z_1(t) = Ke^{\sigma_1 t} [x_1 \sin(w_1 t + \theta_1) + y_1 \cos(w_1 t + \theta_1)] \quad (3.31)$$

where $K > 0$ and θ_1 is the angle between $z_1(0)$ and the line $\mathcal{L}^+ - \mathcal{L}^-$, measured counterclockwise. Since $z_1(0) \in \mathcal{S}_1$, it follows that $c^T z_1(0) \geq 0$ or equivalently

$$z_1(0) = K(x_1 \sin \theta_1 + y_1 \cos \theta_1) \implies c^T z_1(0) = K \sin \theta_1 \geq 0. \quad (3.32)$$

Thus, for a trajectory starting from \mathcal{S}_1 , it follows that $0 \leq \theta_1 < \pi$. Also note that

$$c^T z_1(t) = Ke^{\sigma_1 t} \sin(w_1 t + \theta_1). \quad (3.33)$$

This equation implies that at $\tau_1 = \frac{\pi - \theta_1}{w_1}$ we have $c^T z_1(\tau_1) = 0$ and $c^T A_1 z_1(\tau_1) < 0$. Consequently the trajectory changes mode. The proof for trajectories starting from \mathcal{S}_2 can be done using similar arguments. \square

3.3. Main Stability Results of Bimodal Systems in \mathbb{R}^2

In view of the above, the main result of this chapter is stated and proved as follows.

Theorem 3.1: Consider bimodal system (2.2) in \mathbb{R}^2 , where both modes are observable and the system is well posed. Then, the following hold.

- i) If at least one of the modes have real eigenvalues, then all the trajectories change mode only finite number of times and the system is GAS if and only if real eigenvalues of both modes are negative (stable).*
- ii) If both modes have complex eigenvalues, then all the trajectories change mode infinite number of times and the system is GAS if and only if*

$$\frac{\sigma_1}{w_1} + \frac{\sigma_2}{w_2} < 0, \quad (3.34)$$

where $\{\sigma_i \pm jw_i\}$, $i = 1, 2$ are the eigenvalues of A_1 and A_2 .

Proof 3.1:

- i) Suppose that the eigenvalues of both A_1 and A_2 are all real. Then, in view of Lemma 3.2, any transitive trajectory starting from \mathcal{S}_1 (\mathcal{S}_2) changes mode on \mathcal{L}^- (\mathcal{L}^+) which is in \mathcal{C}^- (\mathcal{C}^+). Then, after first mode change, the trajectories stay in \mathcal{S}_2 (\mathcal{S}_1) without changing mode any more. Thus, after some finite time, all the trajectories enter \mathcal{C}^- or \mathcal{C}^+ and stay in \mathcal{S}_2 or \mathcal{S}_1 without changing mode any more. Consequently, all trajectories $\rightarrow 0$ if and only if all the real eigenvalues of both modes are strictly negative (stable).*

On the other hand, if A_1 has complex eigenvalues $\{\sigma_1 \pm jw_1\}$, then in view of Lemma 3.3, every trajectory starting from \mathcal{S}_1 is transitive. In this case, any transitive trajectory starting from \mathcal{S}_1 changes mode on \mathcal{L}^- which is in \mathcal{C}^- . This implies that (as A_2 has real eigenvalues) the trajectory stays in \mathcal{S}_2 without changing mode any more. Similarly, if a transitive trajectory starts from \mathcal{S}_2 , then after some finite time it will change mode on \mathcal{L}^+ . Since every trajectory starting from \mathcal{S}_1 is transitive, the trajectory will change mode again on \mathcal{L}^- and then stay in \mathcal{S}_2 without changing mode any more. Consequently, again it follows that all trajectories $\rightarrow 0$ if and only if all the real eigenvalues A_2 are strictly negative (stable).

- ii) Suppose that the eigenvalues of A_1 and A_2 are $\{\sigma_i \pm jw_i\}$, $i = 1, 2$. Then Lemma 3.3 implies that all the trajectories change mode infinite number of times. Let us assume*

without loss of generality that $z_1(t)$ is a trajectory given by equation (3.31) which starts from \mathcal{L}^+ . Thus, $\theta_1 = 0$. This trajectory changes mode on \mathcal{L}^- at $\tau_1 = \frac{\pi}{w_1}$ and continues smoothly into \mathcal{S}_2 as

$$z_2(t) = \frac{w_1}{w_2 a_{21}} K e^{(\sigma_1 \frac{\pi}{w_1})} e^{(\sigma_1 t)} [x_2 \sin(w_2 t + \theta_2) + y_2 \cos(w_2 t + \theta_2)] \quad (3.35)$$

where $\theta_2 = 0$. Then it changes mode again on \mathcal{L}^+ at $\tau_2 = \frac{\pi}{w_2}$. Thus, we have

$$z_2(\tau_2) = K e^{(\sigma_1 \frac{\pi}{w_1})} e^{(\sigma_1 \frac{\pi}{w_2})} [x_2 \sin(w_2 \tau_2 + \theta_1) + y_2 \cos(w_2 \tau_2 + \theta_2)]. \quad (3.36)$$

Note that the ratio $\frac{\|z_2(\tau_2)\|}{\|z_1(0)\|} = e^{(\sigma_1 \frac{\pi}{w_1} + \sigma_1 \frac{\pi}{w_2})}$. Consequently, the trajectory decays to the origin if and only if

$$\frac{\|z_2(\tau_2)\|}{\|z_1(0)\|} < 1 \iff \frac{\sigma_1}{w_1} + \frac{\sigma_2}{w_2} < 0. \quad (3.37)$$

□

Remark 3.2: The second item of the above theorem is previously proven by [Camlibel et al., 2003] and by [Iwatani and Hara, 2006]. It is included in the above theorem for the sake of completeness. Note that A_1 has no real eigenvalues and real part of complex eigenvalue could be unstable (strictly positive). But this does not change the result, because every trajectory stays only once and finite period of time in \mathcal{S}_1 . Also note that the line $\ker c^T$ behaves as a fixed direction.

4. STABILITY RESULTS FOR BIMODAL SYSTEMS IN \mathbb{R}^3

In this chapter, the necessary and sufficient conditions for GAS of a class of well-posed BPLS in \mathbb{R}^3 are presented. This class includes the cases where both modes have only real eigenvalues and the cases where one mode has a pair of complex eigenvalues and the other mode has only real eigenvalues with geometric multiplicity equal to one. As in the previous chapter, it is first shown that by an appropriate choice of eigenvectors the trajectories of BPLS in \mathbb{R}^3 can be classified as transitive and non-transitive. It is also shown that if a mode has only real eigenvalues, then the initial conditions of the trajectories, which start from \mathcal{H} and smoothly continue in this mode and never change mode, constitute a convex cone in $\mathbb{R}^3 \cap \mathcal{H}$. Continuing along the same lines, it is further shown that if a mode has a pair of complex eigenvalues and if $\sigma_i - \lambda_i > 0$, then all the trajectories which start from \mathcal{H} and smoothly continue in this mode change mode in a finite time interval. Then, a second classification is presented as follows.

- The trajectories which change mode only finite number of times as $t \rightarrow \infty$.
- The trajectories which change mode infinite number of times as $t \rightarrow \infty$.

The case where both modes have only real eigenvalues is considered first. Since all the trajectories change mode only finite number of times as $t \rightarrow \infty$ in this case, it turns out that BPLS in \mathbb{R}^3 are GAS if and only if the real eigenvalues of both modes are negative. This is the first main result of this chapter. In the final section, the case where only one mode has a pair of complex eigenvalues is investigated. For this investigation, a constant B , which is called the coupling constant, is introduced. It is shown that if $B \leq 0$, then BPLS is GAS if and only if the real eigenvalues of both modes are negative. Then, it is proven that if $B > 0$, then there exists a unique pair of fixed directions. It is further shown that, BPLS is GAS if and only if the real eigenvalues of both modes are negative and the trajectories starting from the fixed direction decay to the origin. Then it is shown that the trajectories starting from the fixed direction decay to the origin if and only if the convergence rate of the fixed direction is strictly

less than one. A sketch of an algorithm which calculates the fixed directions and the convergence rate is provided in Remark 4.4.

At the end of the chapter, it is demonstrated by examples that the coupling constant can be changed without changing the eigenvalues of subsystems and this change can make BPLS GAS or unstable.

4.1. Both Modes With Real Eigenvalues

Consider the system defined by equation (1.1), where $x, c \in \mathbb{R}^3$ and A_1 and A_2 are 3×3 matrices. Suppose that the eigenvalues of A_1 and A_2 are $\{\lambda_1, \lambda_2, \lambda_3\}$ and $\{\mu_1, \mu_2, \mu_3\}$, respectively. Let us also assume without loss of generality that $\lambda_1 \leq \lambda_2 \leq \lambda_3$ and $\mu_1 \leq \mu_2 \leq \mu_3$.

Lemma 4.1: Let $\{r_1, r_2, r_3\}$ and $\{s_1, s_2, s_3\}$ denote the eigenvectors or generalized eigenvectors of A_1 and A_2 , respectively. Then, eigenvectors (or generalized eigenvectors) can be uniquely chosen such that

$$\text{If } \lambda_1 < \lambda_2 < \lambda_3 \text{ then } c^T r_i = 1 \text{ for } i = 1, 2, 3, \quad (4.1)$$

$$\text{If } \lambda_1 = \lambda_2 < \lambda_3 \text{ then } c^T r_i = 1, \text{ for } i = 1, 3 \text{ and } c^T r_2 = 0, \quad (4.2)$$

$$\text{If } \lambda_1 < \lambda_2 = \lambda_3 \text{ then } c^T r_i = 1, \text{ for } i = 1, 2 \text{ and } c^T r_3 = 0, \quad (4.3)$$

$$\text{If } \lambda_1 = \lambda_2 = \lambda_3 \text{ then } c^T r_i = 0 \text{ for } i = 2, 3 \text{ and } c^T r_1 = 1, \quad (4.4)$$

and similarly

$$\text{If } \mu_1 < \mu_2 < \mu_3 \text{ then } c^T s_i = -1 \text{ for } i = 1, 2, 3 \quad (4.5)$$

$$\text{If } \mu_1 = \mu_2 < \mu_3 \text{ then } c^T s_i = -1, \text{ for } i = 1, 3 \text{ and } c^T s_2 = 0 \quad (4.6)$$

$$\text{If } \mu_1 < \mu_2 = \mu_3 \text{ then } c^T s_i = -1, \text{ for } i = 1, 2 \text{ and } c^T s_3 = 0 \quad (4.7)$$

$$\text{If } \mu_1 = \mu_2 = \mu_3 \text{ then } c^T s_i = 0 \text{ for } i = 2, 3 \text{ and } c^T s_1 = -1 \quad (4.8)$$

Proof 4.1: Note that in view of Assumptions 1-2 the system matrices can be written as follows.

$$A_1 = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ 0 & a_{32} & a_{33} \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 0 & \mu_1\mu_2\mu_3 \\ 1 & 0 & -\mu_1\mu_2 - \mu_1\mu_3 - \mu_2\mu_3 \\ 0 & 1 & \mu_1 + \mu_2 + \mu_3 \end{bmatrix} \quad (4.9)$$

$$c^T = [0 \ 0 \ 1] \quad (4.10)$$

where $a_{21}, a_{32} > 0$. If $\lambda_1 < \lambda_2 < \lambda_3$ and $\mu_1 < \mu_2 < \mu_3$, then using straightforward calculations the eigenvectors r_1, r_2, r_3 and s_1, s_2, s_3 can be calculated easily as follows.

$$r_1 = \begin{bmatrix} \frac{(a_{11}-\lambda_3)(a_{11}-\lambda_2)+a_{12}a_{21}}{\frac{a_{32}a_{21}}{\lambda_1-a_{33}}} \\ \frac{a_{32}}{1} \\ 1 \end{bmatrix}, r_2 = \begin{bmatrix} \frac{(a_{11}-\lambda_1)(a_{11}-\lambda_3)+a_{12}a_{21}}{\frac{a_{32}a_{21}}{\lambda_2-a_{33}}} \\ \frac{a_{32}}{1} \\ 1 \end{bmatrix} \quad (4.11)$$

$$r_3 = \begin{bmatrix} \frac{(a_{11}-\lambda_1)(a_{11}-\lambda_2)+a_{12}a_{21}}{\frac{a_{32}a_{21}}{\lambda_3-a_{33}}} \\ \frac{a_{32}}{1} \\ 1 \end{bmatrix} \quad (4.12)$$

$$s_1 = \begin{bmatrix} -\mu_2\mu_3 \\ \mu_2 + \mu_3 \\ -1 \end{bmatrix}, s_2 = \begin{bmatrix} -\mu_1\mu_3 \\ \mu_1 + \mu_3 \\ -1 \end{bmatrix}, s_3 = \begin{bmatrix} -\mu_1\mu_2 \\ \mu_1 + \mu_2 \\ -1 \end{bmatrix} \quad (4.13)$$

Since the pairs (c^T, A_1) and (c^T, A_2) are observable, A_1 and A_2 are cyclic matrices. This implies that, if the algebraic degree of the eigenvalues change, then we have to use generalized eigenvectors which are given as follows.

$$\text{If } \lambda_1 = \lambda_2 < \lambda_3 \text{ then } r_2 = \begin{bmatrix} \frac{a_{11}-\lambda_3}{a_{32}a_{21}} & \frac{1}{a_{32}} & 0 \end{bmatrix}^T, \quad (4.14)$$

$$\text{If } \lambda_1 < \lambda_2 = \lambda_3 \text{ then } r_3 = \begin{bmatrix} \frac{a_{11}-\lambda_1}{a_{32}a_{21}} & \frac{1}{a_{32}} & 0 \end{bmatrix}^T, \quad (4.15)$$

$$\text{If } \lambda_1 = \lambda_2 = \lambda_3 \text{ then } r_2 = \begin{bmatrix} \frac{a_{11}-\lambda_1}{a_{32}a_{21}} & \frac{1}{a_{32}} & 0 \end{bmatrix}^T, r_3 = \begin{bmatrix} \frac{1}{a_{32}a_{21}} & 0 & 0 \end{bmatrix}^T, \quad (4.16)$$

Similarly, for the other mode we have

$$\text{if } \mu_1 = \mu_2 < \mu_3 \text{ then } s_2 = [\mu_3 \quad -1 \quad 0]^T, \quad (4.17)$$

$$\text{if } \mu_1 < \mu_2 = \mu_3 \text{ then } s_3 = [\mu_1 \quad -1 \quad 0]^T, \quad (4.18)$$

$$\text{if } \mu_1 = \mu_2 = \mu_3 \text{ then } s_2 = [\mu_1 \quad -1 \quad 0]^T, s_3 = [-1 \quad 0 \quad 0]^T. \quad (4.19)$$

□

In order to give a general framework, let $z_i(t)$, $i = 1, 2$ denote the trajectories which smoothly continues into the i^{th} mode. Then, the behavior of such trajectories in the i^{th} mode can be written as

$$z_1(t) = \{ a_1 e^{\lambda_1 t} r_1 + a_2 e^{\lambda_2 t} p_2(t) + a_3 e^{\lambda_3 t} p_3(t) \}, \quad (4.20)$$

$$z_2(t) = \{ d_1 e^{\mu_1 t} s_1 + d_2 e^{\mu_2 t} q_2(t) + d_3 e^{\mu_3 t} q_3(t) \}, \quad (4.21)$$

where a_1, a_2, a_3 and d_1, d_2, d_3 are real constants and $r_1, p_2(t), p_3(t)$ and $s_1, q_2(t), q_3(t)$ are vector functions in \mathbb{R}^3 given according to the algebraic degree of the eigenvalues as follows.

$$\text{if } \lambda_1 < \lambda_2 < \lambda_3, \text{ then } p_2(t) := r_2, p_3(t) := r_3 \quad (4.22)$$

$$\text{if } \lambda_1 = \lambda_2 < \lambda_3, \text{ then } p_2(t) := r_1 t + r_2, p_3(t) := r_3 \quad (4.23)$$

$$\text{if } \lambda_1 < \lambda_2 = \lambda_3, \text{ then } p_2(t) := r_2, p_3(t) := r_2 t + r_3 \quad (4.24)$$

$$\text{if } \lambda_1 = \lambda_2 = \lambda_3, \text{ then } p_2(t) := r_1 t + r_2, p_3(t) := r_1 \frac{t^2}{2} + r_2 t + r_3, \quad (4.25)$$

and

$$\text{if } \mu_1 < \mu_2 < \mu_3, \text{ then } q_2(t) := s_2, q_3(t) := s_3, \quad (4.26)$$

$$\text{if } \mu_1 = \mu_2 < \mu_3, \text{ then } q_2(t) := s_1 t + s_2, q_3(t) := s_3, \quad (4.27)$$

$$\text{if } \mu_1 < \mu_2 = \mu_3, \text{ then } q_2(t) := s_2, q_3(t) := s_2 t + s_3, \quad (4.28)$$

$$\text{if } \mu_1 = \mu_2 = \mu_3, \text{ then } q_2(t) := s_1 t + s_2, q_3(t) := s_1 \frac{t^2}{2} + s_2 t + s_3. \quad (4.29)$$

In view of these equations, if a trajectory $z_1(t)$ starts from \mathcal{S}_1 , it follows that $c^T z_1(0) = a_1 + a_2 c^T r_2 + a_3 c^T r_3 \geq 0$. At this point, let us define $K_1 := \sqrt{a_2^2 + a_3^2}$, $\sin \theta_1 := \frac{a_2}{K_1}$, $\cos \theta_1 := \frac{a_3}{K_1}$, $\alpha_1 = \frac{a_1}{K_1}$. Thus, $z_1(t)$ can be written as

$$z_1(t) = K_1 e^{\lambda_1 t} [\alpha_1 r_1 + p_2(t) e^{(\lambda_2 - \lambda_1)t} \sin \theta_1 + p_3(t) e^{(\lambda_3 - \lambda_1)t} \cos \theta_1], \quad (4.30)$$

$$c^T z_1(t) = K_1 e^{\lambda_1 t} [\alpha_1 + (c^T p_2(t)) e^{(\lambda_2 - \lambda_1)t} \sin \theta_1 + (c^T p_3(t)) e^{(\lambda_3 - \lambda_1)t} \cos \theta_1]. \quad (4.31)$$

Let $f_1(t)$ be defined as

$$f_1(t) := [\alpha_1 + (c^T p_2(t)) e^{(\lambda_2 - \lambda_1)t} \sin \theta_1 + (c^T p_3(t)) e^{(\lambda_3 - \lambda_1)t} \cos \theta_1]. \quad (4.32)$$

Since $c^T z_1(t) = K_1 e^{\lambda_1 t} f_1(t)$, the sign of $f_1(t)$ and the sign of $c^T z_1(t)$ are the same. Similarly, if a trajectory $z_2(t)$ starts from \mathcal{S}_2 , it follows that $c^T z_2(0) = -d_1 + d_2 c^T s_2 + d_3 c^T s_3 \leq 0$. Let $K_2 := \sqrt{d_2^2 + d_3^2}$, $\sin \theta_2 := \frac{d_2}{K_2}$, $\cos \theta_2 := \frac{d_3}{K_2}$, and $\beta_1 = \frac{d_1}{K_2}$. Then, $z_2(t)$ and $c^T z_2(t)$ can be written as follows.

$$z_2(t) = K_2 e^{\mu_1 t} [\beta_1 s_1 + q_2(t) e^{(\mu_2 - \mu_1)t} \sin \theta_2 + q_3(t) e^{(\mu_3 - \mu_1)t} \cos \theta_2], \quad (4.33)$$

$$c^T z_2(t) = K_2 e^{\mu_1 t} [-\beta_1 + (c^T q_2(t)) e^{(\mu_2 - \mu_1)t} \sin \theta_2 + (c^T q_3(t)) e^{(\mu_3 - \mu_1)t} \cos \theta_2]. \quad (4.34)$$

Let $f_2(t)$ be defined as

$$f_2(t) := [-\beta_1 + (c^T q_2(t)) e^{(\mu_2 - \mu_1)t} \sin \theta_2 + (c^T q_3(t)) e^{(\mu_3 - \mu_1)t} \cos \theta_2] \quad (4.35)$$

Since $c^T z_2(t) = K_2 e^{\mu_1 t} f_2(t)$, the sign of $f_2(t)$ and the sign of $c^T z_2(t)$ are the same.

4.1.1. Smooth Continuation of Trajectories

In view of equations (4.31) and (4.34), a trajectory $z_i(t)$ with initial condition not on \mathcal{H} , smoothly continues into \mathcal{S}_i if $f_i(0) > 0$ for $i = 1$ and $f_i(0) < 0$ for $i = 2$. For trajectories starting from \mathcal{H} , the following result holds.

Lemma 4.2: Suppose that $c^T z_1(0) = 0$ ($c^T z_2(0) = 0$). Then, $z_1(t)$ ($z_2(t)$) smoothly continues into \mathcal{S}_1 (\mathcal{S}_2) if and only if $-\phi_i \leq \theta_i < \pi - \phi_i$, for $i = 1, 2$, where

$$\cot \phi_1 = \frac{\lambda_2 - \lambda_1}{\lambda_3 - \lambda_1} \text{ if } \lambda_3 > \lambda_2 > \lambda_1 \quad (4.36)$$

$$\cot \phi_1 = \frac{1}{\lambda_3 - \lambda_1} \text{ if } \lambda_3 > \lambda_2 = \lambda_1, \quad (4.37)$$

$$\cot \phi_1 = \lambda_3 - \lambda_1 \text{ if } \lambda_3 = \lambda_2 > \lambda_1, \quad (4.38)$$

$$\cot \phi_1 = \infty \text{ } (\phi_1 = 0) \text{ if } \lambda_3 = \lambda_2 = \lambda_1 \quad (4.39)$$

$$\cot \phi_2 = \frac{\mu_2 - \mu_1}{\mu_3 - \mu_1} \text{ if } \mu_3 > \mu_2 > \mu_1, \quad (4.40)$$

$$\cot \phi_2 = \frac{1}{\mu_3 - \mu_1} \text{ if } \mu_3 > \mu_2 = \mu_1, \quad (4.41)$$

$$\cot \phi_2 = \mu_3 - \mu_1 \text{ if } \mu_3 = \mu_2 > \mu_1, \quad (4.42)$$

$$\cot \phi_2 = \infty \text{ } (\phi_2 = 0) \text{ if } \mu_3 = \mu_2 = \mu_1. \quad (4.43)$$

Proof 4.2: The proof of the Lemma is given in Appendix. \square

4.1.2. Bases for \mathcal{H}

For trajectories starting from \mathcal{H} , we have $f_i(0) = 0$. Thus, equations (4.32) and (4.35) imply that $\alpha_1 = -(c^T r_2) \sin \theta_1 - (c^T r_3) \cos \theta_1$ and $\beta_1 = (c^T s_2) \sin \theta_2 + (c^T s_3) \cos \theta_2$ and

$$z_1(0) = K_1 (\alpha_1 r_1 + r_2 \sin \theta_1 + r_3 \cos \theta_1), \quad (4.44)$$

$$z_2(0) = K_2 (\beta_1 s_1 + s_2 \sin \theta_2 + s_3 \cos \theta_2) \quad (4.45)$$

In view of these observations, there are two different bases for \mathcal{H} as given below.

$$\hat{r}_2 := r_2 - (c^T r_2) r_1, \quad \hat{r}_3 := r_3 - (c^T r_3) r_1, \quad (4.46)$$

$$\hat{s}_2 := s_2 + (c^T s_2) s_1, \quad \hat{s}_3 := s_3 + (c^T s_3) s_1 \quad (4.47)$$

It can be easily seen by Lemma 4.1 and equation (4.22-4.29) that $c^T \hat{r}_2 = c^T \hat{r}_3 = 0$ for each case of multiplicities of eigenvalues of A_1 and the pair $\{\hat{r}_2, \hat{r}_3\}$ is a basis for \mathcal{H} . Same is true for $\{\hat{s}_2, \hat{s}_3\}$. Therefore, any initial condition on \mathcal{H} can be expressed as

$$K_1 (\hat{r}_2 \sin \theta_1 + \hat{r}_3 \cos \theta_1) \text{ or } K_2 (\hat{s}_2 \sin \theta_2 + \hat{s}_3 \cos \theta_2) \quad (4.48)$$

where $-\phi_i \leq \theta_i \leq 2\pi - \phi_i$. Accordingly, $v_1(\theta_1)$ and $v_2(\theta_2)$ are defined as follows.

$$v_1(\theta_1) = \hat{r}_2 \sin \theta_1 + \hat{r}_3 \cos \theta_1 \text{ and } v_2(\theta_2) = \hat{s}_2 \sin \theta_2 + \hat{s}_3 \cos \theta_2. \quad (4.49)$$

Definition 4.1: $v_1(\theta_1)$ and $v_2(\theta_2)$ will be called directions in \mathcal{H} , in the rest of the thesis. The directions for which $\eta v_1(\theta_1) = v_2(\theta_2)$ hold for some constant $\eta > 0$, will be called equivalent directions and denoted as $v_1(\theta_1) \simeq v_2(\theta_2)$.

4.1.3. Characterization of Transitive and Non-transitive Trajectories

Definition 4.2: Let $z_i(t)$ be a trajectory which starts from \mathcal{S}_i . If there exists a finite $\tau_i > 0$ such that $c^T z_i(\tau_i) = 0$ and $z_i(t)$ changes mode at $t = \tau_i$, then $z_i(t)$ is called a transitive trajectory. Otherwise, it is called a non-transitive trajectory.

Theorem 4.1: Let $z_i(t)$ be a trajectory starting from \mathcal{S}_i . Also let $f_1(t_{\min})$ and $f_2(t_{\max})$ denote the first local minimum of $f_1(t)$ and the first local maximum of $f_2(t)$, respectively. Then the following hold.

i) The trajectories are non-transitive if any one of the following conditions holds.

- a) $-\phi_i \leq \theta_i \leq \frac{\pi}{2}$,
- b) $\frac{3\pi}{2} < \theta_i < 2\pi - \phi_i$ and $f_1(t_{\min}) \geq 0$ for $i = 1$ or $f_2(t_{\max}) \leq 0$ for $i = 2$.

ii) The trajectories are transitive if any one of the following conditions holds.

- a) $\frac{\pi}{2} < \theta_i \leq \frac{3\pi}{2}$,
- b) $\frac{3\pi}{2} < \theta_i < 2\pi - \phi_i$ and $f_1(t_{\min}) < 0$ for $i = 1$ or $f_2(t_{\max}) > 0$ for $i = 2$.

Proof 4.1: The proof of the theorem is given in appendix. □

Remark 4.1: Theorem 4.1 gives a complete classification of all the initial conditions in \mathcal{S}_i from which transitive or non-transitive trajectories start. The inequalities involving $f_1(t_{\min})$ or $f_2(t_{\max})$ can be written more explicitly (as a function of system data) by solving for t_{\min} or t_{\max} via equations $\frac{d}{dt}f_1(t_{\min}) = 0$ or $\frac{d}{dt}f_2(t_{\max}) = 0$ and substituting the result in f_1 or f_2 . Thus, for a trajectory starting from \mathcal{S}_1 , it follows that

$$f_1(t_{\min}) = \alpha_1 + (1 - \cot \phi_1) (-\tan \theta_1 \cot \phi_1)^{\frac{(\lambda_2 - \lambda_1)}{(\lambda_3 - \lambda_2)}} \sin \theta_1, \text{ if } \lambda_1 < \lambda_2 < \lambda_3, \quad (4.50)$$

$$f_1(t_{\min}) = \alpha_1 - \sin \theta_1 \cot \phi_1 [1 - \ln (-\tan \theta_1 \cot \phi_1)], \text{ if } \lambda_1 = \lambda_2 < \lambda_3, \quad (4.51)$$

$$f_1(t_{\min}) = \alpha_1 - \tan \phi_1 \cos \theta_1 e^{-(\cot \phi_1 \tan \theta_1 + 1)}, \text{ if } \lambda_1 < \lambda_2 = \lambda_3, \quad (4.52)$$

$$f_1(t_{\min}) = 2\alpha_1 \cos \theta_1 - \sin^2 \theta_1, \text{ if } \lambda_1 = \lambda_2 = \lambda_3. \quad (4.53)$$

and for a trajectory starting from \mathcal{S}_2 , we get

$$f_2(t_{\max}) = \beta_1 + (1 - \cot \phi_2) (-\tan \theta_2 \cot \phi_2)^{\frac{(\mu_2 - \mu_1)}{(\mu_3 - \mu_2)}} \sin \theta_2, \text{ if } \mu_1 < \mu_2 < \mu_3 \quad (4.54)$$

$$f_2(t_{\max}) = \beta_1 - \sin \theta_2 \cot \phi_2 [1 - \ln (-\tan \theta_2 \cot \phi_2)], \text{ if } \mu_1 = \mu_2 < \mu_3 \quad (4.55)$$

$$f_2(t_{\max}) = \beta_1 - \tan \phi_2 \cos \theta_2 e^{-(\cot \phi_2 \tan \theta_2 + 1)}, \text{ if } \mu_1 < \mu_2 = \mu_3, \quad (4.56)$$

$$f_2(t_{\max}) = 2\beta_1 \cos \theta_2 - \sin^2 \theta_2, \text{ if } \mu_1 = \mu_2 = \mu_3. \quad (4.57)$$

Note that the equations for $f_1(t_{\min})$ and $f_2(t_{\max})$ directly follow from the facts that $\frac{3\pi}{2} < \theta_i < 2\pi - \phi_i$, and $-\tan \theta_i \cot \phi_i > 0$.

Definition 4.3: Let \mathcal{S}_i^n and \mathcal{S}_i^t ($i = 1, 2$) denote the set of initial conditions in \mathcal{S}_i ($i = 1, 2$) from which non-transitive trajectories and transitive trajectories start, respectively.

In view of Theorem (4.1), it is clear that $\mathcal{S}_i = \mathcal{S}_i^n \cup \mathcal{S}_i^t$ ($i = 1, 2$). In what follows, a final classification of the trajectories of bimodal systems (described by equations (1.1) and (4.9) is given as follows.

- Trajectories which change mode only finite number of times as $t \rightarrow \infty$. These trajectories will referred to as finitely mode changing trajectories in the sequel.

- Trajectories which change mode infinite number of times as $t \rightarrow \infty$, which will referred to as infinitely mode changing trajectories in the sequel.

This classification is originally given in [Eldem and Şahan, 2014] with the Conjecture 1.1. Furthermore, the Conjecture 1.1 given in introduction implies that, without loss of any generality, only the trajectories which start from \mathcal{H} can be considered for GAS.

4.1.4. Trajectories Starting from \mathcal{H}

In the sequel, some preliminary results, towards the classification of the trajectories of bimodal system (outlined at the end of the previous section), are given. Some of these results will also be used in the next section. In view of Definition 4.3, it is first shown that $\mathcal{S}_i^n \cap \mathcal{H}$ is a closed convex linear cone and $\mathcal{S}_i^t \cap \mathcal{H}$ is an open convex linear cone.

Corollary 4.1: Suppose that $z_i(t)$ is a trajectory starting from \mathcal{H} . Then, the following holds.

- i) $z_i(t)$ is a non-transitive trajectory if and only if $z_i(0) = K_i v_i(\theta_i)$ where $-\phi_i \leq \theta_i \leq \frac{\pi}{2}$ and $K_i > 0$, for $i = 1, 2$ and

$$\mathcal{S}_i^n \cap \mathcal{H} = \left\{ K_i v_i(\theta_i) \mid -\phi_i \leq \theta_i \leq \frac{\pi}{2} \text{ and } K_i > 0 \right\}. \quad (4.58)$$

- ii) $z_i(t)$ is a transitive trajectory if and only if $z_i(0) = K_i v_i(\theta_i)$ where $\frac{\pi}{2} < \theta_i < \pi - \phi_i$ and $K_i > 0$, for $i = 1, 2$ and

$$\mathcal{S}_i^t \cap \mathcal{H} = \left\{ K_i v_i(\theta_i) \mid \frac{\pi}{2} < \theta_i < \pi - \phi_i \text{ and } K_i > 0 \right\}. \quad (4.59)$$

Proof 4.1: The proof is based on the fact that items i.b and ii.b of Theorem 4.1 are not applicable to the trajectories starting from \mathcal{H} , because $\frac{d}{dt} f_i(t) |_{t=0} < 0$ and therefore smooth continuation into \mathcal{S}_i is not possible. Furthermore, the interval in item ii.a of

Theorem 4.1 reduces to $\frac{\pi}{2} < \theta_1 < \pi - \phi_i$ because of the same reason. The descriptions for $\mathcal{S}_i^n \cap \mathcal{H}$ and $\mathcal{S}_i^t \cap \mathcal{H}$ obviously follows. \square

In view of the above result, let $\mathcal{L}(x)$ denote the half line along the direction of vector x .

Definition 4.4: Note that the interval $-\phi_1 \leq \theta_1 \leq \frac{\pi}{2}$ describes a closed convex cone in $\mathcal{S}_1 \cap \mathcal{H}$ bounded by the half-lines $\mathcal{L}(\hat{r}_2)$ and \mathcal{L}^+ . This cone will be called the invariant cone of $\mathcal{S}_1 \cap \mathcal{H}$ and denoted as \mathcal{C}^+ . Similarly, $-\phi_2 \leq \theta_2 \leq \frac{\pi}{2}$ describes a closed convex cone in $\mathcal{S}_2 \cap \mathcal{H}$ bounded by the half-lines $\mathcal{L}(\hat{s}_2)$ and \mathcal{L}^- . This cone will be called the invariant cone of $\mathcal{S}_2 \cap \mathcal{H}$ and denoted as \mathcal{C}^- .

Remark 4.2: Suppose that $z_i(t)$ be a transitive trajectory starting from $\mathcal{S}_i \cap \mathcal{H}$, and $\tau_i > 0$ such that $c^T z_i(\tau_i) = 0$. Then, simultaneously solving equations $f_i(\tau_i) = 0$ and $f_i(0) = 0$ for each case of multiplicity of eigenvalues, τ_i can be expressed as an implicit function of θ_i as given by the following equations. If $i = 1$,

$$-\cot \theta_1 = \frac{e^{(\lambda_2 - \lambda_1)\tau_1} - 1}{e^{(\lambda_3 - \lambda_1)\tau_1} - 1} \text{ if } \lambda_1 < \lambda_2 < \lambda_3, \quad (4.60)$$

$$-\cot \theta_1 = \frac{\tau_1}{e^{(\lambda_3 - \lambda_1)\tau_1} - 1} \text{ if } \lambda_1 = \lambda_2 < \lambda_3, \quad (4.61)$$

$$-\cot \theta_1 = \frac{1 - e^{(\lambda_1 - \lambda_3)\tau_1}}{\tau_1} \text{ if } \lambda_1 < \lambda_2 = \lambda_3, \quad (4.62)$$

$$-\cot \theta_1 = \frac{2}{\tau_1} \text{ if } \lambda_1 = \lambda_2 = \lambda_3 \quad (4.63)$$

and if $i = 2$,

$$-\cot \theta_2 = \frac{(e^{(\mu_2 - \mu_1)\tau_2} - 1)}{(e^{(\mu_3 - \mu_1)\tau_2} - 1)} \text{ if } \mu_1 < \mu_2 < \mu_3 \quad (4.64)$$

$$-\cot \theta_2 = \frac{\tau_2}{e^{(\mu_3 - \mu_1)\tau_2} - 1} \text{ if } \mu_1 = \mu_2 < \mu_3 \quad (4.65)$$

$$-\cot \theta_2 = \frac{1 - e^{(\mu_1 - \mu_3)\tau_2}}{\tau_2} \text{ if } \mu_1 < \mu_2 = \mu_3 \quad (4.66)$$

$$-\cot \theta_2 = \frac{2}{\tau_2} \text{ if } \mu_1 = \mu_2 = \mu_3 \quad (4.67)$$

Note that the equations above also imply that $\tau_i \rightarrow 0$ as $\theta_i \rightarrow \pi - \phi_i$ and $\tau_i \rightarrow \infty$ as $\theta_i \rightarrow \frac{\pi}{2}$. This is expected since at $\theta_i = \frac{\pi}{2}$ the initial condition is on the line $\mathcal{L}(\hat{r}_2)$ if

$i = 1$ and on the line $\mathcal{L}(\hat{s}_2)$ if $i = 2$ and the trajectories starting from these lines are non-transitive ($\tau_i = \infty$). On the other hand, at the limit as $\theta_i \rightarrow \pi - \phi_i$, the initial condition is on the line \mathcal{L}^- if $i = 1$ and on the line \mathcal{L}^+ if $i = 2$ and smooth continuation into \mathcal{S}_i is not possible ($\tau_i = 0$).

Lemma 4.3: Let $z_i(t)$ be a transitive trajectory starting from $\mathcal{S}_i \cap \mathcal{H}$ with initial condition $z_1(0) = \hat{r}_2 \sin \theta_1 + \hat{r}_3 \cos \theta_1$ for $i = 1$ and $z_2(0) = \hat{s}_2 \sin \theta_2 + \hat{s}_3 \cos \theta_2$ for $i = 2$. Then, at $t = \tau_i$ where the trajectory changes mode, the following hold.

i)

$$\mathbf{z}_1(\tau_1) = e^{\lambda_3 \tau_1} N_1 \left\{ \hat{\mathbf{r}}_2 \sin \hat{\theta}_1 + \hat{\mathbf{r}}_3 \cos \hat{\theta}_1 \right\} \quad (4.68)$$

$$\mathbf{z}_2(\tau_2) = e^{\mu_3 \tau_2} N_2 \left\{ \hat{\mathbf{s}}_2 \sin \hat{\theta}_2 + \hat{\mathbf{s}}_3 \cos \hat{\theta}_2 \right\} \quad (4.69)$$

where

$$N_1 := \sqrt{e^{2(\lambda_2 - \lambda_3)\tau_1} \sin^2 \theta_1 + \cos^2 \theta_1}, \text{ if } \lambda_1 < \lambda_2 < \lambda_3 \text{ or } \lambda_1 = \lambda_2 < \lambda_3, \quad (4.70)$$

$$N_1 := \sqrt{e^{2(\lambda_1 - \lambda_3)\tau_1} \sin^2 \theta_1 + \cos^2 \theta_1}, \text{ if } \lambda_1 < \lambda_2 = \lambda_3, \quad (4.71)$$

$$N_1 := \sqrt{(-\sin \theta_1)^2 + (\cos \theta_1)^2} = 1, \text{ if } \lambda_1 = \lambda_2 = \lambda_3, \quad (4.72)$$

$$N_2 := \sqrt{e^{2(\mu_2 - \mu_3)\tau_2} \sin^2 \theta_2 + \cos^2 \theta_2}, \text{ if } \mu_1 < \mu_2 < \mu_3 \text{ or } \mu_1 = \mu_2 < \mu_3, \quad (4.73)$$

$$N_2 := \sqrt{e^{2(\mu_1 - \mu_3)\tau_2} \sin^2 \theta_2 + \cos^2 \theta_2}, \text{ if } \mu_1 < \mu_2 = \mu_3, \quad (4.74)$$

$$N_2 := \sqrt{(-\sin \theta_2)^2 + (\cos \theta_2)^2} = 1, \text{ if } \mu_1 = \mu_2 = \mu_3. \quad (4.75)$$

and $\hat{\theta}_1$ is given as

$$\cot \hat{\theta}_1 = e^{(\lambda_3 - \lambda_2)\tau_1} \cot \theta_1, \text{ if } \lambda_1 < \lambda_2 < \lambda_3 \text{ or } \lambda_1 = \lambda_2 < \lambda_3, \quad (4.76)$$

$$\cot \hat{\theta}_1 = e^{(\lambda_3 - \lambda_1)\tau_1} \cot \theta_1, \text{ if } \lambda_1 < \lambda_2 = \lambda_3, \quad (4.77)$$

$$\cot \hat{\theta}_1 = -\cot \theta_1, \text{ if } \lambda_1 = \lambda_2 = \lambda_3. \quad (4.78)$$

and $\hat{\theta}_2$ satisfy the following equations.

$$\cot \hat{\theta}_2 = e^{(\mu_3 - \mu_2)\tau_2} \cot \theta_2, \text{ if } \mu_1 < \mu_2 < \mu_3 \text{ or } \mu_1 = \mu_2 < \mu_3, \quad (4.79)$$

$$\cot \hat{\theta}_2 = e^{(\mu_3 - \mu_1)\tau_2} \cot \theta_2, \text{ if } \mu_1 < \mu_2 = \mu_3, \quad (4.80)$$

$$\cot \hat{\theta}_2 = -\cot \theta_2, \text{ if } \mu_1 = \mu_2 = \mu_3. \quad (4.81)$$

ii) τ_i and $\hat{\theta}_i$ are decreasing functions of θ_i . Furthermore, transitive trajectories of each mode hit \mathcal{H} and change mode in the open conic regions bounded by the lines \mathcal{L}^- and $\mathcal{L}(-\hat{r}_2)$ for the first mode and \mathcal{L}^+ and $\mathcal{L}(-\hat{s}_2)$ for the second mode. More precisely, the following hold.

- a) $\hat{\theta}_i \rightarrow \pi - \phi_i$ as $\theta_i \rightarrow \pi - \phi_i$ (equivalently as $\tau_i \rightarrow 0$) for all possible algebraic degrees of the eigenvalues,
- b) $\hat{\theta}_i \rightarrow \frac{3\pi}{4}$ as $\theta_i \rightarrow \frac{\pi}{2}$, (equivalently as $\tau_i \rightarrow \infty$) if $\lambda_1 < \lambda_2 < \lambda_3$,
- c) $\hat{\theta}_i \rightarrow \pi$ as $\theta_i \rightarrow \frac{\pi}{2}$, (equivalently as $\tau_i \rightarrow \infty$) if $\lambda_1 = \lambda_2 < \lambda_3$ or if $\lambda_1 < \lambda_2 = \lambda_3$,
- d) $\hat{\theta}_i \rightarrow \frac{3\pi}{2}$ as $\theta_i \rightarrow \frac{\pi}{2}$, (equivalently as $\tau_i \rightarrow \infty$) if $\lambda_1 = \lambda_2 = \lambda_3$.

Proof 4.3: The proof of the Lemma is given in Appendix. □

4.1.5. First Main Result

First main result of this chapter is about BPLS in \mathbb{R}^3 where both of the modes have only real eigenvalues. This result is based on the classification of the trajectories of BPLS in \mathbb{R}^3 given above.

Theorem 4.2: Consider the bimodal system given by equations (1.1) - (4.9) with Assumptions 1-2. Then, the following hold.

- i) All the trajectories change mode only finite number of times.
- ii) Bimodal system is GAS if and only if all the real eigenvalues of both modes are negative.

Proof 4.2: Without loss of generality, the proof is given only for the trajectories starting from $\mathcal{S}_1 \cap \mathcal{H}$.

- i) It was shown above that the trajectories starting from \mathcal{C}^+ do not change mode. The trajectories starting from outside of \mathcal{C}^+ are transitive. If $-\hat{r}_2$ is in \mathcal{C}^- , then in view of Lemma 4.3 all transitive trajectories starting from outside of \mathcal{C}^+ hit \mathcal{C}^- and stay in \mathcal{C}^- without changing mode any more. If $-\hat{r}_2$ is outside of \mathcal{C}^- , then some of the transitive trajectories starting from $\mathcal{S}_1 \cap \mathcal{H}$ hit outside of \mathcal{C}^- . Since the trajectories starting from outside of \mathcal{C}^- are transitive, they continue in \mathcal{S}_2 and then hit $\mathcal{S}_1 \cap \mathcal{H}$ and change mode. Since $-\hat{r}_2$ is outside of \mathcal{C}^- , it follows that $-\hat{s}_2$ and $-\hat{s}_3$ are in \mathcal{C}^+ . Consequently, these trajectories stay in \mathcal{S}_1 without changing mode any more. In summary, all the trajectories change mode finite number of times.
- ii) Since all trajectories change mode finite number of times, it follows that after some finite time $T \geq 0$ all trajectories enter either \mathcal{C}^+ or \mathcal{C}^- and stay in \mathcal{S}_1 or \mathcal{S}_2 , respectively for all $t \geq T$. Therefore, bimodal systems is GAS if and only if real eigenvalues of both modes are negative (stable). \square

4.2. One Mode with Complex Eigenvalues

In this section, BPLS in \mathbb{R}^3 where one of the modes have a pair of complex eigenvalues, are considered. Assumptions 1-2 hold and it is further assumed without loss of generality that

Assumption 3: the eigenvalues of A_1 are $\{\lambda_1, \sigma_1 \pm j\omega_1\}$ where λ_1 , σ_1 and ω_1 are real numbers such that $\omega_1 > 0$ and the eigenvalues of A_2 are $\{\mu_1, \mu_2, \mu_3\}$ where μ_1, μ_2 , and μ_3 are real numbers such that $\mu_1 \leq \mu_2 \leq \mu_3$.

The reason why additional assumptions are needed is explained in detail in Remark 4.7.

The geometry of initial conditions and well-posedness conditions are same as in the previous section. Suppose that the eigenvectors of A_1 are $\{r_1, x_1 \pm j y_1\}$. Then, using the following equation

$$A_1 [r_1 \ x_1 \ y_1] = [r_1 \ x_1 \ y_1] \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \sigma_1 & \omega_1 \\ 0 & -\omega_1 & \sigma_1 \end{bmatrix}, \quad (4.82)$$

the eigenvectors can be uniquely chosen as follows (Lemma 2 in [Eldem and Şahan, 2014]).

$$\mathbf{r}_1 = \begin{bmatrix} \frac{(a_{11}-\sigma_1)^2 + \omega_1^2 + a_{12}a_{21}}{a_{32}a_{21}} \\ \frac{\lambda_1 - a_{33}}{a_{32}} \\ 1 \end{bmatrix}, \mathbf{x}_1 = \begin{bmatrix} \frac{(a_{11}-\lambda_1)(a_{11}-\sigma_1) + a_{12}a_{21}}{a_{32}a_{21}} \\ \frac{\sigma_1 - a_{33}}{a_{32}} \\ 1 \end{bmatrix}, \quad (4.83)$$

$$\mathbf{y}_1 = \begin{bmatrix} \frac{\omega_1(a_{11}-\lambda_1)}{a_{32}a_{21}} \\ \frac{\omega_1}{a_{32}} \\ 0 \end{bmatrix}. \quad (4.84)$$

Let $z_1(t)$ denote the trajectories starting from \mathcal{S}_1 and smoothly continuing to \mathcal{S}_1 . Then, the behavior of such trajectories in the first mode can be written as

$$z_1(t) = K_1 e^{\lambda_1 t} \left\{ \alpha_1 r_1 + e^{(\sigma_1 - \lambda_1)t} [\sin(\omega_1 t + \theta_1) x_1 + \cos(\omega_1 t + \theta_1) y_1] \right\}, \quad (4.85)$$

where $K_1 > 0$ is a real constant. For any initial condition in \mathcal{S}_1 , we must have $c^T z_1(0) \geq 0$. Since

$$c^T z_1(t) = K_1 e^{\lambda_1 t} \left\{ \alpha_1 + e^{(\sigma_1 - \lambda_1)t} \sin(\omega_1 t + \theta_1) \right\}, \quad (4.86)$$

it follows that $c^T z_1(0) = \alpha_1 + \sin \theta_1 \geq 0$. The properties of trajectories starting from \mathcal{S}_1 are investigated in detail in [Eldem and Şahan, 2014]. These properties are summarized in the following Lemma.

Lemma 4.4: Suppose that $(\sigma_1 - \lambda_1) > 0$. Then, the following hold for the trajectories starting from \mathcal{S}_1 .

- i) All trajectories starting from \mathcal{S}_1 are transitive except the ones with initial condition $z_1(0) = K_1 r_1$ where $K_1 > 0$ is a real number. In particular, all trajectories starting from $\mathcal{S}_1 \cap \mathcal{H}$ are transitive (Lemma (4.1) in [Eldem and Şahan, 2014]).*

ii) For a trajectory starting from $S_1 \cap \mathcal{H}$, $z_1(0)$ can be written as $z_1(0) = K_1 v_1(\theta_1)$ where $v_1(\theta_1) := \hat{x}_1 \sin(\theta_1) + y_1 \cos(\theta_1)$, $\hat{x}_1 := (x_1 - r_1)$, $-\phi_1 \leq \theta_1 < \pi - \phi_1$, and

$$b_1 := \cot \phi_1 := \frac{\sigma_1 - \lambda_1}{\omega_1}, \quad (4.87)$$

(Lemma 3 in [Eldem and Şahan, 2014]).

iii) For a trajectory starting from $S_1 \cap \mathcal{H}$ with initial condition $z_1(0) = K_1 v_1(\theta_1)$, there exists $\tau_1 > 0$ such that $c^T z_1(\tau_1) = 0$ and the trajectory changes mode where

$$z_1(\tau_1) = e^{\sigma_1 \tau_1} K_1 v_1(\theta_1 + \omega_1 \tau_1) \text{ and } e^{(\sigma_1 - \lambda_1) \tau_1} \sin(\theta_1 + \omega_1 \tau_1) = \sin \theta_1. \quad (4.88)$$

Furthermore,

$$\text{if } -\phi_1 \leq \theta_1 \leq 0, \quad \text{then } \pi - \phi_1 \leq \theta_1 + \omega_1 \tau_1 \leq \pi, \quad (4.89)$$

$$\text{if } 0 \leq \theta_1 \leq \pi - \phi_1, \text{ then } \pi \leq \theta_1 + \omega_1 \tau_1 < \pi + \phi_1, \quad (4.90)$$

(Lemma 5.1 in [Eldem and Şahan, 2014] and Figure 4.1).

iv) Let $F_1(\theta_1) := \theta_1 + \omega_1 \tau_1$. Then,

a)

$$\frac{d\mathcal{F}_1}{d\theta_1} = \frac{\cot \theta_1 + b_1}{\cot(\theta_1 + \omega_1 \tau_1) + b_1} \leq 0, \quad (4.91)$$

where the equality holds only at $\theta_1 = -\phi_1$.

b) $F_1(\theta_1)$ is a nonexpansive map i.e.,

$$\sup_{-\phi_1 \leq \theta_1 < \pi - \phi_1} \left| \frac{d\mathcal{F}_1}{d\theta_1} \right| \leq 1 \text{ and } \lim_{\theta_1 \rightarrow \pi - \phi_1} \frac{d\mathcal{F}_1}{d\theta_1} = -1, \quad (4.92)$$

(Lemma 6.1 in [Eldem and Şahan, 2014]).

c) If $\theta_1 \in [-\phi_1, 0]$, then $\left| e^{b_1 \omega_1 \tau_1} \frac{d\mathcal{F}_1(\theta_1)}{d\theta_1} \right| \leq 1$ and if $\theta_1 \in [0, \pi - \phi_1]$, then $\left| e^{b_1 \omega_1 \tau_1} \frac{d\mathcal{F}_1(\theta_1)}{d\theta_1} \right| \geq 1$ where equalities hold only at $\theta_1 = 0$ (proof of Lemma 6.1 in [Eldem and Şahan, 2014]).

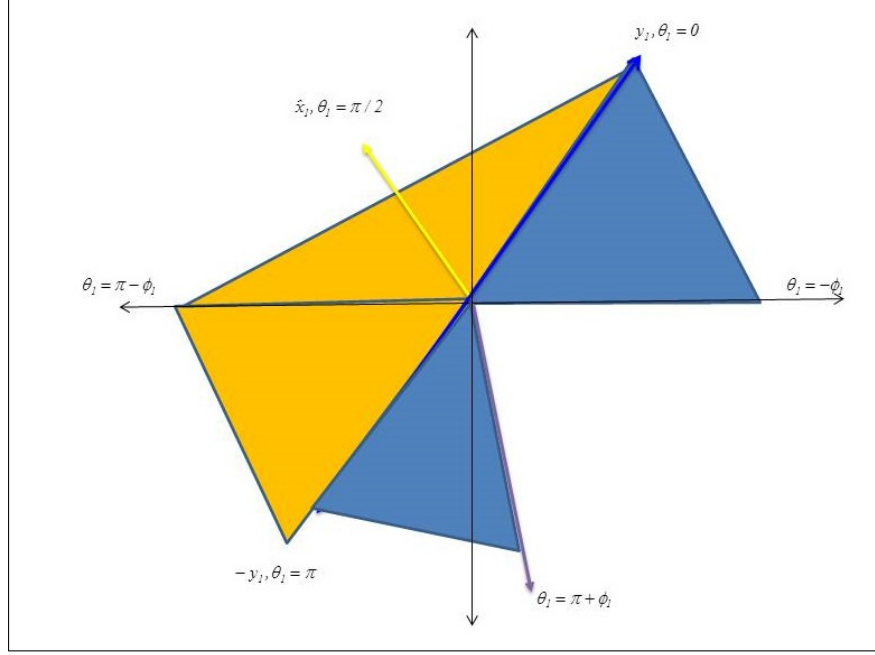


Figure 4.1: Transitive trajectories behavior in the first mode

Assumption 4: $\sigma_1 - \lambda_1 > 0$ for the first mode and $\mu_1 = \mu_2 = \mu_3$ for the second mode.

It is assumed that Assumptions 1 - 4 always hold in the rest of the dissertation.

Definition 4.5: $v_1(\theta_1)$ given in item ii. of Lemma 4.4 will be called directions in \mathcal{H} .

Recall that τ_1 and $\theta_1 + \omega_1 \tau_1$ given in Lemma 4.4 are functions of θ_1 . In order to simplify the notation, the following notation is used.

$$\hat{\theta}_1 := \theta_1 + \omega_1 \tau_1 \quad (4.93)$$

4.2.1. Mode Change

There are two different bases for \mathcal{H} , namely $\{\hat{x}_1, y_1\}$ and $\{\hat{s}_2, \hat{s}_3\}$. When a trajectory changes mode, one has to switch from one basis to the other in order to describe how the trajectory evolves after mode change. Suppose that $z_1(t)$ be a trajectory starting from $\mathcal{S}_1 \cap \mathcal{H}$ with initial condition

$$z_1(0) = v_1(\theta_1) = \hat{x}_1 \sin \theta_1 + y_1 \cos \theta_1, \quad (4.94)$$

where $-\phi_1 \leq \theta_1 < \pi - \phi_1$. In view of Lemma 4.4.iii, there exists $\tau_1 > 0$ such that $c^T z_1(\tau_1) = 0$ and the trajectory changes mode. Using Lemma 4.4-iii, it follows that

$$z_1(\tau_1) = e^{\sigma_1 \tau_1} v_1(\theta_1 + \omega_1 \tau_1). \quad (4.95)$$

Similarly, if $z_2(t)$ is a transitive trajectory starting from $\mathcal{S}_2 \cap \mathcal{H}$ with initial condition $z_2(0) = v_2(\theta_2)$ where $v_2(\theta_2) = \hat{s}_2 \sin \theta_2 + \hat{s}_3 \cos \theta_2$, then Corollary 4.1 implies that $\frac{\pi}{2} < \theta_2 < \pi$ and there exists $\tau_2 > 0$ such that $c^T z_2(\tau_2) = 0$. Using Lemma 4.3-i, and noting that $\mu_1 = \mu_2 = \mu_3$ one gets

$$z_2(\tau_2) = e^{\mu_3 \tau_2} N_2 \left\{ \hat{s}_2 \sin \hat{\theta}_2 + \hat{s}_3 \cos \hat{\theta}_2 \right\} = e^{\mu_3 \tau_2} \left\{ -\hat{s}_2 \sin \theta_2 + \hat{s}_3 \cos \theta_2 \right\}, \quad (4.96)$$

where

$$v_2(\hat{\theta}_2) = \hat{s}_2 \sin \hat{\theta}_2 + \hat{s}_3 \cos \hat{\theta}_2. \quad (4.97)$$

The following result describes a trajectory during its first and second mode changes.

Lemma 4.5: Let $z_1(t)$ be a trajectory starting from $\mathcal{S}_1 \cap \mathcal{H}$ with initial condition $z_1(0) = v_1(\theta_1)$ where $-\phi_1 \leq \theta_1 < \pi - \phi_1$. Also let $B := \frac{(\lambda_1 - a_{11} - \mu_1 a_{21})}{\omega_1(1+b_1^2)}$. Then, at $t = \tau_1$ the trajectory changes mode and we have

i)

$$z_1(\tau_1) = \eta_{12}(\theta_1 + \omega_1 \tau_1) e^{\sigma_1 \tau_1} (\hat{s}_2 \sin \theta_2 + \hat{s}_3 \cos \theta_2), \quad (4.98)$$

where

$$\theta_2 = \cot^{-1} \left\{ -\frac{\frac{\omega_1(b_1^2+1)}{a_{21}} \left\{ 1 + (\cot \hat{\theta}_1 + b_1) B \right\}}{(\cot \hat{\theta}_1 + b_1)} \right\}, \quad (4.99)$$

and

$$\eta_{12}(\hat{\theta}_1) = -\frac{\omega_1 (\cos \hat{\theta}_1 + b_1 \sin \hat{\theta}_1)}{a_{32} \sin \theta_2}. \quad (4.100)$$

ii) Let $z_2(t)$ be a trajectory starting from $\mathcal{S}_2 \cap \mathcal{H}$ with initial condition $z_2(0) = \eta_{12}(\hat{\theta}_1) e^{\sigma_1 \tau_1} v_2(\theta_2)$. If $z_2(t)$ is a transitive trajectory ($\frac{\pi}{2} < \theta_2 < \pi$) (or equivalently out side of the cone \mathcal{C}^-), then

$$z_2(\tau_2) = \eta_{12}(\hat{\theta}_1) e^{\mu_3 \tau_2} e^{\sigma_1 \tau_1} \eta_{21}(\theta_2) (\hat{x}_1 \sin \theta_{11} + y_1 \cos \theta_{11}), \quad (4.101)$$

where

$$\eta_{21}(\theta_2) = \frac{-\frac{\alpha_{32}}{\omega_1} \sin \theta_2}{(\cos \theta_{11} + b_1 \sin \theta_{11})}, \quad (4.102)$$

and

$$\theta_{11} = \cot^{-1} \left\{ \frac{-\left(\cot \hat{\theta}_1 + b_1\right)}{\left(2B \left(\cot \hat{\theta}_1 + b_1\right) + 1\right)} - b_1 \right\}. \quad (4.103)$$

iii) Furthermore, $-\phi_1 < \theta_{11} < -\psi_1$ where $\cot \psi_1 = \frac{1}{B} + b_1$ and $-\hat{s}_2 \simeq v_1(-\psi_1)$.

Proof 4.5: The proof of the Lemma is given in Appendix. □

4.2.2. Stability

In view of the preceding result, an initial condition $z_1(0) = v_1(\theta_1)$ in $\mathcal{S}_1 \cap \mathcal{H}$ is transformed to

$$e^{\mu_3 \tau_2} e^{\sigma_1 \tau_1} \eta_{21}(\theta_2) \eta_{12}(\hat{\theta}_1) (\hat{x}_1 \sin \theta_{11} + y_1 \cos \theta_{11}), \quad (4.104)$$

with the following steps.

$$v_1(\theta_1) \rightarrow e^{\sigma_1 \tau_1} v_1(\hat{\theta}_1) \rightarrow \eta_{12}(\hat{\theta}_1) e^{\sigma_1 \tau_1} v_2(\theta_2) \quad (4.105)$$

$$\rightarrow \eta_{12}(\hat{\theta}_1) e^{\mu_3 \tau_2} e^{\sigma_1 \tau_1} v_2(\hat{\theta}_2) \quad (4.106)$$

$$\rightarrow e^{\mu_3 \tau_2} e^{\sigma_1 \tau_1} \eta_{21}(\theta_2) \eta_{12}(\hat{\theta}_1) v_1(\theta_{11}) \quad (4.107)$$

where $N_2 = 1$. Since change of basis is continuous, it follows that all the maps described above are continuous. Therefore,

$$\mathcal{T}_1 : \theta_1 \rightarrow \theta_{11}; \mathcal{T}_1(\theta_1) := \theta_{11}, \quad (4.108)$$

is also a continuous map, which can be interpreted as Poincare full map from $[-\phi_1, \pi - \phi_1) (\mathcal{S}_1 \cap \mathcal{H})$ into $[-\phi_1, \pi - \phi_1) (\mathcal{S}_1 \cap \mathcal{H})$. Note that this map can be defined only for trajectories which change mode at least two times. For infinitely mode changing trajectories we define $\mathcal{T}_1^k(\theta_1)$ as follows.

$$\mathcal{T}_1^k(\theta_1) := \mathcal{T}_1(\mathcal{T}_1(\dots\mathcal{T}_1(\theta_1)\dots)) \text{ (k times)}, \quad (4.109)$$

and

$$\mathcal{T}_1^k : \theta_1 \rightarrow \theta_{1k}; \mathcal{T}_1^k(\theta_1) := \theta_{1k}. \quad (4.110)$$

Using similar reasoning, a transitive trajectory with initial condition $z_2(0) = v_2(\theta_2)$ in $\mathcal{S}_2 \cap \mathcal{H}$ is transformed to

$$e^{\mu_3\tau_2} e^{\sigma_1\tau_1} \eta_{21}(\theta_2) \eta_{12}(\hat{\theta}_1) v_2(\hat{\theta}_2) \quad (4.111)$$

with the following steps.

$$v_2(\theta_2) \rightarrow e^{\mu_3\tau_2} v_2(\hat{\theta}_2) \rightarrow \eta_{21}(\theta_2) e^{\mu_3\tau_2} v_1(\theta_1) \quad (4.112)$$

$$\rightarrow \eta_{21}(\theta_2) e^{\mu_3\tau_2} e^{\sigma_1\tau_1} v_1(\hat{\theta}_1) \quad (4.113)$$

$$\rightarrow e^{\mu_3\tau_2} e^{\sigma_1\tau_1} \eta_{21}(\theta_2) \eta_{12}(\hat{\theta}_1) v_2(\theta_{21}) \quad (4.114)$$

Therefore, \mathcal{T}_2 can be defined as follows

$$\mathcal{T}_2 : \theta_2 \rightarrow \theta_{21}; \mathcal{T}_2(\theta_2) := \theta_{21} \quad (4.115)$$

which is also a continuous map from $[-\phi_2, \pi - \phi_2) (\mathcal{S}_2 \cap \mathcal{H})$ into $[-\phi_2, \pi - \phi_2) (\mathcal{S}_1 \cap \mathcal{H})$. Again, this map can be defined only for trajectories which change mode at least two times. For trajectories which change mode infinite number of times as $t \rightarrow \infty$, we similarly define $\mathcal{T}_2^k(\theta_2)$ as follows.

$$\mathcal{T}_2^k(\theta_2) := \mathcal{T}_2(\mathcal{T}_2(\cdots \mathcal{T}_2(\theta_2) \cdots)) \text{ (k times),} \quad (4.116)$$

and

$$\mathcal{T}_2 : \theta_2 \rightarrow \theta_{2k}; \quad \mathcal{T}_2^k(\theta_2) := \theta_{2k}. \quad (4.117)$$

The following definitions are necessary to be consistent with the terminology used in [Eldem and Şahan, 2014].

Definition 4.6: Let θ_i^* be a fixed point of $\mathcal{T}_i(\theta_i)$ or equivalently $\mathcal{T}_i(\theta_i^*) = \theta_i^*$ for $i = 1, 2$. Then, $v_i(\theta_i^*)$ is called a fixed direction. A fixed direction $v_i(\theta_i^*)$ is said to be attractive in an interval I_i containing θ_i^* , if for any θ_i in I_i and for every $\varepsilon > 0$, there exists a positive integer k such that $|\mathcal{T}_i^k(\theta_i) - \theta_i^*| < \varepsilon$ for $i = 1, 2$. If I_i consists of only one point θ_i^* , then the fixed direction is said to be repulsive.

In view of this definition, let $z_i^*(t)$ be a trajectory starting from a fixed direction $v_i(\theta_i^*)$. To be more specific, $z_i^*(0) = K_i v_i(\theta_i^*)$ where $K_i > 0$ is a real constant. Then, since $\mathcal{T}_i(\theta_i^*) = \theta_i^*$, in view of Lemma 4.5, $K_i v_i(\theta_i^*)$ is mapped as follows

$$K_i v_i(\theta_i^*) \rightarrow \gamma^*(\theta_i^*) K_i v_i(\theta_i^*), \quad (4.118)$$

where

$$\gamma^*(\theta_1^*) := e^{\mu\tau_2^*} e^{\sigma_1\tau_1^*} \eta_{21}(\theta_2^*) \eta_{12}(\theta_1^* + \omega_1\tau_1^*). \quad (4.119)$$

Definition 4.7: In the sequel, $\gamma^*(\theta_i^*)$ will be called convergence rate as in [Eldem and Şahan, 2014].

It is clear from the development above that if $z_1^*(t)$ is a trajectory starting from a fixed direction $v_1(\theta_1^*)$, then there exists a unique fixed direction in $\mathcal{S}_2 \cap \mathcal{H}$ such that

$$z_1^*(\tau_1^*) = \eta_{12} (\theta_1^* + \omega_1 \tau_1^*) e^{\sigma_1 \tau_1^*} v_2(\theta_2^*). \quad (4.120)$$

In view of the definition of $\mathcal{T}_1(\theta_1^*)$, $z_1^*(\tau_1^*)$ is necessarily in the transitive region of $\mathcal{S}_2 \cap \mathcal{H}$ or equivalently is outside the cone \mathcal{C}^- . This means that fixed directions exist as a pair (θ_1^*, θ_2^*) . Therefore, we use the following notation for the convergence rate.

$$\gamma^*(\theta_1^*) = \gamma^*(\theta_1^*, \theta_2^*). \quad (4.121)$$

In order to complete the picture, we also define below attractiveness of invariant cone \mathcal{C}^- of $\mathcal{S}_2 \cap \mathcal{H}$ given in Definition 4.4.

Definition 4.8: \mathcal{C}^- is said to be attractive in an interval I_2 containing \mathcal{C}^- , if for any $\theta_2 \in I_2$ there exists a finite nonnegative integer k such that $0 \leq \mathcal{T}_2^k(\theta_2) \leq \frac{\pi}{2}$ or $\mathcal{T}_2^k(\theta_2) \rightarrow \frac{\pi}{2}$ as $k \rightarrow \infty$. In the latter case, the boundary $\mathcal{L}(v_2(\frac{\pi}{2}))$ will also be called degenerate fixed direction. If $I_2 = [0, \frac{\pi}{2}]$, then \mathcal{C}^- is said to be repulsive.

Lemma 4.6: Let $v_i(\theta_i^*)$ be a fixed direction which is attractive in an interval I_i . Then, the following hold.

- i) If $z_i^*(t)$ is a trajectory starting from the fixed direction $v_i^*(\theta_i^*)$, then $z_i^*(t) \rightarrow 0$ as $t \rightarrow \infty$ if and only if $\gamma^*(\theta_1^*, \theta_2^*) < 1$ for $i = 1, 2$.
- ii) Suppose that $v_i(\theta_i^*)$ is attractive in an interval I_i containing $\mathbf{v}_i(\theta_i^*)$. If $z_i(t)$ is a trajectory starting from $\mathcal{S}_i \cap \mathcal{H}$ with initial condition $z_i(0) = K_i v_i(\theta_i)$ where $\theta_i \in I_i$, then $z_i(t) \rightarrow 0$ as $t \rightarrow \infty$ if and only if $\gamma^*(\theta_1^*, \theta_2^*) < 1$ for $i = 1, 2$.
- iii) Suppose that \mathcal{C}^- is attractive in an interval I_2 containing \mathcal{C}^- . Let $z_2(0) = v_2(\theta_2)$ where $\theta_2 \in I_2$. Then $z_2(t) \rightarrow 0$ as $t \rightarrow \infty$ if and only if $\mu < 0$.

Proof 4.6:

- i) Let $z_i^*(0) = K_i v_i(\theta_i^*)$. Since $\mathcal{T}_i(\theta_i^*) = \theta_i^*$, it follows that after one loop around \mathcal{H} , $K_i v_i(\theta_i^*)$ is mapped to $\gamma^*(\theta_1^*, \theta_2^*) K_i v_i(\theta_i^*)$. Furthermore, for any integer $k > 0$, we have $\mathcal{T}_i^k(\theta_i^*) = \theta_i^*$. This implies that $K_i v_i(\theta_i^*)$ is mapped to $[\gamma^*(\theta_1^*, \theta_2^*)]^k K_i v_i(\theta_i^*)$ after k loops around \mathcal{H} . Consequently, as $t \rightarrow \infty$ we have $k \rightarrow \infty$ and $z_i^*(t) \rightarrow 0$ if and only if $\gamma^*(\theta_1^*, \theta_2^*) < 1$.

ii) The proof for the second part follows similar lines as the proof of Theorem 11 in [Eldem and Şahan, 2014] and it is based on the fact that, for any $\theta_i \in I_i$ and for every $\varepsilon > 0$, there exists a positive integer k such that $|\mathcal{T}_i^k(\theta_i) - \theta_i^*| < \varepsilon$ for $i = 1, 2$. The details are omitted here.

iii) If $0 \leq \mathcal{T}_2^k(\theta_2) \leq \frac{\pi}{2}$ for some finite integer $k > 0$, then Corollary 4.1.1 implies that $z_2(t)$ will always be in the cone \mathcal{C}^- . Consequently, $z_2(t) \rightarrow 0$ as $t \rightarrow \infty$ if and only if $\mu < 0$. Suppose that $\mathcal{T}_2^k(\theta_2) \rightarrow \frac{\pi}{2}$ as $k \rightarrow \infty$. Then, by Lemma 4.3.2.d $\tau_2 \rightarrow \infty$ as $\theta_2 \rightarrow \frac{\pi}{2}$. This implies that the time which the trajectory spends in \mathcal{S}_2 will increase without bound. Hence, $z_2(t) \rightarrow 0$ as $t \rightarrow \infty$ if and only if $\mu < 0$.

□

Corollary 4.2: A general expression for $\gamma^*(\theta_1^*)$ is

$$\gamma^*(\theta_1^*, \theta_2^*) = -e^{\mu_3 \tau_2^*} e^{\sigma_1 \tau_1^*} \frac{(\cos(\theta_1^* + \omega_1 \tau_1^*) + b_1 \sin(\theta_1^* + \omega_1 \tau_1^*))}{(\cos \theta_1^* + b_1 \sin \theta_1^*)}, \quad (4.122)$$

and if $\theta_1^* \neq 0$, the expression above can be written as

$$\gamma^*(\theta_1^*, \theta_2^*) = \frac{-e^{\mu_3 \tau_2^*} e^{\lambda_1 \tau_1^*}}{\frac{dF_1}{d\theta_1}}. \quad (4.123)$$

Proof 4.2: Using the definition of convergence rate we get

$$\gamma^*(\theta_1^*, \theta_2^*) := e^{\mu_3 \tau_2^*} e^{\sigma_1 \tau_1^*} \eta_{21}(\theta_2^*) \eta_{12}(\theta_1^* + \omega_1 \tau_1^*). \quad (4.124)$$

We substitute for $\eta_{21}(\theta_2^*)$ and $\eta_{12}(\theta_1^* + \omega_1 \tau_1^*)$ given in Lemma 4.5 and use Lemma 4.4.iv, we get

$$\gamma^*(\theta_1^*, \theta_2^*) = -e^{\mu_3 \tau_2} e^{\sigma_1 \tau_1} \frac{(\cos(\theta_1^* + \omega_1 \tau_1) + b_1 \sin(\theta_1^* + \omega_1 \tau_1))}{(\cos \theta_1^* + b_1 \sin \theta_1^*)}, \quad (4.125)$$

$$= -e^{\mu_3 \tau_2} e^{\sigma_1 \tau_1} \frac{\sin(\theta_1^* + \omega_1 \tau_1)}{\sin \theta_1^*} \frac{1}{\frac{dF_1}{d\theta_1}}, \quad (4.126)$$

$$= \frac{-e^{\mu_3 \tau_2} e^{\lambda_1 \tau_1}}{\frac{dF_1}{d\theta_1}}. \quad (4.127)$$

□

Lemma 4.7: The derivative of θ_{11} with respect to θ_1 is positive in domain of \mathcal{T}_1 and

$$\frac{d\theta_{11}}{d\theta_1} = \frac{d\mathcal{T}_1}{d\theta_1} = -\frac{dF_1}{d\theta_1} \left(\frac{\sin \theta_{11}}{\sin \hat{\theta}_1} \right)^2 \left(\frac{1}{\left(2B \left(\cot \hat{\theta}_1 + b_1 \right) + 1 \right)} \right)^2. \quad (4.128)$$

Proof 4.7: Calculating the derivative of both sides of equation (4.103), one gets

$$-\frac{1}{\sin^2 \theta_{11}} \frac{d\theta_{11}}{d\theta_1} = \frac{\frac{dF_1}{d\theta_1} \frac{1}{(\sin \hat{\theta}_1)^2}}{\left(2B \left(\cot \hat{\theta}_1 + b_1 \right) + 1 \right)^2}. \quad (4.129)$$

This implies that equation (4.128) holds. Since $\frac{dF_1}{d\theta_1} < 0$, it follows that $\frac{d\theta_{11}}{d\theta_1} > 0$. \square

4.2.3. Second Main Result

In this section three different geometric structures, which arise by changing the sign of the coupling constant B , are considered as explained below.

Note that the tangent of the angle between $\mathcal{L}(s_2)$ and \mathcal{L}^- is $-1/\mu$, whereas the tangent of the angle between $\mathcal{L}(y_1)$ and \mathcal{L}^+ is $\frac{a_{21}}{(a_{11}-\lambda_1)}$. Thus, if $B < 0$ then $\lambda_1 - a_{11} - \mu a_{21} < 0$ and $\frac{a_{21}}{(a_{11}-\lambda_1)} < \frac{-1}{\mu}$. This implies that $-y_1$ is in the non-transitive cone \mathcal{C}^- . Similarly, if $B > 0$, then $\lambda_1 - a_{11} - \mu a_{21} > 0$ and this implies that $-y_1$ is outside the non-transitive cone \mathcal{C}^- . For the case $B = 0$, it follows that y_1 and s_2 are on the same line. Thus, B determines how the modes are coupled on \mathcal{H} . The three different geometric structures are given as follows.

- $-y_1$ is in the cone \mathcal{C}^- ($B < 0$).
- y_1 and s_2 are on the same line ($B = 0$).
- $-y_1$ is outside the cone \mathcal{C}^- ($B > 0$).

Definition 4.9: The constant defined as

$$B := \frac{(\lambda_1 - a_{11} - \mu a_{21})}{\omega_1 (1 + b_1^2)} \quad (4.130)$$

will be called the coupling constant in the remaining part of the thesis.

Theorem 4.3: Given the bimodal system Σ_0 with Assumptions 1-4 the following hold.

i) If $B \leq 0$, then the cone \mathcal{C}^- is attractive in the interval $[0, \pi)$ or equivalently in $\mathcal{S}_2 \cap \mathcal{H}$. The system is GAS if and only if all the real eigenvalues of both modes are negative.

ii) If $B > 0$, then there exists a unique $\bar{\theta}_1 \in [0, \pi - \phi_1)$ such that the trajectories with initial condition $z_1(0) = K_1 v_1(\theta_1)$, where $\theta_1 \geq \bar{\theta}_1$, change mode once and enter the cone \mathcal{C}^- . Furthermore, the following hold.

a) There exists a unique pair of fixed directions $v_1(\theta_1^)$ and $v_2(\theta_2^*)$, where $\theta_1^* \in (-\phi_1, -\psi_1)$ and $\theta_2^* \in (\frac{\pi}{2}, \pi)$, which are attractive in $[-\phi_1, \bar{\theta}_1)$ and $(\frac{\pi}{2}, \pi)$, respectively. The transitive cone \mathcal{C}^- is repulsive.*

b) The system is GAS if and only if real eigenvalues of both modes are negative and $\gamma^(\theta_1^*, \theta_2^*) < 1$.*

Proof 4.3: Consider a trajectory starting from $\mathcal{S}_1 \cap \mathcal{H}$.

i) If $B < 0$, then $v_1(\pi) = -y_1 \in \mathcal{C}^-$. Suppose that $z_1(0) = v_1(\theta_1)$ where $\theta_1 \in [0, \pi - \phi_1)$. Then, by Lemma 4.4.iii, $z_1(t)$ changes mode at $t = \tau_1$ where $\hat{\theta}_1 \in (\pi - \phi_1, \pi]$. Since $v_1(\pi - \phi_1) \simeq v_2(0)$ by equation (B1. 14) and $v_1(\pi) = -y_1$ are in \mathcal{C}^- , it follows that $z_1(t)$ will enter the cone \mathcal{C}^- after one mode change and stay in \mathcal{S}_2 for all $t \geq \tau_1$.

On the other hand, suppose that $z_1(0) = v_1(\theta_1)$ where $\theta_1 \in [-\phi_1, 0)$. Then, since $\mathcal{F}_1 : \theta_1 \rightarrow \hat{\theta}_1$ is a continuous map, there exists $\bar{\theta}_1 \in [-\phi_1, 0)$ such that $v_1(\hat{\theta}_1(\bar{\theta}_1)) \simeq v_2(\frac{\pi}{2}) = s_2$. Consequently, since $\hat{\theta}_1$ is a decreasing function of θ_1 , any trajectory starting from $v_1(\theta_1)$ where $\theta_1 \geq \bar{\theta}_1$, will enter the cone \mathcal{C}^- after one mode change and stay in \mathcal{S}_2 for all $t \geq \tau_1$.

If $\theta_1 < \bar{\theta}_1$, and $\cot \theta_{11} + b_1 > 0$, then it follows that $\theta_{11} \in [0, \pi - \phi_1)$. Consequently, such trajectories will enter \mathcal{C}^- in the next mode change and stay there. Suppose that $\theta_1 < \bar{\theta}_1$ and $\cot \theta_{11} + b_1 < 0$ ($\theta_{11} \in [-\phi_1, 0)$). Since $\cot \hat{\theta}_1 + b_1 > 0$ by Lemma 4.4.iii and since $\cot \hat{\theta}_2 = -\cot \theta_2$ by Lemma 4.3.i, equations (B1. 14) and (B1. 16) imply that

$$\frac{\cot \theta_1 + b_1}{\cot \theta_{11} + b_1} = -\frac{(\cot \theta_1 + b_1) \left(2B \left(\cot \hat{\theta}_1 + b_1 \right) + 1 \right)}{\left(\cot \hat{\theta}_1 + b_1 \right)}, \quad (4.131)$$

$$= -\frac{d\mathcal{F}_1}{d\theta_1} \left(2B \left(\cot \hat{\theta}_1 + b_1 \right) + 1 \right). \quad (4.132)$$

In view of Lemma 4.5.ii, it follows that $\theta_{11} \in (-\phi_1, -\psi_1)$. Furthermore, since $\theta_1 \in [-\phi_1, 0)$, it also follows that both $\cot \theta_1 + b_1 < 0$ and $\cot \theta_{11} + b_1 < 0$ hold. Since $\frac{d\mathcal{F}_1}{d\theta_1} < 0$ and $B < 0$, then

$$0 < \left(2B \left(\cot \hat{\theta}_1 + b_1 \right) + 1 \right) < 1. \quad (4.133)$$

Since $\left| \frac{d\mathcal{F}_1}{d\theta_1} \right| < 1$, by Lemma 4.4.iv.b, the equation above implies that

$$0 \leq \frac{\cot \theta_1 + b_1}{\cot \theta_{11} + b_1} < 1 \Rightarrow |\cot \theta_{11} + b_1| > |\cot \theta_1 + b_1|, \quad (4.134)$$

$$\Rightarrow \cot \theta_{11} + b_1 < \cot \theta_1 + b_1 \rightarrow \theta_{11} > \theta_1. \quad (4.135)$$

Then, since $\mathcal{T}_1^k(\theta_1) = \theta_{1k}$, it follows that θ_{1k} increases towards $\bar{\theta}_1$ as k increases. Consequently, for some finite integer k , we get $v_1(\hat{\theta}_{1k}(\theta_{1k})) \in \mathcal{C}^-$. This implies that \mathcal{C}^- is attractive in $\mathcal{S}_2 \cap \mathcal{H}$ or equivalently attractive in the interval $[0, \pi)$. Therefore, bimodal system is GAS if and only if all the real eigenvalues of both modes are negative. On the other hand, if $B = 0$ then $v_1(\pi) = -y_1$ and $-y_1$ and s_2 are equivalent directions. As in the previous case, if $z_1(0) = v_1(\theta_1)$ where $\theta_1 \in [0, \pi - \phi_1)$, then $z_1(t)$ will enter the cone \mathcal{C}^- after one mode change and stay in \mathcal{S}_1 for all $t \geq \tau_1$. If $z_1(0) = v_1(\theta_1)$ where $\theta_1 \in [-\phi_1, 0)$, then $\hat{\theta}_1 \in (\pi, \pi + \phi_1)$ by item iii of Lemma 4.4 and this implies that $z_1(\tau_1)$ is outside \mathcal{C}^- . Consequently, in view of Lemma 4.5.i, there exists $\theta_2 \in (\frac{\pi}{2}, \pi)$ such that $z_1(\tau_1) = e^{\sigma_1 \tau_1} \eta_{12} \left(\hat{\theta}_1 \right) v_2(\theta_2)$. This implies that if $\theta_1 \in [-\phi_1, 0)$ then θ_1 is in the domain of \mathcal{T}_1 . Since $B = 0$, by equations (B1. 14) and (B1. 16), one gets

$$\frac{\cot \theta_1 + b_1}{\cot \theta_{11} + b_1} = -\frac{\cot \theta_1 + b_1}{\cot \hat{\theta}_1 + b_1} = -\frac{d\mathcal{F}_1}{d\theta_1}. \quad (4.136)$$

As $\left| \frac{d\mathcal{F}_1}{d\theta_1} \right| < 1$ by the Lemma 4.4.iv.b, it follows that

$$\cot \theta_{11} + b_1 < \cot \theta_1 + b_1 \Rightarrow \theta_{11} > \theta_1. \quad (4.137)$$

Then, in view of Lemma 4.6.iii, $\mathcal{T}_1^k(\theta_1) \rightarrow 0$ as $k \rightarrow \infty$. This implies that $\mathcal{T}_2^k(\theta_2) \rightarrow \frac{\pi}{2}$ as $k \rightarrow \infty$. Therefore, \mathcal{C}^- is again attractive in the interval $[0, \pi)$, and the system is GAS if and only if all the real eigenvalues of both modes are negative.

ii) Suppose that $B > 0$, i.e., $-\mathbf{y}_1$ is outside the cone \mathcal{C}^- . Since $\mathcal{F}_1(\theta_1)$ is a continuous function which maps $[0, \pi - \phi_1)$, to $(\pi - \phi_1, \pi]$ and $\frac{d\mathcal{F}_1}{d\theta_1} < 0$, there exists a unique $\bar{\theta}_1 \in [0, \pi - \phi_1)$ such that $v_1(\hat{\theta}_1(\bar{\theta}_1)) \simeq s_2$. Therefore, if $z_1(0) = v_1(\theta_1)$ where $\theta_1 \geq \bar{\theta}_1$, then $z_1(t)$ will enter the cone \mathcal{C}^- after one mode change and stay in \mathcal{S}_2 for all $t \geq \tau_1$.

a) On the other hand, for any $\theta_1 \in [-\phi_1, \bar{\theta}_1)$, it follows that $z_1(\tau_1)$ is outside the cone \mathcal{C}^- . Then, Lemma 4.5.ii implies that if $\theta_1 \in [-\phi_1, \bar{\theta}_1)$, it follows that $\mathcal{T}_1(\theta_1) \in (-\phi_1, -\psi_1)$. Calculating the derivative of $\frac{\cot \theta_1 + b_1}{\cot \theta_{11} + b_1}$ with respect to θ_1 and substituting for $\frac{d\theta_{11}}{d\theta_1}$ from equation (4.99), we get

$$\frac{d}{d\theta_1} \left(\frac{\cot \theta_1 + b_1}{\cot \theta_{11} + b_1} \right) = \frac{-1}{\sin^2 \theta_1 (\cot \theta_{11} + b_1)} \left\{ 1 - \frac{\left(\frac{d\mathcal{F}_1}{d\theta_1} e^{b_1 \omega_1 \tau_1} \right)^2}{2B (\cot \hat{\theta}_1 + b_1) + 1} \right\} \quad (4.138)$$

Since $\left| e^{b_1 \omega_1 \tau_1} \frac{d\mathcal{F}_1(\theta_1)}{d\theta_1} \right| \leq 1$ by item iv-c of Lemma 4.4, and $B > 0$, it follows that

$$\frac{\left(\frac{d\mathcal{F}_1}{d\theta_1} e^{b_1 \omega_1 \tau_1} \right)^2}{2B (\cot \hat{\theta}_1 + b_1) + 1} < 1 \quad (4.139)$$

Since $\theta_{11} \in (-\phi_1, -\psi_1)$, then $\cot \theta_{11} + b_1 < 0$. Consequently, $\frac{d}{d\theta_1} \left(\frac{\cot \theta_1 + b_1}{\cot \theta_{11} + b_1} \right) > 0$. This implies that $\frac{\cot \theta_1 + b_1}{\cot \theta_{11} + b_1}$ is an increasing function of θ_1 . Also note that $\frac{\cot \theta_1 + b_1}{\cot \theta_{11} + b_1} = 0$ at $\theta_1 = -\phi_1$ and $\cot \theta_1 + b_1 = -1/B$ at $\theta_1 = -\psi_1$. Thus, equations (B1. 14) and (B1. 16) imply that

$$\frac{\cot \theta_1 + b_1}{\cot \theta_{11} + b_1} = \frac{\left(2B (\cot \hat{\theta}_1 + b_1) + 1 \right)}{B (\cot \hat{\theta}_1 + b_1)} = 2 + \frac{1}{B (\cot \hat{\theta}_1 + b_1)} > 1. \quad (4.140)$$

Therefore, there exists a unique $\theta_1^* \in (-\phi_1, -\psi_1)$, such that $\frac{\cot \theta_1^* + b_1}{\cot \theta_{11}^* + b_1} = 1$. Equivalently, $\mathcal{T}_1(\theta_1^*) = \theta_1^*$. Moreover, $\frac{\cot \theta_1 + b_1}{\cot \theta_{11} + b_1} < 1$ for any $\theta_1 < \theta_1^* \implies \theta_{11} = \mathcal{T}_1(\theta_1) > \theta_1$. Similarly, for any $0 > \theta_1 > \theta_1^*$, it follows that $\frac{\cot \theta_1 + b_1}{\cot \theta_{11} + b_1} > 1 \implies \theta_{11} = \mathcal{T}_1(\theta_1) < \theta_1$. This implies that $v_1(\theta_1^*)$ is attractive at least in the interval $[-\phi_1, 0)$. Furthermore, if $0 \leq \theta_1 < \bar{\theta}_1$, then by Lemma 4.5.ii $\mathcal{T}_1(\theta_1) < \theta_1$ as $\mathcal{T}_1(\theta_1) \in (-\phi_1, -\psi_1)$. Consequently, $v_1(\theta_1^*)$ is attractive in the interval $[-\phi_1, \bar{\theta}_1)$ and all other trajectories enter the cone \mathcal{C}^- after one mode change. Since the fixed directions exist in pairs, there exists a unique $\theta_2^* \in (\frac{\pi}{2}, \pi)$, such that $\mathcal{T}_2(\theta_2^*) = \theta_2^*$ and $v_2(\theta_2^*)$ is attractive in the interval $(\frac{\pi}{2}, \pi)$. This implies that \mathcal{C}^- is repulsive.

b) In view of the above, Lemma 4.6.ii and iii imply that bimodal system is GAS if and only if real eigenvalues of both modes are negative and $\gamma^*(\theta_1^*, \theta_2^*) < 1$. \square

Remark 4.3: Note that if $B = 0$, then the interval $[-\phi_1, 0)$ is invariant under \mathcal{T}_1 . More precisely, if $\theta_1 \in [-\phi_1, 0)$, then $\mathcal{T}_1(\theta_1) \in [-\phi_1, 0)$. Furthermore, all the trajectories starting from $v_1(\theta_1)$ where $\theta_1 \in [-\phi_1, 0)$ converge to the trajectories starting from $v_1(0)$. However, $v_1(0)$ can not be regarded as a fixed direction as 0 is not in the domain of $\mathcal{T}_1(\theta_1)$. In such cases, the pair $(v_1(0), v_2(\frac{\pi}{2}))$ is said to be a degenerate pair of fixed directions.

Remark 4.4: Note that negative real eigenvalues for both modes is an obvious necessary condition for GAS of BPLS. However, it also becomes a sufficient condition for the case where $B \leq 0$ (even if $\sigma_1 > 0$). This is also pointed out in [Carmona et al., 2005](Proposition 10) for the case where the vector field is continuous on \mathcal{H} and both modes have complex eigenvalues. If $B > 0$, then negative real eigenvalues is just a necessary condition. Therefore, the convergence rate of the unique pair of fixed directions has to be calculated for the assessment of GAS. This calculation can be done only numerically because the relation between θ_1 and τ_1 is implicitly known and it is nonlinear. Given the problem data, one can write a simple algorithm which calculates both the fixed directions and the convergence rate $\gamma^*(\theta_1^*, \theta_2^*)$, numerically. Such an algorithm may start from $-\phi_1$ as $\theta_1^* > -\phi_1$ and basically calculate $\mathcal{T}_1^k(-\phi_1)$ for some appropriate integer k such that $|\theta_1^* - \mathcal{T}_1^k(-\phi_1)|$ is sufficiently small (for most of the cases $k = 25$ will be enough). The first iteration of this algorithm involves the

following steps.

- Given $-\phi_1$ calculate the time $\tau_1(-\phi_1)$ at which a trajectory starting from $v_1(-\phi_1)$ will hit \mathcal{H} and change mode. This is equivalent to solving the following equation (given in Lemma 4.4.iii) for $\tau_1(-\phi_1)$, numerically.

$$e^{(\sigma_1 - \lambda_1)\tau_1} \sin \hat{\theta}_1 = \sin \theta_1. \quad (4.141)$$

- This trajectory will hit $\mathcal{S}_2 \cap \mathcal{H}$ along a direction $v_2(\theta_2)$ which is outside of \mathcal{C}^- . Calculate θ_2 using equation (B1. 14).
- Calculate the time $\tau_2(\theta_2)$ at which a trajectory starting from $v_2(\theta_2)$ will hit $\mathcal{S}_1 \cap \mathcal{H}$ along a direction $v_1(\theta_{11})$. Note that by Lemma 4.3.i, we have $\tau_2(\theta_2) = -2/\cot(\theta_2)$.
- Calculate θ_{11} using equation (B1. 16).
- Go back to the first step and calculate the time $\tau_1(\theta_{11})$ at which a trajectory starting from $v_1(\theta_{11})$ will hit \mathcal{H} .

After k such iterations, $\mathcal{T}_1^k(-\phi_1)$ will get sufficiently close to θ_1^* . More precisely, $|\mathcal{T}_1^k(-\phi_1) - \mathcal{T}_1^{k-1}(-\phi_1)|$ will be sufficiently small (less than 10^{-6}). Then, these estimates of fixed points can be used together with equation (B1. 24) to calculate the convergence rate $\gamma^*(\theta_1^*, \theta_2^*)$.

Remark 4.5: It is well known in the literature that the dynamics of bimodal systems in \mathbb{R}^3 are quite unpredictable under certain conditions. For instance, both A_1 and A_2 can have eigenvalues with negative real parts, but bimodal system may be unstable (see [Carmona et al., 2005], [Carmona et al., 2006]). Conversely, both modes or one of the modes could be unstable, but bimodal system may be GAS. Another unpredictable dynamic behavior of bimodal systems in \mathbb{R}^3 is demonstrated below. Basically, it is shown that the coupling constant can be changed without changing the eigenvalues of subsystems, and this can make a bimodal system unstable or GAS (see Example 1 and Example 2 below). It should also be noted here that such behavior can not be observed if the vector field is continuous in \mathcal{H} . Because in this case, both modes are in observable canonical form and $a_{11} = 0$ and $a_{21} = 1$. Therefore, in view of Definition 4.9, B can not be changed without changing the eigenvalues of subsystem matrices.

Remark 4.6: Some of the tools used in this dissertation have appeared previously in the literature (without using the same terminology and without the same emphasis). For instance, fixed directions are used by [Carmona et al., 2005] in order to determine two zonal invariant cones of BPLS in \mathbb{R}^3 with continuous vector fields. In [Carmona et al., 2005] fixed direction is defined as a half line in \mathcal{H} which is invariant under Poincaré full map or as the fixed point of a map, which is similar to $\mathcal{T}_1(\theta_1)$ in our set up. (see Proposition 13 in [Carmona et al., 2005]). It is further pointed out by [Carmona et al., 2005] that the stability of the trajectories starting from fixed directions imply the stability of all trajectories in the corresponding invariant two zonal cone (see Remark 14 in [Carmona et al., 2005]). However, it is also stated that this does not guarantee GAS of BPLS. This fact is demonstrated by an example (see example given by equation (2) in [Carmona et al., 2005]). Thus, the main result (Theorem 2 in [Carmona et al., 2005]) gives the sufficient conditions for the existence of invariant cones and the stability of the trajectories living in these cones. In a later paper by [Carmona et al., 2006], a special class of BPLS in \mathbb{R}^3 , where the vector field is again continuous on \mathcal{H} and subsystem matrices are Hurwitzian with complex eigenvalues, is investigated. In the second item of the first main result of this paper, it is stated that the dynamics on the cone could be stable, unstable or periodic. The authors further comment that "without additional knowledge on parameter values it is not possible to discriminate the three cases " (see Theorem 1 and the following discussion in [Carmona et al., 2006]). The second main result of this paper gives a set of sufficient condition for GAS of BPLS in \mathbb{R}^3 (see Theorem 2 in [Carmona et al., 2006]). The case in Theorem 2.c of [Carmona et al., 2006], is investigated in detail in a more general setting, where the vector field is not necessarily continuous and the subsystem matrices are not necessarily Hurwitzian, by [Eldem and Şahan, 2014]. Thus, the basic differences between the approach and the results in this dissertation and in the papers by [Carmona et al., 2005], [Carmona et al., 2006]) can be summarized as follows.

- *In this dissertation the case, where only one mode has complex eigenvalues and the vector field is not necessarily continuous on \mathcal{H} , is considered . In [Carmona et al., 2005], [Carmona et al., 2006] the case, where both modes have complex eigenvalues and the vector field is continuous on \mathcal{H} , is studied.*
- *Fixed directions are used in order to classify the trajectories BPLS in this*

dissertation. In [Carmona et al., 2005], [Carmona et al., 2006] fixed directions are used to detect invariant cones.

- In this dissertation, verifiable necessary and sufficient conditions for GAS of BPLS is given . In [Carmona et al., 2005], [Carmona et al., 2006] provide only sufficient conditions for certain cases in [Carmona et al., 2006]) and for the other cases they claim that additional information is necessary. In fact, a procedure to obtain this additional information is given in this dissertation. This procedure involves the calculation of the intervals where the fixed directions are attractive and showing that the fixed directions are unique in these intervals (see Remarks 4.4 and 4.7).
- Finally, the effect of discontinuity of the vector field is emphasized in this dissertation and it is demonstrated that a change in the coupling constant (without changing the eigenvalues subsystem matrices) can make BPLS unstable or GAS. This is not possible in the setup used in [Carmona et al., 2005], [Carmona et al., 2006], because they assume that the vector field is continuous (see Remark 4.5).

In [Iwatani and Hara, 2006], separate conditions for necessity and for sufficiency of GAS of bimodal systems in \mathbb{R}^n , where $n > 2$, are proposed (Theorem 19 in [Iwatani and Hara, 2006]). The necessary conditions given by [Iwatani and Hara, 2006] are obvious (real eigenvalues are negative). However, the sufficient conditions are very restrictive, as they require that the observability index of one of the subsystems to be ≤ 2 . This result can not be used in our set up, because it is assumed that the observability index is three for both subsystems.

It is shown by [Eldem and Şahan, 2014] that the GAS of BPLS in \mathbb{R}^3 , where both modes have complex eigenvalues, reduces to the GAS of BPLS in \mathbb{R}^2 , if BPLS in \mathbb{R}^3 has a certain structure. This structure is similar to the case where $B = 0$ in Theorem 1. A complete set of verifiable necessary and sufficient conditions for the GAS of BPLS in \mathbb{R}^3 is first given in this dissertation, for the class of BPLS being considered.

At this stage, it is necessary to add a few remarks in order to shed some light on the non-trivial part of the problem considered in this dissertation. These remarks also explain the reason why the second part of Assumption 4 is used in the dissertation.

Remark 4.7: Note that the calculation of the convergence rate $\gamma^(\theta_1^*, \theta_2^*)$ is crucial for*

the assessment of GAS of BPLS in \mathbb{R}^3 . This means that θ_1^* and θ_2^* and the intervals where $v_1(\theta_1^*)$ and $v_2(\theta_2^*)$ are attractive must be calculated first. Then, one has to prove that $v_1(\theta_1^*)$ and $v_2(\theta_2^*)$ are unique fixed directions within these intervals and this constitutes the non-trivial part of the problem. In view of Lemma 4.7, this is equivalent to proving the uniqueness of the fixed point of a non-decreasing function $\mathcal{T}_1(\theta_1)$ (or $\mathcal{T}_2(\theta_2)$) which maps a closed interval into itself. The existence of a fixed point of $\mathcal{T}_1(\theta_1)$ is obvious, since $\mathcal{T}_1(\theta_1)$ (or $\mathcal{T}_2(\theta_2)$) continuously maps a closed interval into itself. However, the proof of uniqueness is non-trivial. If there are more than one pair of fixed directions of BPLS in a given interval of θ_1 or a pair of fixed directions are repulsive, then the calculation of $v_1(\theta_1^*)$, $v_2(\theta_2^*)$, and $\gamma^*(\theta_1^*, \theta_2^*)$ could be very difficult, if not impossible. The reason for using Assumption 4, is because we can prove uniqueness only for the case where the second mode has a single eigenvalue μ , with the algebraic degree equal to three and geometric degree is equal to one. Then, it becomes possible to use the algorithm described in Remark 4.4, to calculate $\gamma^*(\theta_1^*, \theta_2^*)$, numerically. The other cases in this set up are $\{\mu_1 = \mu_2 < \mu_3\}$, $\{\mu_1 < \mu_2 = \mu_3\}$, $\{\mu_1 < \mu_2 < \mu_3\}$, where $\{\mu_1 \leq \mu_2 \leq \mu_3\}$ denote the eigenvalues of the second mode. The uniqueness of fixed directions for these cases (with $B > 0$) are still open problems. In this respect, the case where both modes have complex eigenvalues is also an open problem. The basic difficulty in these problems is to show that the fixed directions are unique in certain intervals as pointed out above. This difficulty can either be handled by an additional assumption on system parameters or by proving uniqueness via a different approach.

In the examples below, the convergence rate $\gamma^*(\theta_1^*, \theta_2^*)$ is calculated by an algorithm (written in MATLAB) which is constructed according to the outline given Remark 4.4.

Example 4.1: Consider the following bimodal system. The eigenvalues of A_1 are $\{-\frac{1}{3}, -\frac{1}{40} + \frac{1}{5}i, -\frac{1}{40} - \frac{1}{5}i\}$. A_2 is in observable canonical form as given by equation (4.9) with eigenvalues $\{-3, -3, -3\}$.

$$A_1 = \begin{bmatrix} -\frac{551}{1560} & -\frac{19651}{307320} & \frac{27}{260} \\ \frac{197}{195} & \frac{59233}{153660} & -\frac{557}{260} \\ 0 & \frac{20617}{310472} & -\frac{655}{1576} \end{bmatrix}. \quad (4.142)$$

Both modes of this system are stable, but bimodal system is unstable with $\gamma^*(\theta_1^*, \theta_2^*) = 1.5103$. The coupling constant is $B = 4.5171$. We can change the coupling constant while keeping the eigenvalues of both modes fixed. Consider the bimodal system given below.

$$A_1 = \begin{bmatrix} -\frac{481}{1832} & -\frac{289}{60} & -\frac{67}{20} \\ \frac{5057}{52441} & -\frac{32173}{27480} & -\frac{1897}{4580} \\ 0 & \frac{229}{120} & \frac{21}{20} \end{bmatrix}, \quad (4.143)$$

where the eigenvalues of A_1 are again $\{-\frac{1}{3}, -\frac{1}{40} + \frac{1}{5}i, -\frac{1}{40} - \frac{1}{5}i\}$ and A_2 is in observable canonical form as given by equation (4.9) with the same eigenvalues $\{-3, -3, -3\}$. This system is stable with $\gamma^*(\theta_1^*, \theta_2^*) = 0.0978$ and with the coupling constant $B = 0.3236$.

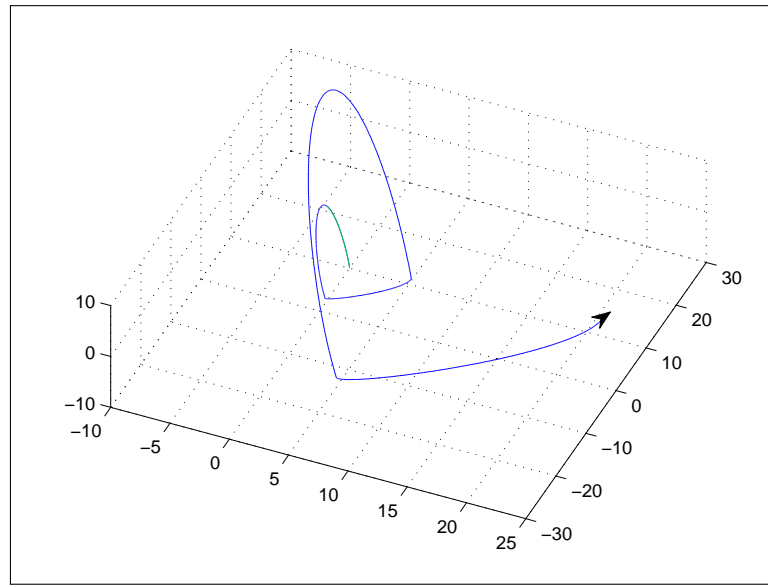


Figure 4.2: Unstable bimodal systems

Example 4.2: In this example we consider a bimodal system where the first mode has unstable complex eigenvalues as given below.

$$A_1 = \begin{bmatrix} -\frac{2839}{1070} & -\frac{293}{130} & \frac{227}{26} \\ \frac{91988}{11449} & \frac{16241}{2782} & -\frac{41044}{1391} \\ 0 & \frac{107}{130} & -\frac{233}{65} \end{bmatrix}, \quad (4.144)$$

where the eigenvalues of A_1 are $\{-1, \frac{3}{10} + 4i, \frac{3}{10} - 4i\}$. A_2 is in observable canonical form as before with eigenvalues $\{-3, -3, -3\}$. This system is unstable with $\gamma^*(\theta_1^*, \theta_2^*) = 2.8819$ and the coupling constant is $B = 5.8241$ (See Figure 4.2). We can change the coupling constant while keeping the eigenvalues of both modes fixed with A_1 as given below.

$$A_1 = \begin{bmatrix} \frac{679}{1287} & -\frac{22952734}{36407943} & -\frac{5812}{19305} \\ \frac{28289}{12870} & \frac{2455607449}{364079430} & -\frac{64412}{19305} \\ 0 & \frac{15911818890}{800267521} & -\frac{2170413}{282890} \end{bmatrix}. \quad (4.145)$$

This system is stable with $\gamma^*(\theta_1^*, \theta_2^*) = 0.4466$ and $B = 1.1456$ (See Figure 4.3).

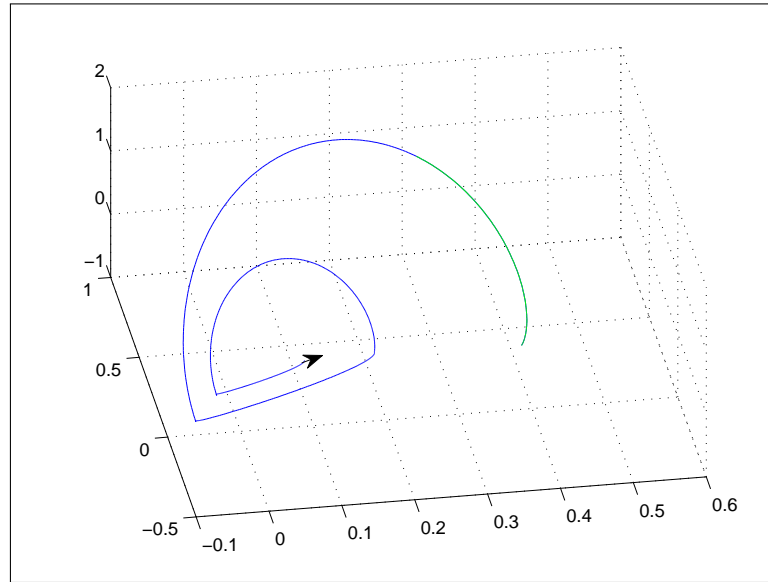


Figure 4.3: Stable bimodal system

Example 4.3: Suppose that A_1 is as given above with eigenvalues $\{-1, \frac{3}{10} + 4i, \frac{3}{10} - 4i\}$ and A_2 is in observable canonical form. The following Table 4.1 shows the effect of the changing the eigenvalues of A_2 on $\gamma^*(\theta_1^*, \theta_2^*)$ and B .

It is intuitively expected that decreasing the eigenvalues of A_2 should move bimodal system towards GAS. However, just the opposite occurs in the above list. More precisely, as the eigenvalues of A_2 are decreased, convergence rate increases and bimodal system first becomes marginally stable with $\gamma^*(\theta_1^*, \theta_2^*) = 1$ and then

Table 4.1: Coupling constant and stability relation

First mode eigenvalues	Second mode eigenvalues	$\gamma^*(\theta_1^*, \theta_2^*)$	B
$\frac{3}{10} + 4i, \frac{3}{10} - 4i, -1$	$-3, -3, -3$	0.446	1.1456
$\frac{3}{10} + 4i, \frac{3}{10} - 4i, -1$	$-5; -5; -5$	0.8312	2.1397
$\frac{3}{10} + 4i, \frac{3}{10} - 4i, -1$	$-5.8741; -5.8741; -5.8741$	1	2.5741
$\frac{3}{10} + 4i, \frac{3}{10} - 4i, -1$	$-6, -6, -6$	1.0241	2.6367
$\frac{3}{10} + 4i, \frac{3}{10} - 4i, -1$	$-7, -7, -7$	1.2179	3.1337

further decreasing the eigenvalues of A_2 causes bimodal system to be unstable. This may be due to the fact that the coupling constant increases. Furthermore, when $\gamma^*(\theta_1^*, \theta_2^*) = 1$, we have infinite number of periodic solutions which are closed space curves as depicted in Figure 4.4.

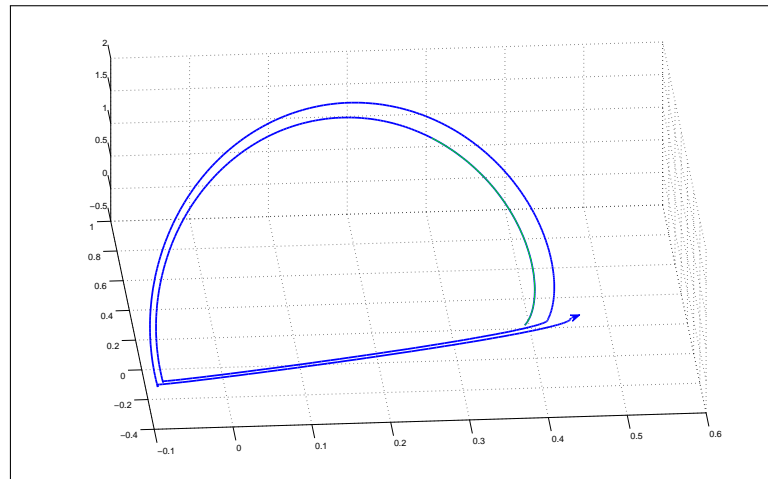


Figure 4.4: Marginally stable bimodal system

5. CONCLUSIONS

It is shown that the conjecture given in [Eldem and Şahan, 2014] hold for bimodal systems being considered in this dissertation. It is clear that the classification of the trajectories as finitely mode changing and infinitely mode changing plays a crucial role in GAS. In addition, it is demonstrated by the examples above that, GAS also depends on how the two subsystems are coupled on the plane $\ker c^T$. The next step along this line is to consider the cases where real eigenvalues of the second mode have different multiplicities and the cases where $\sigma_1 - \lambda_1 \leq 0$ for the first mode. In this way, GAS of bimodal systems in \mathbb{R}^3 , where at least one of the modes have only real eigenvalues, will be completed. Another topic for future work could be to consider the cases where both modes have complex eigenvalues with coupling conditions different from bimodal systems investigated in [Eldem and Şahan, 2014].

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BIOGRAPHY

Işıl ÖNER was born in Ordu, Turkey, in 1977. She got a Bsc degree in Department of Mathematics Education from Ondokuz Mayıs University, in 1998. Between 1998 and 2004 she worked as a mathematics teachers in state and private schools. She received a M.Sc degree in Mathematics from Gebze Institute of Technology, in 2004. She has been working as a research assistant at Gebze Institute of Technology, since 2004.

APPENDICES

Appendix A: Publications Based on the Thesis

Eldem V., Öner, I., (2014), “A note on the stability of bimodal systems in \mathbb{R}^3 with discontinuous vector fields”, International Journal of Control, DOI :10.1080/00207179.2014.971522.

Eldem V., Öner, I., (2013), “Classification of the trajectories of bimodal systems in \mathbb{R}^3 ”, 4th International Conference on Control and Optimization with Industrial Applications, Borovets, Bulgaria, 10-12 July.

Öner, I., Eldem V., (2014), “ \mathbb{R}^3 te İki Modlu Sistemlerin Kararlılık Analizi”, 2014 Otomatik Kontrol Ulusal Toplantısı, 11-13 Eylül.

Appendix B: Proofs

Proof of Lemma 4.2 : Only the proof for $z_1(t)$ is given here. The proof for $z_2(t)$ follows similar lines. Since $c^T z_1(0) = 0$, it follows that $z_1(t)$ smoothly continues into \mathcal{S}_1 if either $\left[\frac{d}{dt}(c^T z_1(t))\right]_{t=0} > 0$ or if $\left[\frac{d}{dt}(c^T z_1(t))\right]_{t=0} = 0$ and $\left[\frac{d^2}{dt^2}(c^T z_1(t))\right]_{t=0} > 0$. Since $c^T z_1(0) = 0$ it follows that $f_1(0) = 0$. Therefore, in order to prove the hypothesis, it is enough to check time derivatives of $f_1(t)$ at $t = 0$. Towards this end, note that

$$\left[\frac{d}{dt}f_1(t)\right]_{t=0} = (\lambda_2 - \lambda_1) \sin \theta_1 + (\lambda_3 - \lambda_1) \cos \theta_1, \text{ if } \lambda_1 < \lambda_2 < \lambda_3, \quad (\text{B1. 1})$$

$$= \sin \theta_1 + (\lambda_3 - \lambda_1) \cos \theta_1, \text{ if } \lambda_1 = \lambda_2 < \lambda_3, \quad (\text{B1. 2})$$

$$= (\lambda_3 - \lambda_1) \sin \theta_1 + \cos \theta_1, \text{ if } \lambda_1 < \lambda_2 = \lambda_3 \quad (\text{B1. 3})$$

$$= \sin \theta_1, \text{ if } \lambda_1 = \lambda_2 = \lambda_3. \quad (\text{B1. 4})$$

In view of the definition of $\cot \phi_1$ given above, it follows that

$$\left[\frac{d}{dt}f_1(t)\right]_{t=0} = L \sin(\theta_1 + \phi_1), \quad (\text{B1. 5})$$

for some appropriate constant $L > 0$ for each case of multiplicities of the eigenvalues. Thus, it follows that $\sin(\theta_1 + \phi_1) \geq 0$ if and only if $\left[\frac{d}{dt}(f_1(t))\right]_{t=0} \geq 0$ or equivalently $\left[\frac{d}{dt}(c^T z_1(t))\right]_{t=0} \geq 0$. This implies that $-\phi_1 \leq \theta_1 \leq \pi - \phi_1$. Note that $0 \leq \phi_1 < \frac{\pi}{2}$ where the equality holds for the case $\lambda_1 = \lambda_2 = \lambda_3$. Furthermore, as $\sin(\theta_1 + \phi_1) = 0$ at $\theta_1 = -\phi_1$, or at $\theta_1 = \pi - \phi_1$, the second derivative is calculated at $\theta_1 = -\phi_1$ or at $\theta_1 = \pi - \phi_1$. Since $\left[\frac{d}{dt}(f_1(t))\right]_{t=0} = 0$, it follows that

$$\left[\frac{d^2}{dt^2}f_1(t)\right]_{t=0} = (\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)\cos\theta_1, \text{ if } \lambda_1 < \lambda_2 < \lambda_3, \quad (\text{B1. 6})$$

$$= (\lambda_3 - \lambda_1)^2 \cos\theta_1, \text{ if } \lambda_1 = \lambda_2 < \lambda_3, \quad (\text{B1. 7})$$

$$= (\lambda_3 - \lambda_1)\cos\theta_1, \text{ if } \lambda_1 < \lambda_2 = \lambda_3, \quad (\text{B1. 8})$$

$$= \cos\theta_1, \text{ if } \lambda_1 = \lambda_2 = \lambda_3. \quad (\text{B1. 9})$$

Since $0 \leq \phi_1 < \frac{\pi}{2}$, the equations above imply that $\left[\frac{d^2}{dt^2}f_1(t)\right]_{t=0} > 0$ at $\theta_1 = -\phi_1$ and $\left[\frac{d^2}{dt^2}f_1(t)\right]_{t=0} < 0$ at $\theta_1 = \pi - \phi_1$. Consequently, if $\theta_1 = -\phi_1$, then the trajectories smoothly continue into \mathcal{S}_1 . If $\theta_1 = \pi - \phi_1$, then smooth continuation into \mathcal{S}_1 is not possible. Thus, smooth continuation into \mathcal{S}_1 is possible if and only if $-\phi_1 \leq \theta_1 < \pi - \phi_1$. \square

Proof of the Theorem 4.1 : Only the proof for $i = 1$ provided here. The proof for $i = 2$ follows similar lines.

i) Let $z_1(t)$ be a trajectory given by equation 4.30. Then, the following hold.

a) If $0 \leq \theta_1 \leq \frac{\pi}{2}$, then $\cos\theta_1 \geq 0$ and $\sin\theta_1 \geq 0$. In view of equation (4.22-4.29) and Lemma 4.1, $c^T p_2(t) \geq 0$ and $c^T p_3(t) \geq 0$ for all $t \geq 0$. Thus, it follows that $(c^T p_2(t)) e^{(\lambda_2 - \lambda_1)t} \geq 0$ and $(c^T p_3(t)) e^{(\lambda_3 - \lambda_1)t} \geq 0$ for all $t \geq 0$. Then, using equation (4.31) one gets $c^T z_1(t) \geq 0$ for all $t \geq 0$. Therefore, such trajectories do not change mode. If $-\phi_1 \leq \theta_1 < 0$, then $\sin(\theta_1 + \phi_1) \geq 0$, where the equality holds only at $\theta_1 = -\phi_1$. Furthermore, calculating the time derivative of $f_1(t)$, for $\lambda_1 < \lambda_2 < \lambda_3$, yields

$$\frac{d}{dt}f_1(t) = (\lambda_2 - \lambda_1) e^{(\lambda_2 - \lambda_1)t} \sin\theta_1 + (\lambda_3 - \lambda_1) e^{(\lambda_3 - \lambda_1)t} \cos\theta_1, \quad (\text{B1. 10})$$

and for $\lambda_1 < \lambda_2 = \lambda_3$

$$\frac{d}{dt} f_1(t) = [(\sin \theta_1 + t \cos \theta_1) (\lambda_3 - \lambda_1) + \cos \theta_1] e^{(\lambda_3 - \lambda_1)t} \geq 0. \quad (\text{B1. 11})$$

For the other cases it follows that

$$\frac{d}{dt} f_1(t) = \sin \theta_1 + (\lambda_3 - \lambda_1) e^{(\lambda_3 - \lambda_1)t} \cos \theta_1, \text{ if } \lambda_1 = \lambda_2 < \lambda_3, \quad (\text{B1. 12})$$

$$\frac{d}{dt} f_1(t) = \sin \theta_1 + t \cos \theta_1, \text{ if } \lambda_1 = \lambda_2 = \lambda_3. \quad (\text{B1. 13})$$

The equations above imply that $[\frac{d}{dt} f_1(t)]_{t=0} > 0$ if and only if $\sin(\theta_1 + \phi_1) > 0$. Consequently, in view of equations (B1. 10) and (B1. 13) $\frac{d}{dt} f_1(t) > 0$ for all $t \geq 0$. Thus, $c^T z_1(t) \geq 0$ for all $t \geq 0$. On the other hand, if $[\frac{d}{dt} f_1(t)]_{t=0} = 0$, the second time derivative of $f_1(t)$ is calculated and for $\lambda_1 < \lambda_2 < \lambda_3$, this yields

$$\frac{d^2}{dt^2} f_1(t) = (\lambda_3 - \lambda_1) e^{(\lambda_2 - \lambda_1)t} ((\lambda_3 - \lambda_1) e^{(\lambda_3 - \lambda_1)t} - (\lambda_2 - \lambda_1)) \cos \theta_1, \quad (\text{B1. 14})$$

and for $\lambda_1 < \lambda_2 = \lambda_3$,

$$\frac{d^2}{dt^2} f_1(t) = [t(\lambda_3 - \lambda_1) + 1] (\lambda_3 - \lambda_1) e^{(\lambda_3 - \lambda_1)t} \cos \theta_1. \quad (\text{B1. 15})$$

For the other cases, one gets

$$\frac{d^2}{dt^2} f_1(t) = (\lambda_3 - \lambda_1)^2 e^{(\lambda_3 - \lambda_1)t} \cos \theta_1 \text{ if } \lambda_1 = \lambda_2 < \lambda_3, \quad (\text{B1. 16})$$

$$= \cos \theta_1, \text{ if } \lambda_1 = \lambda_2 = \lambda_3. \quad (\text{B1. 17})$$

These equations imply that $[\frac{d^2}{dt^2} f_1(t)]_{t=0} > 0$ and $\frac{d^2}{dt^2} f_1(t) > 0$ for all $t \geq 0$. Consequently, $c^T z_1(t) \geq 0$ for all $t \geq 0$ and such trajectories do not change mode
b) If $\frac{3\pi}{2} < \theta_1 < 2\pi - \phi_1$, then $\cos \theta_1 > 0$ and $\sin(\theta_1 + \phi_1) < 0$. As in the proof of the previous item, $\sin(\theta_1 + \phi_1) < 0$ implies and is implied by the fact that $[\frac{d}{dt} f_1(t)]_{t=0} < 0$. Thus, for smooth continuation into \mathcal{S}_1 , it is necessary that $f_1(0) > 0$. In this case

$f_1(t)$ decreases until it reaches to a minimum at some $t_{\min} > 0$ where $\frac{d}{dt}f_1(t_{\min}) = 0$. Since equations (B1. 14) and (B1. 17) imply that $\frac{d^2}{dt^2}f_1(t_{\min}) > 0$ as $\cos \theta_1 > 0$, it follows that t_{\min} exists and $f_1(t_{\min})$ is the first local minimum. If $c^T z_1(t_{\min}) \geq 0$ then, since $\frac{d^2}{dt^2}f_1(t) \geq 0$ for all $t \geq t_{\min}$, it follows that $c^T z_1(t) \geq 0$ for all $t \geq 0$ and such trajectories do not change mode.

ii) Let $z_1(t)$ be a trajectory given by equation 4.30. Then, the following hold.

a) If $\frac{\pi}{2} < \theta_1 \leq \frac{3\pi}{2}$, then $\cos \theta_1 \leq 0$. If $\cos \theta_1 < 0$ then, in view of equations (B1. 15)-(B1. 17), it follows that $\frac{d^2}{dt^2}f_1(t) < 0$ for all $t \geq 0$ except for the case where $\lambda_1 < \lambda_2 < \lambda_3$. For this case, we have

$$\frac{d^2}{dt^2}f_1(t) = (\lambda_2 - \lambda_1)^2 e^{(\lambda_2 - \lambda_1)t} \sin \theta_1 + (\lambda_3 - \lambda_1)^2 e^{(\lambda_3 - \lambda_1)t} \cos \theta_1, \quad (\text{B1. 18})$$

where

$$(\lambda_2 - \lambda_1)^2 e^{(\lambda_2 - \lambda_1)t} < (\lambda_3 - \lambda_1)^2 e^{(\lambda_3 - \lambda_1)t}, \quad (\text{B1. 19})$$

for all $t \geq 0$. This implies that there exists a finite $T \geq 0$ such that $\frac{d^2}{dt^2}f_1(t) < 0$ for all $t \geq T$. In summary, there exists a finite $T \geq 0$ such that $\frac{d^2}{dt^2}f_1(t) < 0$ for all $t \geq T$ for all the cases of eigenvalues (multiple or distinct) and therefore such trajectories change mode. If $\theta_1 = \frac{3\pi}{2}$, then $\cos \theta_1 = 0 > \sin \theta_1$. Using equation (B1. 10)-(B1. 13) we get $\frac{d}{dt}f_1(t) < 0$ for all $t \geq 0$ and for all cases of eigenvalues (multiple or distinct). This implies that such trajectories change mode.

b) Let us consider the case $\frac{3\pi}{2} < \theta_1 < 2\pi - \phi_1$. In this case $\cos \theta_1 > 0$ and $\sin(\theta_1 + \phi_1) < 0$. This is equivalent to the fact that $\frac{d}{dt}f_1(t)|_{t=0} < 0$. Consequently, smooth continuation into \mathcal{S}_1 is possible if $f_1(0) > 0$. Thus, $f_1(t)$ decreases until it reaches to a minimum at some $t_{\min} > 0$ where $\frac{d}{dt}f_1(t_{\min}) = 0$. Since equations (B1. 14) and (B1. 17) imply that $\frac{d^2}{dt^2}f_1(t_{\min}) > 0$ as $\cos \theta_1 > 0$, it follows that t_{\min} exists and $f_1(t_{\min})$ is the first local minimum. Hence, if $c^T z_1(t_{\min}) < 0$, it is clear that the trajectory changes mode at some $t < t_{\min}$. \square

Proof of the Lemma 4.3 :

i) Only the proof for $z_1(t)$ is given here. The proof for $z_2(t)$ follows similar lines. Let $z_1(t)$ be a transitive trajectory starting from $\mathcal{S}_1 \cap \mathcal{H}$. Then, $c^T z_1(0) = 0$ and there

exists a finite $\tau_1 > 0$ such that $c^T z_1(\tau_1) = 0$ where the trajectory changes mode. Since $c^T z_1(0) = 0$ by equation (4.31) it follows that

$$\alpha_1 = -c^T r_2 \sin \theta_1 - c^T r_3 \cos \theta_1. \quad (\text{B1. 20})$$

Substituting the equation above in equation (4.30), we get

$$\begin{aligned} z_1(\tau_1) &= e^{\lambda_1 \tau_1} \{ \sin \theta_1 (p_2(\tau_1) e^{(\lambda_2 - \lambda_1) \tau_1} - (c^T r_2) r_1) \\ &\quad + \cos \theta_1 (p_3(\tau_1) e^{(\lambda_3 - \lambda_1) \tau_1} - (c^T r_3) r_1) \}. \end{aligned} \quad (\text{B1. 21})$$

Since $c^T z_1(\tau_1) = 0$ equation (4.31) also implies that

$$\alpha_1 = -c^T p_2(\tau_1) \sin \theta_1 e^{(\lambda_2 - \lambda_1) \tau_1} - c^T p_3(\tau_1) \cos \theta_1 e^{(\lambda_3 - \lambda_1) \tau_1}. \quad (\text{B1. 22})$$

Consequently,

$$\begin{aligned} c^T p_2(\tau_1) \sin \theta_1 e^{(\lambda_2 - \lambda_1) \tau_1} &+ c^T p_3(\tau_1) \cos \theta_1 e^{(\lambda_3 - \lambda_1) \tau_1} \\ &= c^T r_2 \sin \theta_1 + c^T r_3 \cos \theta_1. \end{aligned} \quad (\text{B1. 23})$$

Using this equality in the expression above for $z_1(\tau_1)$, one gets

$$\begin{aligned} z_1(\tau_1) &= e^{\lambda_2 \tau_1} \{ \sin \theta_1 (p_2(\tau_1) - c^T p_2(\tau_1) r_1) \\ &\quad + \cos \theta_1 (p_3(\tau_1) - c^T p_3(\tau_1) r_1) e^{(\lambda_3 - \lambda_2) \tau_1} \}. \end{aligned} \quad (\text{B1. 24})$$

Note that if $\lambda_1 < \lambda_2 < \lambda_3$ or if $\lambda_1 = \lambda_2 < \lambda_3$, then it follows that

$$(p_2(\tau_1) - c^T p_2(\tau_1) r_1) = \hat{r}_2 \text{ and } (p_3(\tau_1) - c^T p_3(\tau_1) r_1) = \hat{r}_3, \quad (\text{B1. 25})$$

which implies that

$$z_1(\tau_1) = e^{\lambda_3 \tau_1} \{ \hat{r}_2 \sin \theta_1 e^{(\lambda_2 - \lambda_3) \tau_1} + \hat{r}_3 \cos \theta_1 \}. \quad (\text{B1. 26})$$

Let $N_1 := \sqrt{e^{2(\lambda_2 - \lambda_3)\tau_1} \sin^2 \theta_1 + \cos^2 \theta_1}$. Then, it follows that

$$z_1(\tau_1) = e^{\lambda_3 \tau_1} N_1 \left\{ \hat{r}_2 \sin \hat{\theta}_1 + \hat{r}_3 \cos \hat{\theta}_1 \right\}, \quad (\text{B1. 27})$$

where $\sin \hat{\theta}_1 = \sin \theta_1 e^{(\lambda_2 - \lambda_3)\tau_1} / N_1$ and $\cos \hat{\theta}_1 = \cos \theta_1 / N_1$. Consequently, $\cot \hat{\theta}_1 = e^{(\lambda_3 - \lambda_2)\tau_1} \cot \theta_1$. On the other hand, if $\lambda_1 < \lambda_2 = \lambda_3$ or $\lambda_1 = \lambda_2 = \lambda_3$, then equation (B1. 24) implies that

$$z_1(\tau_1) = e^{\lambda_3 \tau_1} \left\{ (\sin \theta_1 + \tau_1 \cos \theta_1) \hat{r}_2 + \hat{r}_3 \cos \theta_1 \right\}. \quad (\text{B1. 28})$$

If $\lambda_1 < \lambda_2 = \lambda_3$, then using equation (4.60) one obtains $\sin \theta_1 + \tau_1 \cos \theta_1 = \sin \theta_1 e^{(\lambda_1 - \lambda_3)\tau_1}$. Thus, it follows that

$$z_1(\tau_1) = e^{\lambda_3 \tau_1} \left\{ \hat{r}_2 e^{(\lambda_1 - \lambda_3)\tau_1} \sin \theta_1 + \hat{r}_3 \cos \theta_1 \right\}. \quad (\text{B1. 29})$$

Let $N_1 := \sqrt{e^{2(\lambda_1 - \lambda_3)\tau_1} \sin^2 \theta_1 + \cos^2 \theta_1}$. Then,

$$z_1(\tau_1) = e^{\lambda_3 \tau_1} \left\{ \hat{r}_2 \sin \hat{\theta}_1 + \hat{r}_3 \cos \hat{\theta}_1 \right\}, \quad (\text{B1. 30})$$

where $\sin \hat{\theta}_1 = \sin \theta_1 e^{(\lambda_1 - \lambda_3)\tau_1} / N_1$ and $\cos \hat{\theta}_1 = \cos \theta_1 / N_1$. Hence, it follows that $\cot \hat{\theta}_1 = e^{(\lambda_3 - \lambda_1)\tau_1} \cot \theta_1$. If $\lambda_1 = \lambda_2 = \lambda_3$, then via equation (4.60) one has $\sin \theta_1 + \tau_1 \cos \theta_1 = -\sin \theta_1$ which implies that

$$z_1(\tau_1) = e^{\lambda_3 \tau_1} \left\{ -\sin \theta_1 \hat{r}_2 + \hat{r}_3 \cos \theta_1 \right\}, \quad (\text{B1. 31})$$

$$= e^{\lambda_3 \tau_1} N_1 \left\{ \hat{r}_2 \sin \hat{\theta}_1 + \hat{r}_3 \cos \hat{\theta}_1 \right\}, \quad (\text{B1. 32})$$

where $N_1 := \sqrt{(-\sin \theta_1)^2 + (\cos \theta_1)^2} = 1$, $\sin \hat{\theta}_1 = -\sin \theta_1$ and $\cos \hat{\theta}_1 = \cos \theta_1$.

Consequently, it follows that $\cot \hat{\theta}_1 = -\cot \theta_1$.

ii) In order to prove that τ_1 is a decreasing function of θ_1 , implicit differentiation is used in equation (4.60) and the following expressions are obtained.

$$\frac{d\tau_1}{d\theta_1} = \frac{(e^{(\lambda_3-\lambda_1)\tau_1} - 1)}{(\lambda_3 - \lambda_1) e^{(\lambda_2-\lambda_1)\tau_1} (b_1 + \cot \hat{\theta}_1) \sin^2 \theta_1} \text{ if } \lambda_1 < \lambda_2 < \lambda_3, \quad (\text{B1. 33})$$

$$= \frac{(e^{(\lambda_3-\lambda_1)\tau_1} - 1)}{(\lambda_3 - \lambda_1) (b_1 + \cot \hat{\theta}_1) \sin^2 \theta_1} \text{ if } \lambda_1 = \lambda_2 < \lambda_3, \quad (\text{B1. 34})$$

$$= \frac{\tau_1}{e^{(\lambda_1-\lambda_3)\tau_1} (b_1 + \cot \hat{\theta}_1) \sin^2 \theta_1} \text{ if } \lambda_1 < \lambda_2 = \lambda_3, \quad (\text{B1. 35})$$

$$= \frac{-2}{\sin^2 \theta_1 \cot^2 \theta_1} \text{ if } \lambda_1 = \lambda_2 = \lambda_3. \quad (\text{B1. 36})$$

Note that if $\lambda_1 < \lambda_2 < \lambda_3$, then by Lemma 4.2 one gets $\cot \phi_1 = \frac{\lambda_2 - \lambda_1}{\lambda_3 - \lambda_1} < 1$ which implies that $\phi_1 > \frac{\pi}{4}$. Since $\pi - \phi_1 < \hat{\theta}_1 < \frac{3\pi}{4}$, it follows that $b_1 + \cot \hat{\theta}_1 < 0$. Consequently, $\frac{d\tau_1}{d\theta_1} < 0$. For the cases where $\lambda_1 = \lambda_2 < \lambda_3$ or $\lambda_1 < \lambda_2 = \lambda_3$, one has $\pi - \phi_1 < \hat{\theta}_1 < \pi$, which again implies that $b_1 + \cot \hat{\theta}_1 < 0$ and $\frac{d\tau_1}{d\theta_1} < 0$. For the case where $\lambda_1 = \lambda_2 = \lambda_3$ it is obvious that $\frac{d\tau_1}{d\theta_1} < 0$. Now it will be proven that $\hat{\theta}_1$ is also a decreasing functions of θ_1 . Towards this end, note that

$$\frac{d\hat{\theta}_1}{d\theta_1} = \frac{d\hat{\theta}_1}{d\tau_1} \frac{d\tau_1}{d\theta_1}, \quad (\text{B1. 37})$$

where

$$\frac{d\hat{\theta}_1}{d\tau_1} = \frac{(\lambda_3 - \lambda_1) (\cot \theta_1 + b_1) e^{(\lambda_3-\lambda_2)\tau_1} \sin^2 \hat{\theta}_1}{(e^{(\lambda_3-\lambda_1)\tau_1} - 1)} \text{ if } \lambda_1 \leq \lambda_2 < \lambda_3, \quad (\text{B1. 38})$$

$$= \frac{(\cot \theta_1 + b_1) e^{(\lambda_3-\lambda_1)\tau_1} \sin^2 \hat{\theta}_1}{\tau_1} \text{ if } \lambda_1 < \lambda_2 = \lambda_3, \quad (\text{B1. 39})$$

$$= \frac{\sin^2 \hat{\theta}_1 \cot^2 \hat{\theta}_1}{2} \text{ if } \lambda_1 = \lambda_2 = \lambda_3. \quad (\text{B1. 40})$$

Since $\frac{\pi}{2} < \theta_1 < \pi - \phi_1$ for transitive trajectories, it follows that $\cot \theta_1 + b_1 > 0$. This implies that $\frac{d\hat{\theta}_1}{d\tau_1} > 0$. Then, since $\frac{d\tau_1}{d\theta_1} < 0$, it also follows that $\frac{d\hat{\theta}_1}{d\theta_1} = \frac{d\hat{\theta}_1}{d\tau_1} \frac{d\tau_1}{d\theta_1} < 0$.

a) Recall that $\frac{\pi}{2} < \theta_1 < \pi - \phi_1$ for a transitive trajectory. Furthermore by the previous Remark, $\tau_1 \rightarrow 0$ as $\theta_1 \rightarrow \pi - \phi_1$ and $\tau_1 \rightarrow \infty$ as $\theta_1 \rightarrow \frac{\pi}{2}$. Therefore,

$$\lim_{\tau_1 \rightarrow 0} \cot \hat{\theta}_1 = \cot(\pi - \phi_1), \quad (\text{B1. 41})$$

for all possible algebraic degrees of the eigenvalues.

b) On the other hand, if the limit as $\tau_1 \rightarrow \infty$ is considered and equation (4.60) used for $\cot \theta_1$, then one obtains

$$\lim_{\tau_1 \rightarrow \infty} \cot \hat{\theta}_1 = -1, \text{ if } \lambda_1 < \lambda_2 < \lambda_3, \left(\hat{\theta}_1 \rightarrow \frac{3\pi}{4} \right). \quad (\text{B1. 42})$$

c) if $\lambda_1 = \lambda_2 < \lambda_3$ or if $\lambda_1 < \lambda_2 = \lambda_3$ $\left(\hat{\theta}_1 \rightarrow \pi \right)$,

$$\lim_{\tau_1 \rightarrow \infty} \cot \hat{\theta}_1 = -\infty, \quad (\text{B1. 43})$$

d) if $\lambda_1 = \lambda_2 = \lambda_3$, $\left(\hat{\theta}_1 \rightarrow \frac{3\pi}{2} \right)$,

$$\lim_{\tau_1 \rightarrow \infty} \cot \hat{\theta}_1 = 0. \quad (\text{B1. 44})$$

□

Proof of the Lemma 4.5 :

i) Suppose that $z_1(t)$ be a trajectory starting from $\mathcal{S}_1 \cap \mathcal{H}$ with initial condition $z_1(0) = v_1(\theta_1)$ where $-\phi_1 \leq \theta_1 < \pi - \phi_1$. In view of Lemma 4.4-iii, there exists $\tau_1 > 0$ such that $c^T z_1(\tau_1) = 0$ and the trajectory changes mode. Thus, one has

$$z_1(\tau_1) = e^{\sigma_1 \tau_1} v_1(\hat{\theta}_1), \quad (\text{B1. 45})$$

where $\hat{\theta}_1$ in $(\pi - \phi_1, \pi + \phi_1)$. Then, in view of Definition 4.1 there exists a unique θ_2 in $[-\phi_2, 2\pi - \phi_2]$ and $\eta_{12}(\hat{\theta}_1) > 0$ such that $v_1(\hat{\theta}_1) = \eta_{12}(\hat{\theta}_1) v_2(\theta_2)$. This implies that

$$e^{\sigma_1 \tau_1} v_1(\hat{\theta}_1) = \eta_{12}(\hat{\theta}_1) e^{\sigma_1 \tau_1} (\hat{s}_2 \sin \theta_2 + \hat{s}_3 \cos \theta_2), \quad (\text{B1. 46})$$

and

$$\left(\hat{x}_1 \sin \hat{\theta}_1 + y_1 \cos \hat{\theta}_1 \right) = \eta_{12}(\hat{\theta}_1) (\hat{s}_2 \sin \theta_2 + \hat{s}_3 \cos \theta_2). \quad (\text{B1. 47})$$

The equation above can be written as

$$\Gamma_1 \begin{bmatrix} \sin \hat{\theta}_1 \\ \cos \hat{\theta}_1 \end{bmatrix} = \eta_{12}(\hat{\theta}_1) \Gamma_2 \begin{bmatrix} \sin \theta_2 \\ \cos \theta_2 \end{bmatrix}, \quad (\text{B1. 48})$$

where

$$\Gamma_1 = \begin{bmatrix} \hat{x}_1 & y_1 \end{bmatrix} = \frac{\omega_1}{a_{32}} \begin{bmatrix} \frac{b_1(a_{11}-\sigma_1)-\omega_1}{a_{21}} & \frac{(a_{11}-\lambda_1)}{a_{21}} \\ b_1 & 1 \end{bmatrix} \quad (\text{B1. 49})$$

$$\text{and} \quad \Gamma_2 = \begin{bmatrix} \hat{s}_2 & \hat{s}_3 \end{bmatrix} = \begin{bmatrix} \mu_1 & -1 \\ -1 & 0 \end{bmatrix} \quad (\text{B1. 50})$$

$$\eta_{12}(\hat{\theta}_1) \begin{bmatrix} \sin \theta_2 \\ \cos \theta_2 \end{bmatrix} = \Gamma_2^{-1} \Gamma_1 \begin{bmatrix} \sin \hat{\theta}_1 \\ \cos \hat{\theta}_1 \end{bmatrix}. \quad (\text{B1. 51})$$

Let $Q_1 = \begin{bmatrix} 1 & 0 \\ b_1 & 1 \end{bmatrix}$, then it follows that

$$\eta_{12}(\hat{\theta}_1) \begin{bmatrix} \sin \theta_2 \\ \cos \theta_2 \end{bmatrix} = \Gamma_2^{-1} \Gamma_1 Q_1^{-1} Q_1 \begin{bmatrix} \sin \hat{\theta}_1 \\ \cos \hat{\theta}_1 + b_1 \sin \hat{\theta}_1 \end{bmatrix} \quad (\text{B1. 52})$$

Since $0 < \theta_2 < \pi$, using above equations it follows that equation (4.99) and equation (4.100) hold.

ii) For the second part of the proof, let $z_2(t)$ be a transitive trajectory starting from $S_2 \cap \mathcal{H}$ with initial condition $z_2(0) = \eta_{12}(\hat{\theta}_1) e^{\sigma_1 \tau_1} v_2(\theta_2)$. Then Corollary 4.1 implies that $\frac{\pi}{2} < \theta_2 < \pi$ and there exists $\tau_2 > 0$ such that $c^T z_2(\tau_2) = 0$. Using Lemma 4.3.i, one obtains

$$z_2(\tau_2) = \eta_{12}(\hat{\theta}_1) e^{\sigma_1 \tau_1} e^{\mu_3 \tau_2} N_2 \left\{ \hat{s}_2 \sin \hat{\theta}_2 + \hat{s}_3 \cos \hat{\theta}_2 \right\}, \quad (\text{B1. 53})$$

$$= \eta_{12}(\hat{\theta}_1) e^{\sigma_1 \tau_1} e^{\mu_3 \tau_2} \left\{ -\hat{s}_2 \sin \theta_2 + \hat{s}_3 \cos \theta_2 \right\}. \quad (\text{B1. 54})$$

Then, in view of Definition 4.1 and Definition 4.5 there exists a unique θ_{11} in $(-\phi_1, -\psi_1)$ and $\eta_{21}(\theta_2) > 0$ such that $v_2(\theta_2) = \eta_{21}(\theta_2) v_1(\theta_{11})$. Therefore,

$$z_2(\tau_2) = \eta_{12}(\hat{\theta}_1) e^{\sigma_1 \tau_1} e^{\mu_3 \tau_2} \left\{ -\hat{s}_2 \sin \theta_2 + \hat{s}_3 \cos \theta_2 \right\}, \quad (\text{B1. 55})$$

$$= \eta_{12}(\hat{\theta}_1) e^{\sigma_1 \tau_1} e^{\mu_3 \tau_2} \eta_{21}(\theta_2) \left\{ \hat{x}_1 \sin \theta_{11} + y_1 \cos \theta_{11} \right\}, \quad (\text{B1. 56})$$

and

$$\{-\hat{s}_2 \sin \theta_2 + \hat{s}_3 \cos \theta_2\} := \eta_{21}(\theta_2) \{\hat{x}_1 \sin \theta_{11} + y_1 \cos \theta_{11}\}. \quad (\text{B1. 57})$$

The existence of $\eta_{21}(\theta_2) > 0$ follows from the fact that both $\{\hat{s}_2, \hat{s}_3\}$ and $\{\hat{x}_1, y_1\}$ are bases for \mathcal{H} . This equation can be written as

$$\eta_{21}(\theta_2) \Gamma_1 \begin{bmatrix} \sin \theta_{11} \\ \cos \theta_{11} \end{bmatrix} = \Gamma_2 \begin{bmatrix} -\sin \theta_2 \\ \cos \theta_2 \end{bmatrix}. \quad (\text{B1. 58})$$

Let $Q_1 = \begin{bmatrix} 1 & 0 \\ b_1 & 1 \end{bmatrix}$, then it follows that

$$\eta_{21}(\theta_2) \Gamma_1 Q_1^{-1} Q_1 \begin{bmatrix} \sin \theta_{11} \\ \cos \theta_{11} \end{bmatrix} = \Gamma_2 \begin{bmatrix} -\sin \theta_2 \\ \cos \theta_2 \end{bmatrix} \quad (\text{B1. 59})$$

$$\eta_{21}(\theta_2) \Gamma_1 Q_1^{-1} \begin{bmatrix} \sin \theta_{11} \\ \cos \theta_{11} + b_1 \sin \theta_{11} \end{bmatrix} = \begin{bmatrix} \mu_1 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -\sin \theta_2 \\ \cos \theta_2 \end{bmatrix} \quad (\text{B1. 60})$$

Premultiplying the equation above by $Q_2 = \begin{bmatrix} 1 & -\frac{(a_{11}-\lambda_1)}{a_{21}} \\ 0 & 1 \end{bmatrix}$, one gets

$$\eta_{21}(\theta_2) Q_2 \Gamma_1 Q_1^{-1} \begin{bmatrix} \sin \theta_{11} \\ \cos \theta_{11} + b_1 \sin \theta_{11} \end{bmatrix} = Q_2 \begin{bmatrix} \mu_1 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -\sin \theta_2 \\ \cos \theta_2 \end{bmatrix} \quad (\text{B1. 61})$$

$$\implies \eta_{21}(\theta_2) \begin{bmatrix} \sin \theta_{11} \\ \cos \theta_{11} + b_1 \sin \theta_{11} \end{bmatrix} = (Q_2 \Gamma_1 Q_1^{-1})^{-1} Q_2 \begin{bmatrix} -\sin \theta_2 \\ \cos \theta_2 \end{bmatrix} \quad (\text{B1. 62})$$

Consequently,

$$\eta_{21}(\theta_2) = \frac{\frac{a_{32}}{\omega_1} \sin \theta_2}{(\cos \theta_{11} + b_1 \sin \theta_{11})}, \quad (\text{B1. 63})$$

and

$$\theta_{11} = \cot^{-1} \left\{ \frac{-1}{B - \frac{a_{21}}{\omega_1(1+b_1^2)} \cot \theta_2} - b_1 \right\}. \quad (\text{B1. 64})$$

Then using the equation (4.99), it follows that the equation (4.103) holds.

iii) For the last part of the proof of this item recall by Lemma 4.3.ii.d that $\pi - \phi_2 < \hat{\theta}_2 < \frac{3\pi}{2}$ where $v_2(\pi - \phi_2) \simeq v_1(-\phi_1)$ and $v_2\left(\frac{3\pi}{2}\right) \simeq -\hat{s}_2$. Since $\phi_2 = 0$ by Remark 4.2, substituting $\theta_2 = 0$ in equation (B1. 64) it follows that $\theta_{11} = -\phi_1$. On the other hand, substituting $\theta_2 = \frac{3\pi}{2}$ in equation (B1. 64) one gets $\theta_{11} = -\psi_1$, where $v_1(-\psi_1) \simeq v_2\left(\frac{3\pi}{2}\right) \simeq -\hat{s}_2$ and $\cot \psi_1 = \frac{1}{B} + b_1$. Hence, it follows that $-\phi_1 < \theta_{11} < -\psi_1$. \square