FLUCTUATION THEORY OF OMEGA-KILLED LÉVY PROCESSES

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ABSTRACT

FLUCTUATION THEORY OF OMEGA-KILLED LÉVY PROCESSES

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This thesis explores the fluctuation identities for omega-killed (reflected) processes. In the first part, we develop the theory of the so-called $\mathcal{W}_q$ and $\mathcal{Z}_q$ scale functions for the fluctuations of right-continuous discrete time and space killed random walks. Explicit expressions are derived for the resolvents and two-sided exit problems when killing depends on the present level of the process. Similar results in the reflected case are also considered. All the derivations are given in terms of a new generalization of the scale functions, which are obtained using different arguments from the continuous case (spectrally negative Lévy processes). Hence, we spell out the connections between the two cases. We obtain the probability of bankruptcy in the omega model of the actuarial literature for a specific form of the killing function.

In the second part of this thesis, we analyze exit problems for a level-dependent Lévy process, which is exponentially killed with an intensity depending on the present state of the process. By considering level-dependent spectrally negative Lévy processes, we take general premium rate as a function depending on the current level of the processes as well. Further, we derive the respective resolvents for the omega-killed level-dependent process. All exit identities are introduced via a new generalization of scale functions, $\mathcal{W}^{(q)}$ and $\mathcal{Z}^{(q)}$ (counterparts of the scale function from the theory of Lévy processes), which are solutions of Volterra integral equations. Additionally, we obtain similar results for the reflected omega-killed level-dependent Lévy processes. The existence of the solution of the stochastic differential equation for reflected level-dependent Lévy processes is also discussed. Finally, to illustrate our result, the probability of bankruptcy is obtained for an insurance risk process.

Keywords: Fluctuation theory, Omega-killed lévy process, Upwards skip-free random walk, Multi-refracted process, Level-dependent process.

ÖZ

OMEGA-TİPİ DURDURULMUŞ LÉVY SÜREÇLERİNİN GİRİŞ VE ÇIKIŞ HAREKETLERİ

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Bu tez çalışmasında ω-tipi durdurulmuş süreçlerin geçiş problemleri çalışılmıştır. İlk olarak kesikli zaman ve uzay rassal yürüyüş (yukarı sıçramasız rassal yürüyüş) modelinin geçiş problemleri için q-tipi skala fonskiyonları $\mathcal{W}_q$ ve $\mathcal{Z}_q$ teorisi geliştirilmiştir. Böylece bahsedilen rassal yürüyüş mevcut seviyesine bağlı olarak durdurulmuş sürece çevrilip, geçiş problemlerinin açık çözümleri kanıtlanmıştır. Aynı problem çözümleri ayrıca bahsedilen rassal yürüyüşün yansıtlan hali içinde bulunmuştur. Bütün bulunan açık çözümler, sürekli süreçlerden farklı argümanlar kullanılarak elde edilen ve yeni bir genelleme olan skala fonksiyonları cinsinden ifade edilmiştir. Formülize edilen sonuçumuzun bir uygulmasını göstermek amacıyla aktüerya literatüründe omega-model olarak adlandırılan, nihai iflas olasılığı bulunmuştur.

Bu tezin ikinci kısmında ise ω-tipi durdurulmuş seviyeye bağlı Lévy süreçleri için geçiş problemleri analiz edilmiştir. Böylece yukarı sıçramasız Lévy süreci hem seviyeye bağlı olarak durdurulmuş hem de seviyeye bağlı prim oranı kazanımı haline getirilmiştir. Ayrıca ilgili çözenler de türetilmiştir. Yine bu çalışmada da bütün teoremler

Voltera integral denklemi olan skala fonksiyonlarının yeni bir genellemesi cinsinden sunulmuştur. Ayrıca, bütün çıkış problemleri yansıyan ve seviyeye bağlı Lévy süreçleri için de elde edilmiştir. Ek olarak, yansıyan seviyeye bağlı Lévy süreçlerinin stokastik differensiyel denklemlerinin çözümünün varlığı tartışılmıştır. Son olarak, elde edilen sonuçların bir uygulması olarak bir sigorta şirketi için nihai olasılık bahsedilen süreç göz önünde bulundurularak hesaplanmıştır.

Anahtar Kelimeler: Geçiş problemleri, Omega-tipi durdurulmuş lévy süreçleri, yukarı sıçramasız rassal yürüyüş, Çoklu kırılmış süreçler, Seviyeye bağlı süreçler.

To my family,

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LIST OF ABBREVIATIONS

ABBREVIATIONS

a.e.	almost everywhere
a.s.	almost surely
$(a \wedge b)$	minimum of a and b, i.e. $\min\{a, b\}$
$(c \vee d)$	maximum of c and d, i.e. $\max\{c, d\}$
IID	independent identically distributed
LHS	left-hand side
p.g.f	probability generating function
p.m.f	probability mass function
SNLP	Spectrally Negative Lévy Process
SDE	Stochastic Differential Equation
RHS	right-hand side

CHAPTER 1

INTRODUCTION

In the theory of stochastic processes, it is very natural to consider an isolated point ∂, which is interpreted as a cemetery state, to the state space of a process, supposing that if it hits this isolated point ∂, then it stays there forever. Mathematically, the killing concept can be introduced as follows. Let e_q be an independent exponential (time) random variable of rate $q \geq 0$ (for convention $e_0 = \infty$ almost surely) and also be independent of the process of interest $\widetilde{X}_t$. Then, the process killed at e_q (or exponentially killed process) is defined as

$$X_t = \begin{cases} \widetilde{X}_t, & \text{if } t < e_q, \\ \partial, & \text{if } t \geq e_q, \end{cases} \qquad t \geq 0,$$

where ∂ is called a cementary or dummy state. The idea of killing processes presents more handy results for the risk theory since the main interest is related to the first passage time and other respective quantities. In fact, it is possible to extend the results related to these identities (or even the state space) without giving much additional effort, see the excellent monograph with some examples from risk theory in [20]. Nevertheless, the classical killing concepts are not enough to express certain structures, for example, exit identities for the risk model with dividend or tax payment in which it is more convenient to consider dividend or tax payment according to the current balance of a financial company.

In the paper of Albrecher et. al (2011) [1], for example, contrary to the ruin concept in the traditional actuarial applications, an insurance company with a negative surplus is able to continue its business until bankruptcy occurs. They also assumed that the probability of bankruptcy is a function of the level of a negative surplus and then is called *omega model*. This model is further investigated by Gerber et al. (2012) [18]

by focusing on two main objectives: first is to determine the probability of bankruptcy and expected discounted penalty at the time of bankruptcy, and the second is to derive the total time of the process spent below zero. Inspired by the omega model mentioned in the above literature, Li and Palmowski (2019) [26] is introduced a seminal paper that considers level-dependent killing for a spectrally negative Lévy process (SNLP). In that paper, the rate of that random exponential time is not constant anymore but rather a function of the present value (or state) of the process, which is called as ω-killing. The expression of ω-killing could be interpreted as a killing feature that is level (or state) dependent Laplace argument, a discount factor, a bankruptcy rate, or the weight for occupation time up to some exit times. For example, we can consider ω as a weight function to describe a sophisticated discount structure, which is seen as a dynamic discount reflecting path dependence in financial applications. The study presents the fluctuation identities such as two-sided exit below/above and resolvent for ω-killed spectrally negative Lévy processes that generalizes the classical fluctuation theory of SNLP. Particularly if the killing function is chosen as a constant, q, they immediately obtain classical exit identities.

Proposed Research Studies

In this thesis, we study ω-killed processes with different characteristics, see Figure 1.1 for the structure of the thesis. In the first part of this thesis, we examine ω-type killed discrete time/space so-called *q-killed (reflected) upwards skip-free random walk*. Similar to ω-killed SNLP in [26], we consider *q-killing* as a killing feature, which may be interpreted as a (level-dependent) probability generating argument, a bankruptcy probability, or the weight for occupation in different contexts. Due to q-killing, our study generalizes the result of fluctuation theory of upwards skip-free random walk derived in [5]. We derive explicit expressions for the exit problems when the upwards skip-free random walk is killed geometrically with a rate depends on level of the process by using *first-step analysis approach*. All exit identities are presented in terms of q-killed scale functions $\{\mathcal{W}_q(x),\ x \in \mathbb{N}_0\}$ and $\{\mathcal{Z}_q(x),\ x \in \mathbb{N}_0\}$ as recursive equations. Further, we derive the important quantities potential measure (resolvent) and exit identities for the reflected q-killed upwards skip-free random

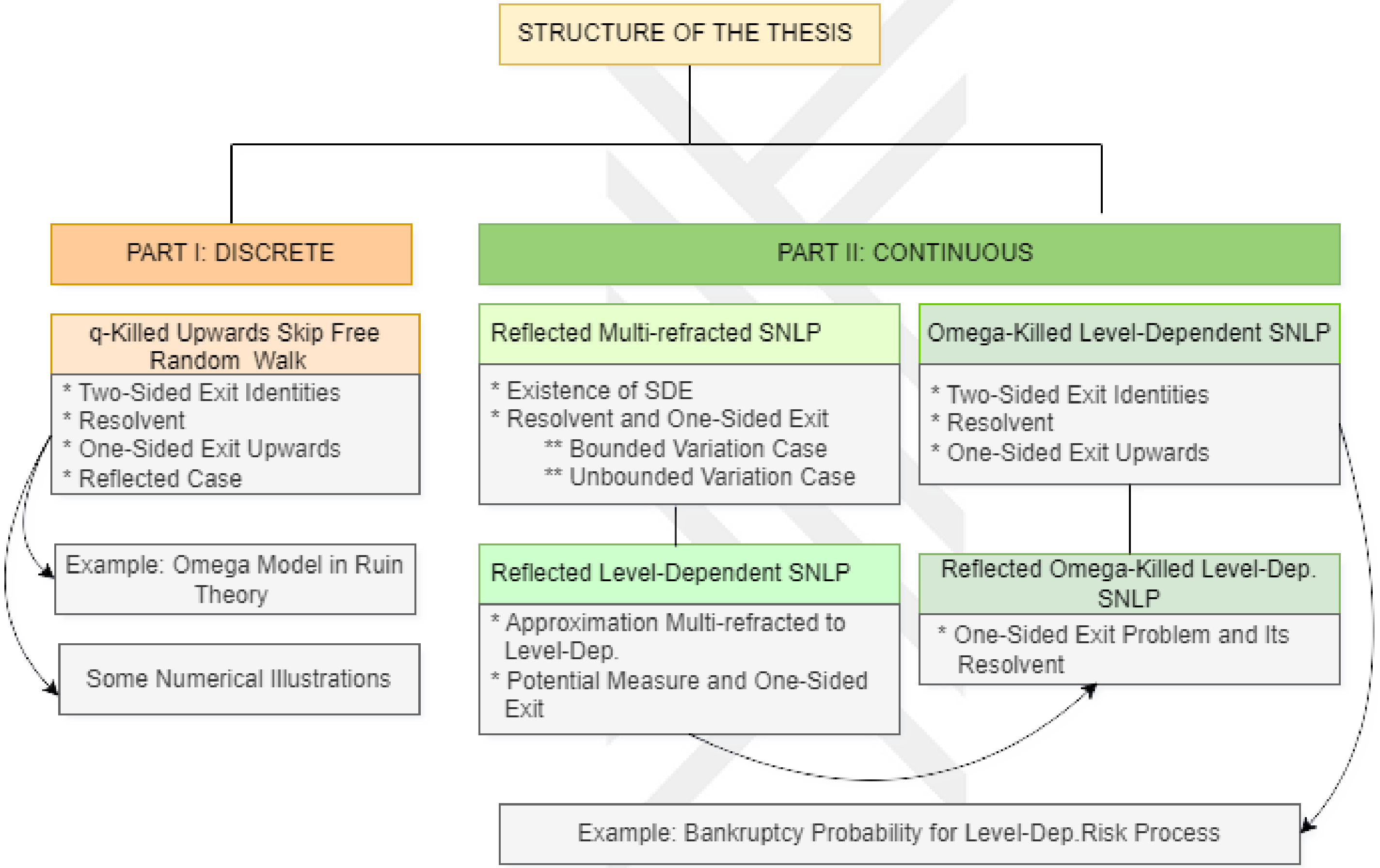


Figure 1.1: Structure of the thesis.

walk which are mostly used in the capital injection in the actuarial context. Lastly in this part of the thesis, we show one of the applications of our results, namely omega model in ruin theory in which we calculate the probability of bankruptcy for a specific function of q-killing. Then, we present some numerical illustration to understand the effect of parameters such as geometric distribution and q-killing.

In the second part of this thesis, we explore ω-killed level-dependent spectrally negative Lévy processes which generalizes the study of level-dependent SNLP introduced in [13] and also in [25, 26, 31]. We start with a combination of the level-dependent and reflected SNLP. For derivation of exit identities for a reflected level-dependent SNLP with a general function of ϕ, we first focus on reflected multi-refracted SNLP. We discuss the existence of reflected multi-refracted SNLP by employing a strong approximation scheme. Then, we derive the potential measure and one-sided exit problem for reflected multi-refracted SNLP, which does not exist in the current literature. We then approximate a general rate function ϕ by a sequence of rate functions $(\phi_n)_{n\geq 1}$ of the form in the multi-refracted case. We define a sequence of multi-refracted SNLP which permits us to prove existence and uniqueness of solution to reflected level-dependent SNLP by limiting argument. Thus, we are able to obtain potential measure and one-sided exit identity for the reflected-level dependent SNLP by using limiting argument of the multi-refracted scale functions with the same fashion in [13]. The results of this section is new in literature and constitute a generalization of the reflected-refracted case in [31].

In this current part of the thesis, our main focus to study ω-killed level-dependent SNLP, which is the natural extension of study [26]. Similar to the discrete part of this thesis, ω-killing transforms level-dependent SNLP to a process that is exponentially killed with an intensity depends on the present value of the process. Moreover, by considering level-dependent SNLP, we model the risk process with the premium rate function depends on the current level of the process as well. We derive exit problems and resolvents in terms of a new generalization of the scale functions, namely omega-killed level-dependent scale functions, i.e. $\{\mathcal{W}^{(\omega)}(x),\ x\in\mathbb{R}\}$ and $\{\mathcal{Z}^{(\omega)}(x),\ x\in\mathbb{R}\}$. Additionally, we present the results for the reflected case: potential measure and one-sided exit problem for the reflected ω-killed level-dependent SNLP. We finalize this study with an substantial example for finding the probability of bankruptcy if the surplus process is modeled by ω-killed level-dependent SNLP. The primary method throughout all proofs in this part depends on the Poisson observation method and classical fluctuation identities.

Contributions and Novelties

Our derivation of fluctuation theory both in the discrete and continuous case described in section ‘Proposed Research Studies’ uses in some cases on a number of results available in literature which are explained and presented in some subsections of the thesis. To separate these from the new results, our contributions (new results) can be summarized as follows:

- In Section 2.2, we prove the exit identities for (reflected) upwards skip-free random walk. The result of this study is obtained which are non-direct analogous of the ω-killed spectrally negative Lévy processes [12, 26]. The highlights of this study are:
    - The results of Section 2.2 is a generalization of the first passage problems for upwards skip-free random walk given in [5].
    - Due to their recursive nature of q-killed scale functions (see Eq. (2.2.2) and Eq. (2.2.3)), they are straightforward to solve when it is compared with continuous time Lévy setup.
    - To the best of our knowledge, it is the first time the resolvent identity given in Section 2.2.3 is solved by first-step analysis and the exit identities for reflected upwards skip-free random walk given in Section 2.2.4 is presented in terms of scale functions.
- In Section 3.6, we derive the potential measure and one-sided exit problem for reflected level-dependent SNLP. Although reflected refracted processes are studied in [31], the fluctuation theory of reflected multi-refracted and reflected level-dependent processes does not exist in the literature. In fact, findings of Section 3.6 are a non-trivial extension of the study in [31]. Main highlights of this current study are:

  - We deal with complexity of the existence of the stochastic differential equation (SDE) given in Eq. (3.6.1) with strong approximation.
  - It is the first time in literature reflected version of multi-refracted and analogously level-dependent SNLP are considered.

- In Section 3.7, we study fluctuation theory of ω-killed (reflected) level-dependent SNLP, which is a novel concept in the literature. The results are generalizations of the fluctuation theory of ω-killed SNLP given in [26] and level-dependent SNLP in [13], refracted SNLP in [25], and refracted-reflected SNLP in [31].

The Outline of the Thesis

The organization of this thesis as follows. We start with overviewing upwards skip-free random walk and its fluctuation theory in Chapter 2. Then, in Section 2.2, we present our study, namely (reflected) q-killed upwards skip-free random walk and its exit identities. After studying the discrete time/space model, we move on to the continuous time/space setup in Chapter 3, where we first review the fluctuation theory of various processes: starting with the basic properties of the Lévy processes, and gradually summarizing the background concepts from refracted to level-dependent SNLP. We devote Section 3.6 for our results related to the potential measure and one-sided exit problem of the reflected level-dependent SNLP. Finally, in Section 3.7, we present our study of fluctuation theory for ω-killed level-dependent SNLP along with the reflected case. We outline our findings in Chapter 4.

CHAPTER 2

OMEGA-TYPE KILLED UPWARDS SKIP-FREE RANDOM WALK AND ITS FLUCTUATION THEORY

In probability theory applications, one of the main concerns is a quantity that changes randomly at time (or in a space), which is modeled by stochastic processes. For instance, the motion of a particle that moves in discrete jumps with certain probabilities from one point to another can be described as a path of a stochastic process known as a random walk. Mathematically, a random walk is a stochastic process constructed by successive summation of independent identically distributed (IID) random variables. If a random walk goes upwards with unit size, and downwards with any arbitrary magnitude, it is called as *upwards skip-free random walk (or right-continuous random walk)* which is well-studied in actuarial and financial applications.

In this chapter, firstly we aim to construct general framework for an upwards skip-free random walk and its fluctuations when the time and space of the process are both discrete. The results on this section can be found in great details in [5, 9, 16, 21, 30, 38] and references therein. Secondly, we introduce our results related to the exit identities for q-killed upwards skip-free random walk which generalizes the results of classical exit problems for the upwards-skip free random walk given in the following section.

2.1 Upwards Skip-Free Random Walk

Consider a discrete time/space random walk $X = (X_n)_{n\geq 0}$ defined by

$$X_n = X_0 + cn - \sum_{i=1}^{n} C_i, \quad n \in \mathbb{N}_0 = \{0, 1, 2, \ldots\},$$

where, in the actuarial context, $X_0 \in \mathbb{Z}$, is the initial capital of the insurance firm, $c \in \mathbb{N}$ is the premium income rate and the claims $\{C_i\}_{i\in\mathbb{N}}$, are jumps that takes values in $\mathbb{N}_0$. We assume that $\{C_i\}_{i\geq 1}$ are independent, identically distributed (IID) random variables with probability mass function (p.m.f) $p_k = \mathbb{P}(C_1 = k)$ for $k \in \mathbb{N}_0$.

Throughout this chapter, the law of X such that $X_0 = x$ is denoted by $\mathbb{P}_x$ and the corresponding expectation by $\mathbb{E}_x$. We write $\mathbb{P}$ and $\mathbb{E}$ when $x = 0$.

The above discrete time risk model can be simplified to *upwards skip-free random walk*, when $c = 1$, as

$$X_n = X_0 + n - \sum_{i=1}^{n} C_i, \quad n \in \mathbb{N}_0. \tag{2.1.1}$$

Provided that $p_0 > 0$, the probability generating function (p.g.f) of the claims is given

$$\tilde{p}(z) := \mathbb{E}[z^{C_1}] = \sum_{k=0}^{\infty} z^k p_k, \quad z \in (0,1].$$

Then for $n \in \mathbb{N}$, we have

$$\mathbb{E}[z^{\sum_{i=1}^{n} C_i}] = [\tilde{p}(z)]^n = (p_0 + (1-p_0)\tilde{p}_{C|C\geq 1}(z))^n,$$

which shows that the total claim arising from n time periods and hence, $\sum_{i=1}^{n} C_i$ has compound binomial distribution. This explains the name of *compound binomial model* in which at each discrete time, a positive claim either occurs with probability $(1-p_0)$ or not occur with probability p_0, independently of the size of positive claims.

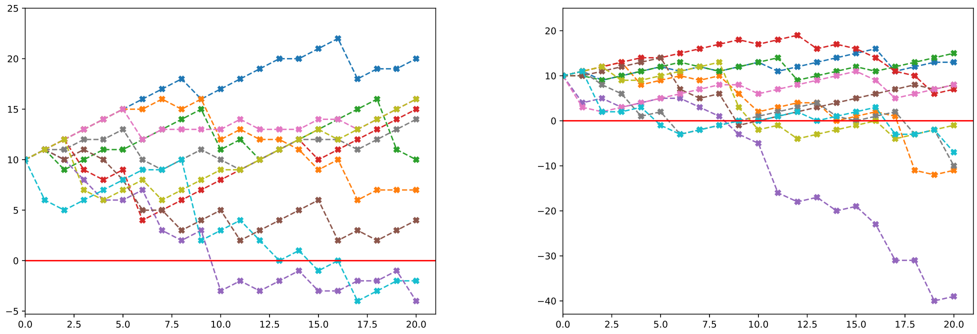
In Figure 2.1, we demonstrate the different paths of upwards skip-free random walk when the claims are distributed as geometric with parameter $p = 0.4$ (see Figure 2.1a) and negative binomial with parameters $r = 3,\ p = 0.4$ (see Figure 2.1b). These are the most commonly used distributions for the compound binomial model in the actuarial context.

By independence of the $\{C_i\}_{i\geq 1}$, it can also be written for $n \in \mathbb{N}_0$, and $\nu \in (0, z/\tilde{p}(z))$

$$\mathbb{E}[z^{\sum_{i=1}^{n}(C_i-1)}] = \left(\frac{\tilde{p}(z)}{z}\right)^n \implies \sum_{m=0}^{\infty} \nu^m \mathbb{E}[z^{\sum_{i=1}^{m}(C_i-1)}] = \frac{1}{1-\nu\tilde{p}(z)/z}.$$

Further, let φ_ν be the largest root of the equation $z/\tilde{p}(z) = \nu$, i.e.

$$\varphi_\nu = \sup\left\{z : \frac{z}{\tilde{p}(z)} = \nu\right\} = \sup\left\{z : \mathbb{E}z^{1-C_1} = \nu\right\}.$$



(a) Geometric(0.4)

(b) Neg. Binomial(3, 0.4)

Figure 2.1: Simulation of upwards skip-free random walk under different distribution assumption.

We also let

$$\tau_b^- = \inf\{n \geq 0 : X_n \leq b\}, \quad \text{and} \quad \tau_a^+ = \inf\{n \geq 0 : X_n \geq a\}, \tag{2.1.2}$$

denote the first passage times below level b and above level a respectively with $\inf \emptyset = \infty$.

In the following sequel, we give the results for one-sided upwards exit problem, which introduce the Lundberg root φ_ν (analogue of $\Phi(q)$ from the fluctuation theory of SNLP which will be used later in Chapter 3) two-sided exit problems where discrete scale functions $W_\nu(x)$ and $Z_\nu(x)$ appear.

2.1.1 Exit Identities for Upwards Skip-Free Random Walk

In this subsection explicit results for the exit identities of upwards-skip free random walk, i.e. two-sided and one-sided exit problems and the resolvent measure are given.

Note that for one-sided exit (upwards) problem, we can say that due to the upwards skip-free and invariant property of p.g.f of τ_a^+, given below, implies multiplicative structure.

Theorem 1 (One-Sided Upwards Exit [5]). *For $x \leq a$, and $\nu \in (0, 1]$*

$$\mathbb{E}_x\left[\nu^{\tau_a^+} \mathbf{1}_{(\tau_a^+ < \infty)}\right] = \varphi_\nu^{a-x}. \tag{2.1.3}$$

In the context of Lévy processes, the two-sided upwards passage problem is expressed in terms of the $W^{(q)}$ scale functions (will be introduced in Chapter 3). In our setting of right-continuous random walk X, the upwards and downwards exit problems are handled by defining the discrete analogue scale functions W_ν of $W^{(q)}$.

For the right-continuous random walk X, discrete-time scale function $W_\nu(x)$ is given below.

Theorem 2 (Two-Sided Exit Above [5])**.** *For* $0 \leq x \leq a$*,*

$$\mathbb{E}_x\left[\nu^{\tau_a^+}\mathbf{1}_{(\tau_a^+<\tau_{-1}^-)}\right] = \frac{W_\nu(x)}{W_\nu(a)}, \tag{2.1.4}$$

where, in discrete time/space, we define

$$W_\nu(x) := \frac{1}{p_0\mathbb{E}\left[\nu^{\tau_x^+}\mathbf{1}_{(\tau_x^+<\tau_{-1}^-)}\right]}, \quad \textit{with} \quad W_\nu(0) = \frac{1}{p_0}. \tag{2.1.5}$$

Note that the choice of normalization $W_\nu(0) = 1/p_0$ is decided arbitrarily in order to obtain the simplest possible form for the z-transform of $W_\nu(\cdot)$. In order to utilize Eq. (2.1.4), we need to calculate $W_\nu(x)$. A recursive approach may be used to compute $W_\nu(x)$, as it is given in the following lemma.

Lemma 3 (Sec. 4 in [5])**.** *Conditioning on the first jump* $C_1 = k$*, Eq. (2.1.4) implies the harmonic recursion (see [30], Equation (3.1)), i.e.* $W_\nu(x)$*, can be calculated recursively by*

$$W_\nu(x) = \nu\sum_{y=-1}^{x} W_\nu(x-y)p_{y+1}, \quad x \in \mathbb{N}_0, \tag{2.1.6}$$

where the initial value of $W_\nu(\cdot)$ *is given by Eq. (2.1.5).*

Alternative to recursive equation of Lemma 3, z-transform (generating function) can be derived by the following proposition.

Proposition 4 (Sec. 4 in [5])**.** *Using the recursion in Eq. (2.1.6), taking* z*-transform of* $W_\nu(x)$ *gives*

$$\widetilde{W_\nu}(z) := \sum_{x=0}^{\infty} z^x W_\nu(x) = \frac{1}{\tilde{p}(z) - z/\nu}, \quad z \in (0, \varphi_\nu). \tag{2.1.7}$$

We conclude this section with the following proposition.

Proposition 5 (Prop. 1 in [5])**.** *For every* $\nu \in (0,1]$, $\left\{\nu^{n\wedge\tau_{-1}^-}W_\nu(X_{n\wedge\tau_{-1}^-})\right\}_{n\in\mathbb{N}_0}$ *is a martingale under* $\mathbb{P}_x$*.*

Next, we use the scale function $W_\nu(\cdot)$ and introduce another scale function namely $Z_\nu(\cdot)$, in order to provide an explicit expression for the two-sided downwards exit problem.

Theorem 6 (Two-Sided Exit Below [5])**.** *For* $\nu, \xi \in (0,1]$ *and* $a \geq 0$*, we have*

$$\mathbb{E}_x\left[\nu^{\tau_{-1}^-}\xi^{-X_{\tau_{-1}^-}}\mathbf{1}_{(\tau_{-1}^-<\tau_a^+)}\right] = Z_\nu(x,\xi) - \frac{W_\nu(x)}{W_\nu(a)}Z_\nu(a,\xi), \tag{2.1.8}$$

where

$$Z_\nu(x,\xi) = \left(\tilde{p}(\xi) - \frac{\xi}{\nu}\right)\sum_{k=0}^{\infty}\xi^k W_\nu(x+k),$$

for $\xi \in (0, \varphi_\nu),\ \nu \in (0,1],\ x \in \mathbb{N}_0$*.*

Note that when $\xi = 1$, Eq. (2.1.8) reduces to

$$\mathbb{E}_x\left[\nu^{\tau_{-1}^-}\mathbf{1}_{(\tau_{-1}^-<\tau_a^+)}\right] = Z_\nu(x) - \frac{W_\nu(x)}{W_\nu(a)}Z_\nu(a),$$

where $Z_\nu(x,1) = Z_\nu(x)$ with

$$Z_\nu(x) = 1 + \left(\frac{1}{\nu} - 1\right)\sum_{y=0}^{x-1} W_\nu(y), \quad \nu \in (0,1],\ x \in \mathbb{N}_0. \tag{2.1.9}$$

We also give an example to illustrate how scale functions may be utilized to describe potential measures (or sometimes called resolvents). The resolvents are usually used to explain the position of a random walk (or more generally of a Lévy process) right before the exit from some interval. For more details for the discrete time/space case, we refer to [5, 30]. For the analogue in the case of spectrally negative Lévy processes, we refer to [Chapter 8.4 in [24] and Theorem 2.7 in [22]].

Theorem 7 (Potential Measure [5])**.** *For* $a \in \mathbb{N}$*, the resolvent of the process X killed on exiting* $I_a := \{0, \ldots, a-1\}$*, denoted by* X'*, is given as*

$$\begin{aligned} U_\nu(x,y) &= \sum_{n=0}^{\infty}\nu^n\mathbb{P}_x[X'_n = y] = \sum_{n=0}^{\infty}\nu^n\mathbb{P}_x[X_n = y,\ n < \tau_{-1}^- \wedge \tau_a^+] \\ &= \nu^{-1}\left(\frac{W_\nu(a-1-y)W_\nu(x)}{W_\nu(a)} - W_\nu(x-y-1)\right), \end{aligned} \tag{2.1.10}$$

where $\{x,y\} \subset I_a$*, and* $\nu \in (0,1]$*.*

Remark 8. *As pointed out in [5], in the discrete time/space setup the method for deriving Eq. (2.1.4) and Eq. (2.1.8) is significantly different from the continuous time/space setup (where one has to make sure that the process drifts to ∞ and use change of measure – see proof of Theorem 8.1 in [24]). However, the expressions in the two-sided exit problems, as well as the resolvent measure are analogous with the spectrally negative case.*

Note that the role of the variable $\nu \in (0, 1]$ in the above transforms can also be thought of as a survival probability at each time period for the so-called killed (stopped) random walk. That is, at each time point, the random walk X may be 'killed' with some probability $1 - \nu \in [0, 1)$ and understood to be absorbed into some 'cemetery' type state.

In the following section, we shall consider the case where the 'killing' (survival) probability at each time point is no longer constant but depends on the level of the process; in the continuous-time case, this is known as omega-killing, see [26].

2.2 q-Killed Upward Skip-Free Random Walk

In this section, we derive explicit expressions for exit problems when the right-continuous random walk defined in Chapter 2 is weighted by a killing function $q(\cdot)$, which depends on the level of the process. We shall use the expression *q-killing* for such a killing feature, which may be interpreted as a (level-dependent) probability generating argument, a bankruptcy probability or the weight for occupation in different context. Although in continuous time/space, exit problems have been studied in [26] (for SNLP) and in [12] (for Markov additive processes), whilst ruin related quantities are studied in [1, 11], in discrete time/space studies for the above problems are missing. We point out that, as seen in the following sections, the techniques that have to be employed and the results for the exit problems within a discrete setting are considerably different with the ones in the continuous setting. This comes under no coincidence, since in the discrete time setting (unlikely with the continuous setting- Lévy processes) one may use the so-called *first-step analysis* to derive the exit problems and/or the potential measures. The first-step analysis approach leans on finding the position of the process right after the first jump epoch. By doing so, we obtain recursive equations, which we can solve them subsequently. Although this approach has been used in the literature in wide range (see for example [37] and references therein), we believe that, to the best of our knowledge, it is the first time in fluctuation theory of discrete time/space process is considered in this study especially in finding the potential measure in terms of a new generalizations of scale functions. Moreover, it is worth pointing out that the discrete set-up has significance over their more popular continuous time models, which are tractability in practice due to simplicity, whilst from a theoretical point of view one can replace the Wiener-Hopf factorisation by the conceptually simpler factorisation of Laurent series (see for e.g. [5, 6]).

To formulate our problem mathematically, let us assume that $q : \mathbb{Z} \to (0,1]$ non-negative function that represents killing mechanism and $X = \{X_n\}_{n\in\mathbb{N}_0}$ be an upwards skip-free random walk defined on the filtered probability space $(\Omega, \mathbf{F}, (\mathcal{F}_n)_{n\in\mathbb{N}_0}, \mathbb{P})$ (as it is given in Eq. (2.1.1) in the Section 2)

$$X_n = X_0 + n - \sum_{i=1}^{n} C_i, \quad n \in \mathbb{N}_0. \tag{2.2.1}$$

We denote the first passage times below level b and above level a respectively as follows

$$\tau_b^- = \inf\{n \geq 0 : X_n \leq b\} \quad \text{and} \quad \tau_a^+ = \inf\{n \geq 0 : X_n \geq a\}.$$

Moreover, let the law of X, such that $X_0 = x$, be denoted by $\mathbb{P}_x$ with corresponding expectation $\mathbb{E}_x$. We will write $\mathbb{P}$ and $\mathbb{E}$ when $x = 0$. As mentioned before, our main interest in this section is to derive closed form expressions for the first passage times and resolvent measures for ω-type killed upwards skip-free random walk, as well as their reflections. Further, as will be seen next, all the identities can be expressed in terms of two families of scale functions $\{\mathcal{W}_q(x),\ x \in \mathbb{N}_0\}$ and $\{\mathcal{Z}_q(x),\ x \in \mathbb{N}_0\}$, which we call as q-scale functions and satisfies the following recursive equations

$$\mathcal{W}_q(x) = \sum_{k=-1}^{x} p_{k+1} q(x-k) \mathcal{W}_q(x-k), \tag{2.2.2}$$

$$\mathcal{Z}_q(x) = \sum_{k=-1}^{x} p_{k+1} q(x-k) \mathcal{Z}_q(x-k) + \sum_{k=x+1}^{\infty} p_{k+1} q(x-k), \tag{2.2.3}$$

respectively and $\mathcal{W}_q(x) = 0$ and $\mathcal{Z}_q(x) = 1$ for $x < 0$. As mentioned before, the fact that the scale functions in the discrete setting satisfy recursive type expressions is not surprising. This is a consequence of the so-called 'first-step analysis' and the properties of the random walk. In fact, similar recursive expressions to those above have already been identified – in the case of a constant killing function – within the literature (see for example [5] and [30]). In these papers, the recursions are only briefly discussed or used to determine the form of the corresponding z-transforms as a means of determining the scale functions themselves. In the more general setting of this study however, it turns out to be the only way to characterise the scale functions, since it is not possible to obtain their z-transforms and thus, are employed to prove all of the following results.

2.2.1 Two-Sided Exit Upwards

In this section, we aim to derive explicit expression for the two-sided upwards exit problem, which will be given in terms of the q-killed scale function.

Theorem 9 (Two-sided exit upwards)**.** *For integer $x \leq a \in \mathbb{N}_0$, we have that*

$$\mathcal{A}(x,a) = \mathbb{E}_x\Big[\prod_{n=1}^{\tau_a^+} q(X_n)\mathbf{1}_{(\tau_a^+<\tau_{-1}^-)}\Big] = \frac{\mathcal{W}_q(x)}{\mathcal{W}_q(a)}, \tag{2.2.4}$$

where $\mathcal{W}_q(\cdot)$ satisfies the recursive equation given in Eq. (2.2.2).

Proof. For $X_0 = x < 0$, it is straightforward to see, since $\mathcal{W}_q(x) = 0$ for $x \in \mathbb{Z}^-$.

For $x \geq 0$, applying the strong Markov property of X at τ_x^+ and using the fact that X is upwards skip-free random walk, for $0 \leq x \leq a$, we get

$$\begin{aligned}
\mathbb{E}\Big[\prod_{n=1}^{\tau_a^+} q(X_n)\mathbf{1}_{(\tau_a^+<\tau_{-1}^-)}\Big] &= \mathbb{E}\Big[\prod_{n=1}^{\tau_x^+} q(X_n)\mathbb{E}_{X_{\tau_x^+}}\big[\prod_{n=1}^{\tau_a^+} q(X_n)\mathbf{1}_{(\tau_a^+<\tau_{-1}^-)}\big]\mathbf{1}_{(\tau_x^+<\tau_{-1}^-)}\Big] \\
&= \mathbb{E}\Big[\prod_{n=1}^{\tau_x^+} q(X_n)\mathbf{1}_{(\tau_x^+<\tau_{-1}^-)}\Big] \times \mathbb{E}_{X_{\tau_x^+}}\Big[\prod_{n=1}^{\tau_a^+} q(X_n)\mathbf{1}_{(\tau_a^+<\tau_{-1}^-)}\Big] \\
&= \mathbb{E}\Big[\prod_{n=1}^{\tau_x^+} q(X_n)\mathbf{1}_{(\tau_x^+<\tau_{-1}^-)}\Big] \times \mathbb{E}_x\Big[\prod_{n=1}^{\tau_a^+} q(X_n)\mathbf{1}_{(\tau_a^+<\tau_{-1}^-)}\Big],
\end{aligned}$$

or equivalently by using the definition of $\mathcal{A}(x,\cdot)$

$$\mathcal{A}(0,a) = \mathcal{A}(0,x)\mathcal{A}(x,a), \quad \text{for all} \quad 0 \leq x \leq a.$$

Following the same line of logic in [5], we set for $x \in \mathbb{N}_0$,

$$\mathcal{W}_q(x) := \Big(p_0 \mathbb{E}\Big[\prod_{n=1}^{\tau_x^+} q(X_n)\mathbf{1}_{(\tau_x^+ < \tau_{-1}^-)}\Big]\Big)^{-1}.$$

Then, we immediately get that

$$\mathcal{A}(x,a) = \frac{\mathcal{A}(0,a)}{\mathcal{A}(0,x)} = \frac{\mathcal{W}_q(x)}{\mathcal{W}_q(a)}. \tag{2.2.5}$$

Further, by conditioning on the first jump, we write $\mathcal{A}(x,a)$ as

$$\begin{aligned}
\mathcal{A}(x,a) &= \mathbb{E}_x\Big[\prod_{n=1}^{\tau_a^+} q(X_n)\mathbf{1}_{(\tau_a^+ < \tau_{-1}^-)}\Big] \\
&= \sum_{k=0}^{x+1} \mathbb{P}_x(C_1 = k) q(x+1-k) \mathbb{E}_{x+1-k}\Big[\prod_{n=1}^{\tau_a^+} q(X_n)\mathbf{1}_{(\tau_a^+ < \tau_{-1}^-)}\Big] \\
&= \sum_{k=0}^{x+1} p_k q(x+1-k)\mathcal{A}(x+1-k,a) = \sum_{k=-1}^{x} p_{k+1} q(x-k)\mathcal{A}(x-k,a).
\end{aligned}$$

Substituting Eq. (2.2.5) into the above function yields to the following recursive equation

$$\mathcal{W}_q(x) = \sum_{k=-1}^{x} p_{k+1} q(x-k) \mathcal{W}_q(x-k).$$

□

Remark 10. *We would like to highlight the followings.*

(i) The result in Eq. (2.2.4) holds for any $\mathcal{W}_q$ that satisfies Eq. (2.2.2) with arbitrary $\mathcal{W}_q(0)$ and thus, is unique only up to a multiplicative constant. However, it is discussed in [5] how the choice of normalisation, such that the initial value with the convention of $\prod_{n=1}^{0} q(X_n) = 1$

$$\begin{aligned}
\mathcal{W}_q(0) &= \Big(p_0 \mathbb{E}\Big[\prod_{n=1}^{\tau_0^+} q(X_n)\mathbf{1}_{(\tau_0^+ < \tau_{-1}^-)}\Big]\Big)^{-1} \\
&= \Big(p_0 \mathbb{E}\Big[\prod_{n=1}^{0} q(X_n)\mathbf{1}_{(0 < \tau_{-1}^-)}\Big]\Big)^{-1} \\
&= \Big(p_0 \mathbb{P}(0 < \tau_{-1}^-)\Big)^{-1} = 1/p_0,
\end{aligned}$$

which results in a simpler expression for the z-transform of $\mathcal{W}_q$. Although the z-transform is not obtainable in closed form in this study, due to the generality of the 'q-killing' function, we also adopt this normalisation for consistency and comparison of results.

(ii) Letting $q(x) = \nu$, for all $x \in \mathbb{Z}$, in Eq. (2.2.4) of Theorem 9 we recover the results of [5], given by Eqs. (2.1.4) – (2.1.6), respectively.

(iii) Under the discrete-time/space set-up, the numerical calculation of $\mathcal{W}_q$ can be obtained recursively by Eq. (2.2.2) (with initial value $\mathcal{W}_q(0) = 1/p_0$), which differs significantly from the numerical calculation of the corresponding ω-killed scale function in the continuous Lévy set up, where solutions to renewal equations are required (see Eq. (1.2) in [26]).

Mostly due to the practical applications within risk theory and insurance, it is common to consider the lower exit barrier at the level 0 (as in Theorem 9). However, the result can be generalised to consider exit from a general strip $[z, y]$ with $z \leqslant y$, which is presented in the following corollary.

Corollary 11. *For $z \leq x \leq y$, it follows that*

$$\mathbb{E}_x\Big[\prod_{n=1}^{\tau_y^+} q(X_n)\mathbf{1}_{(\tau_y^+<\tau_{z-1}^-)}\Big] = \frac{\mathcal{W}_q(x,z)}{\mathcal{W}_q(y,z)}, \tag{2.2.6}$$

where the scale function $\mathcal{W}_q(\cdot, z)$ satisfies the following recursive equation

$$\mathcal{W}_q(u,z) = \sum_{k=-1}^{u-z} p_{k+1} q(u-k)\mathcal{W}_q(u-k,y), \quad u \geq z, \tag{2.2.7}$$

with $\mathcal{W}_q(u,z) = 0$ for $u < z$.

Proof. We prove the above identity by using similar arguments with Theorem 9. We define $\mathcal{W}_q(x,z) := \Big(p_0\mathbb{E}_z\Big[\prod_{n=1}^{\tau_x^+} q(X_n)\mathbf{1}_{(\tau_x^+<\tau_{z-1}^-)}\Big]\Big)^{-1}$, and set $\mathcal{W}_q(x,z) = 0$ for $x < z$. Then, for $x \geq z$, applying the strong Markov property of X at τ_x^+ and using the fact that X is upwards skip-free random walk, for $z \leq x \leq y$, we have

$$\mathbb{E}_z\Big[\prod_{n=1}^{\tau_y^+} q(X_n)\mathbf{1}_{(\tau_y^+<\tau_{z-1}^-)}\Big] = \mathbb{E}_z\Big[\prod_{n=1}^{\tau_x^+} q(X_n)\mathbb{E}_{X_{\tau_x^+}}\big[\prod_{n=1}^{\tau_y^+} q(X_n)\mathbf{1}_{(\tau_y^+<\tau_{z-1}^-)}\big]\mathbf{1}_{(\tau_x^+<\tau_{z-1}^-)}\Big]$$

$$= \mathbb{E}_z\Big[\prod_{n=1}^{\tau_x^+} q(X_n)\mathbf{1}_{(\tau_x^+<\tau_{z-1}^-)}\Big] \times \mathbb{E}_{X_{\tau_x^+}}\Big[\prod_{n=1}^{\tau_y^+} q(X_n)\mathbf{1}_{(\tau_y^+<\tau_{z-1}^-)}\Big]$$

$$= \mathbb{E}_z\Big[\prod_{n=1}^{\tau_x^+} q(X_n)\mathbf{1}_{(\tau_x^+<\tau_{z-1}^-)}\Big] \times \mathbb{E}_x\Big[\prod_{n=1}^{\tau_y^+} q(X_n)\mathbf{1}_{(\tau_y^+<\tau_{z-1}^-)}\Big].$$

Let us define $\mathcal{A}(x,y,z) := \mathbb{E}_x\Big[\prod_{n=1}^{\tau_y^+} q(X_n)\mathbf{1}_{(\tau_y^+<\tau_{z-1}^-)}\Big]$, then the above equation becomes

$$\mathcal{A}(x,y,z) = \frac{\mathcal{W}_q(x,z)}{\mathcal{W}_q(y,z)}.$$

Further, by conditioning on the first jump, we write $\mathcal{A}(x,y,z)$ as a recursive equation

$$\mathcal{A}(x,y,z) = \sum_{k=0}^{x+1-z} p_k q(x+1-k)\mathcal{A}(x+1-k,y,z)$$

$$= \sum_{k=-1}^{x-z} p_{k+1} q(x-k)\mathcal{A}(x-k,y,z).$$

Substituting the above form of $\mathcal{A}(\cdot,\cdot,z)$ into the above equation, yields to

$$\frac{\mathcal{W}_q(x,z)}{\mathcal{W}_q(y,z)} = \sum_{k=-1}^{x-z} p_{k+1} q(x-k)\frac{\mathcal{W}_q(x-k,z)}{\mathcal{W}_q(y,z)}.$$

Thus, we have the following equation

$$\mathcal{W}_q(x,z) = \sum_{k=-1}^{x-z} p_{k+1} q(x-k)\mathcal{W}_q(x-k,z),$$

as required. □

Remark 12. *As in Theorem 9, we note that the above result holds for any $\mathcal{W}_q$ satisfying Eq. (2.2.7). However, in a similar way to that discussed in Remark 10, for the remainder of this study we will consider the specific normalisation $\mathcal{W}_q(u,u) = 1/p_0$. Finally, we note that $\mathcal{W}_q(u,0) = \mathcal{W}_q(u)$.*

We finalize this section, by showing that the scale function $\mathcal{W}_q(\cdot)$ contains many useful properties such as the martingale property. In the rest of this subsection, we show some martingale properties for the scale function $\mathcal{W}_q(\cdot)$. Based on these, other identities for further exit problems are derived.

Proposition 13. *For every $q : \mathbb{Z} \to (0,1]$, $\big\{\prod_{i=1}^{n\wedge\tau_{z-1}^-} q(X_i)\mathcal{W}_q\big(X_{n\wedge\tau_{z-1}^-}, z\big)\big\}_{n\in\mathbb{N}_0}$ is a martingale under $\mathbb{P}_x$ for all $x,z \in \mathbb{Z}$.*

Proof. Let us first note that by conditioning on the size of the first jump and using Eq. (2.2.7), it follows that for $x \geqslant z$, we have

$$\mathbb{E}_x\Big[q(X_1)\mathcal{W}_q\big(X_1, z\big)\Big] = \sum_{k=-1}^{x-z} p_{k+1} q(x-k)\mathcal{W}_q(x-k, z) = \mathcal{W}_q(x, z). \tag{2.2.8}$$

On the other hand, by conditioning on the filtration $\mathcal{F}_n$, and noticing that $\mathcal{W}_q(X_{\tau_{z-1}^-}, z) = 0$, we find that

$$\begin{aligned}
&\mathbb{E}_x\left[\prod_{i=1}^{(n+1)\wedge\tau_{z-1}^-} q(X_i)\mathcal{W}_q(X_{(n+1)\wedge\tau_{z-1}^-}, z)\mathbf{1}_{\{\tau_{z-1}^- > n\}} \;\Big|\; \mathcal{F}_n\right] \\
&= \prod_{i=1}^{n} q\left(X_i\right)\mathbf{1}_{\{\tau_{z-1}^- > n\}}\mathbb{E}_x\left[q\big(X_{(n+1)\wedge\tau_{z-1}^-}\big)\mathcal{W}_q\big(X_{(n+1)\wedge\tau_{z-1}^-}, z\big) \mid \mathcal{F}_n\right] \\
&= \prod_{i=1}^{n} q(X_i)\mathbf{1}_{\{\tau_{z-1}^- > n\}}\mathbb{E}_{X_n}\left[q(X_1)\mathcal{W}_q(X_1, z) \mid \mathcal{F}_n\right] \\
&= \prod_{i=1}^{n} q(X_i)\mathbf{1}_{\{\tau_{z-1}^- > n\}}\mathcal{W}_q(X_n, z) \\
&= \prod_{i=1}^{n\wedge\tau_{z-1}^-} q\left(X_i\right)\mathcal{W}_q(X_{n\wedge\tau_{z-1}^-}, z).
\end{aligned}$$

This completes the proof. □

Using the above result, we can now derive identities for further exit problems, as they given in the corollary below.

Corollary 14. *For any integer $x \leq a$ and $z \leq b \leq a$, we have*

$$\mathbb{E}_x\Big[\prod_{i=1}^{\tau_{b-1}^-} q(X_i)\mathcal{W}_q(X_{\tau_{b-1}^-}, z)\mathbf{1}_{(\tau_{b-1}^- < \tau_a^+)}\Big] = \mathcal{W}_q(x, z) - \frac{\mathcal{W}_q(x, b)}{\mathcal{W}_q(a, b)}\mathcal{W}_q(a, z).$$

where $\mathcal{W}_q(\cdot, \cdot)$ is defined in Eq. (2.2.7).

Proof. For any integer x, by using optional sampling theorem, we obtain

$$\begin{aligned}
\mathcal{W}_q(x, z) &= \mathbb{E}_x\Big[\prod_{i=1}^{\tau_{b-1}^-\wedge\tau_a^+} q(X_i)\mathcal{W}_q(X_{\tau_{b-1}^-\wedge\tau_a^+}, z)\Big] \\
&= \mathbb{E}_x\Big[\prod_{i=1}^{\tau_{b-1}^-\wedge\tau_a^+} q(X_i)\mathcal{W}_q(X_{\tau_{b-1}^-\wedge\tau_a^+}, z)\mathbf{1}_{(\tau_{b-1}^- < \tau_a^+)}\Big]
\end{aligned}$$

$$+\mathbb{E}_x\Big[\prod_{i=1}^{\tau_{b-1}^-\wedge\tau_a^+} q(X_i)\mathcal{W}_q(X_{\tau_{b-1}^-\wedge\tau_a^+}, z)\mathbf{1}_{(\tau_{b-1}^->\tau_a^+)}\Big]$$

$$=\mathbb{E}_x\Big[\prod_{i=1}^{\tau_{b-1}^-} q(X_i)\mathcal{W}_q(X_{\tau_{b-1}^-}, z)\mathbf{1}_{(\tau_{b-1}^-<\tau_a^+)}\Big]$$

$$+\mathbb{E}_x\Big[\prod_{i=1}^{\tau_a^+} q(X_i)\mathcal{W}_q(X_{\tau_a^+}, z)\mathbf{1}_{(\tau_{b-1}^->\tau_a^+)}\Big].$$

For the second expectation using skip-free property of X and using Eq. (2.2.6), it yields

$$\mathcal{W}_q(x,z)=\mathbb{E}_x\Big[\prod_{i=1}^{\tau_{b-1}^-} q(X_i)\mathcal{W}_q(X_{\tau_{b-1}^-}, z)\mathbf{1}_{(\tau_{b-1}^-<\tau_a^+)}\Big]$$

$$+\mathcal{W}_q(a,z)\mathbb{E}_x\Big[\prod_{i=1}^{\tau_a^+} q(X_i)\mathbf{1}_{(\tau_{b-1}^->\tau_a^+)}\Big]$$

$$=\mathbb{E}_x\Big[\prod_{i=1}^{\tau_{b-1}^-} q(X_i)\mathcal{W}_q(X_{\tau_{b-1}^-}, z)\mathbf{1}_{(\tau_{b-1}^-<\tau_a^+)}\Big]+\mathcal{W}_q(a,z)\frac{\mathcal{W}_q(x,b)}{\mathcal{W}_q(a,b)},$$

which gives the required equation after some arrangement. □

2.2.2 Two-Sided Exit Downwards

In this subsection, we derive fluctuation identity for the two-sided exit from below utilizing the new scale function $\mathcal{Z}_q(x)$.

Theorem 15 (Two-sided exit downwards)**.** *For $x \in \mathbb{Z}$ such that $x \le a$, we have*

$$\mathcal{B}(x,a)=\mathbb{E}_x\Big[\prod_{n=1}^{\tau_{-1}^-} q(X_n)\mathbf{1}_{(\tau_{-1}^-<\tau_a^+)}\Big]=\mathcal{Z}_q(x)-\frac{\mathcal{W}_q(x)}{\mathcal{W}_q(a)}\mathcal{Z}_q(a), \tag{2.2.9}$$

where $\mathcal{W}_q(x)$ and $\mathcal{Z}_q(x)$ satisfy Eqs. (2.2.2) – (2.2.3), respectively.

Proof. For $0 \le x \le a$, using skip-free property and strong Markov properties, we have

$$\mathbb{E}_x\Big[\prod_{n=1}^{\tau_{-1}^-} q(X_n)\mathbf{1}_{(\tau_{-1}^-<\infty)}\Big]=\mathbb{E}_x\Big[\prod_{n=1}^{\tau_{-1}^-} q(X_n)\mathbf{1}_{(\tau_{-1}^-<\tau_a^+)}\Big]$$

$$+\,\mathbb{E}_x\Big[\prod_{n=1}^{\tau_a^+} q(X_n)\mathbf{1}_{(\tau_a^+<\tau_{-1}^-)}\Big]\mathbb{E}_a\Big[\prod_{n=1}^{\tau_{-1}^-} q(X_n)\mathbf{1}_{(\tau_{-1}^-<\infty)}\Big],$$

from which we get that

$$\mathcal{B}(x,a) = \mathbb{E}_x\Big[\prod_{n=1}^{\tau_{-1}^-} q(X_n)\mathbf{1}_{(\tau_{-1}^-<\infty)}\Big] - \mathbb{E}_x\Big[\prod_{n=1}^{\tau_a^+} q(X_n)\mathbf{1}_{(\tau_a^+<\tau_{-1}^-)}\Big]\mathbb{E}_a\Big[\prod_{n=1}^{\tau_{-1}^-} q(X_n)\mathbf{1}_{(\tau_{-1}^-<\infty)}\Big].$$

Defining

$$B(x) := \mathbb{E}_x\Big[\prod_{n=1}^{\tau_{-1}^-} q(X_n)\mathbf{1}_{(\tau_{-1}^-<\infty)}\Big],$$

with $B(x) = 1$ for $x < 0$, then the above equation becomes

$$\mathcal{B}(x,a) = B(x) - \mathcal{A}(x,a)B(a). \tag{2.2.10}$$

Note that $\mathcal{B}(x,a)$ is monotone increasing in a and also we easily see that it is bounded since $0 \le \mathcal{B}(x,a) \le \mathbb{P}_x\big(\tau_{-1}^- < \tau_a^+\big) \le 1$, therefore it follows that $\lim_{a\to\infty}\mathcal{B}(x,a) = B(x)$ exist and finite.

To compute $B(x)$, we can apply similar first-step analysis as in the previous theorem, i.e. conditioning on the first jump such that $x \in \mathbb{N}_0$, we get

$$\begin{aligned} B(x) = \mathbb{E}_x\Big[\prod_{n=1}^{\tau_{-1}^-} q(X_n)\mathbf{1}_{(\tau_{-1}^-<\infty)}\Big] &= \sum_{k=0}^{\infty} p_k q(x+1-k)B(x+1-k) \\ &= \sum_{k=0}^{x+1} p_k q(x+1-k)B(x+1-k) + \sum_{k=x+2}^{\infty} p_k q(x+1-k), \end{aligned}$$

since for $k \ge x+2$, $B(x+1-k) = 1$ as for $k \ge x+2$, $x+1-k < 0$ and thus $B(x+1-k) = \mathbb{E}_{x+1-k}\Big[\prod_{n=1}^{0} q(X_n)\mathbf{1}_{(\tau_{-1}^-<\infty)}\Big]$. Therefore, we have

$$B(x) = \sum_{k=-1}^{x} p_{k+1} q(x-k)B(x-k) + \sum_{k=x+1}^{\infty} p_{k+1} q(x-k). \tag{2.2.11}$$

We now let us further define, for some arbitrary constant $a_B \in [0,\infty)$ the $\mathcal{Z}_q$ function

$$\mathcal{Z}_q(x) := B(x) + a_B\mathcal{W}_q(x). \tag{2.2.12}$$

By using Eq. (2.2.10), we obtain

$$\begin{aligned}\mathcal{B}(x,a) &= \mathcal{Z}_q(x) - a_B\mathcal{W}_q(x) - \frac{\mathcal{W}_q(x)}{\mathcal{W}_q(a)}\{\mathcal{Z}_q(a) - a_B\mathcal{W}_q(a)\}\\ &= \mathcal{Z}_q(x) - \frac{\mathcal{W}_q(x)}{\mathcal{W}_q(a)}\mathcal{Z}_q(a),\end{aligned}$$

as required.

Moreover, solving Eq. (2.2.12) with respect to $B(x)$, substituting the resulting equation into Eq. (2.2.11) and using Eq. (2.2.2), we see that $\mathcal{Z}_q$ also satisfies a recursive expression in the same form of Eq. (2.2.12), which is given by

$$\mathcal{Z}_q(x) = \sum_{k=-1}^{x} p_{k+1}q(x-k)\mathcal{Z}_q(x-k) + \sum_{k=x+1}^{\infty} p_{k+1}q(x-k).$$

Finally, it is easy to see that the result holds for the case $x < 0$ after noting that $\mathcal{Z}_q(x) = 1$ for $x < 0$, which follows from the definitions of $B(x)$ and $\mathcal{W}_q$. $\square$

Remark 16. *Related to the above theorem and its proof, we remark the followings.*

(i) In a similar way to Theorem 9, we point out that due to the form of the expressions defined in the above result, the initial condition $\mathcal{Z}_q(0)$ does not need to be specified in order to compute $\mathcal{B}(x,a)$. However, for the sake of results presented later on this study and for reasons given in [5] and references therein (see also Remark 16(ii) below), we will choose $a_B := p_0(1 - B(0))$ in Eq. (2.2.12), such that $\mathcal{Z}_q(0) = 1$.

(i) It is worth pointing out here that in the above theorem, the function $B(\cdot)$ along with its recursive relationship given in Eq. (2.2.11) is sufficient to compute $\mathcal{B}(x,a)$ since Eq. (2.2.10) holds for any $B(\cdot)$ that satisfies the recursion Eq. (2.2.11). However, it is usually preferable to work with $\mathcal{Z}_q$ as it leads to more concise expressions – this is especially the case when dealing with transforms of these functions – and, as discussed in Remark 16(i), allows us to identify the value of $\mathcal{Z}_q(0)$.

Similar to Corollary 11, it is not difficult to see that the result of Theorem 15 generalized to obtain downwards exit from a general $[z, y]$, as shown in the following corollary.

Corollary 17. *For $z \leq x \leq y$,*

$$\mathbb{E}_x\Big[\prod_{n=1}^{\tau_{z-1}^-} q(X_n)\mathbf{1}_{(\tau_{z-1}^- < \tau_y^+)}\Big] = \mathcal{Z}_q(x,z) - \frac{\mathcal{W}_q(x,z)}{\mathcal{W}_q(y,z)}\mathcal{Z}_q(y,z), \tag{2.2.13}$$

where, for $u \geq z$,

$$\mathcal{Z}_q(u,z) = \sum_{k=-1}^{u-z} p_{k+1}q(u-k)\mathcal{Z}_q(u-k,z) + \sum_{k=u-z+1}^{\infty} p_{k+1}q(u-k), \tag{2.2.14}$$

with $\mathcal{Z}_q(u,0) = \mathcal{Z}_q(u)$.

Proof. For the proof, we utilize the same arguments in the proof of Theorem 15. For $z \leq x \leq y$, using skip-free and the strong Markov properties gives that

$$\begin{aligned}\mathbb{E}_x\Big[\prod_{n=1}^{\tau_{z-1}^-} q(X_n)\mathbf{1}_{(\tau_{z-1}^- < \infty)}\Big] &= \mathbb{E}_x\Big[\prod_{n=1}^{\tau_{z-1}^-} q(X_n)\mathbf{1}_{(\tau_{z-1}^- < \tau_y^+)}\Big] \\ &\quad + \mathbb{E}_x\Big[\prod_{n=1}^{\tau_y^+} q(X_n)\mathbf{1}_{(\tau_y^+ < \tau_{z-1}^-)}\Big]\mathbb{E}_{X_{\tau_y^+}}\Big[\prod_{n=1}^{\tau_{z-1}^-} q(X_n)\mathbf{1}_{(\tau_{z-1}^- < \infty)}\Big],\end{aligned}$$

from which we get that

$$\begin{aligned}\mathcal{B}(x,y,z) := \mathbb{E}_x\Big[\prod_{n=1}^{\tau_{z-1}^-} q(X_n)\mathbf{1}_{(\tau_{z-1}^- < \infty)}\Big] \\ - \mathbb{E}_x\Big[\prod_{n=1}^{\tau_y^+} q(X_n)\mathbf{1}_{(\tau_y^+ < \tau_{z-1}^-)}\Big]\mathbb{E}_y\Big[\prod_{n=1}^{\tau_{z-1}^-} q(X_n)\mathbf{1}_{(\tau_{z-1}^- < \infty)}\Big].\end{aligned}$$

We now define

$$B(x,z) := \mathbb{E}_x\Big[\prod_{n=1}^{\tau_{z-1}^-} q(X_n)\mathbf{1}_{(\tau_{z-1}^- < \infty)}\Big],$$

with $B(x,z) = 1$ for $x < z$, then the above equation becomes

$$\mathcal{B}(x,y,z) = B(x,z) - \mathcal{A}(x,y,z)B(y,z). \tag{2.2.15}$$

Next, by conditioning on the first jump, we get

$$\begin{aligned}B(x,z) &= \mathbb{E}_x\Big[\prod_{n=1}^{\tau_{z-1}^-} q(X_n)\mathbf{1}_{(\tau_{z-1}^- < \infty)}\Big] \\ &= \sum_{k=0}^{\infty} p_k q(x+1-k)B(x+1-k,z)\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=0}^{x+1-z} p_k q(x+1-k) B(x+1-k, z) \\
&\quad + \sum_{k=x+2-z}^{\infty} p_k q(x+1-k),
\end{aligned}
$$

since for $k \geq x+2-z$, $B(x+1-k, z) = 1$ as for $k \geq x+2-z$, $x+1-k < z$. Therefore, we obtain a recursive equation

$$
B(x, z) = \sum_{k=-1}^{x-z} p_{k+1} q(x-k) B(x-k, z) + \sum_{k=x-z+1}^{\infty} p_{k+1} q(x-k). \tag{2.2.16}
$$

For some arbitrary $a_{B_z} \in [0, \infty)$, we further define

$$
\mathcal{Z}_q(x, z) := B(x, z) + a_{B_z} \mathcal{W}_q(x, z). \tag{2.2.17}
$$

Now, we will find a recursive equation for $\mathcal{Z}_q(x, z)$. Solving Eq. (2.2.17) with respect to $B(\cdot, z)$, substituting the resulting equation into Eq. (2.2.16) and using Eq. (2.2.7), we get that

$$
\mathcal{Z}_q(x, z) = \sum_{k=-1}^{x-z} p_{k+1} q(x-k) \mathcal{Z}_q(x-k, z) + \sum_{k=x-z+1}^{\infty} p_{k+1} q(x-k),
$$

which proves the corollary. □

Remark 18. *For the same arguments as in Remark 16(i), in this study we choose to define $\mathcal{Z}_q(\cdot, \cdot)$ such that $\mathcal{Z}_q(x, z) = 1$ for $x \leq z$.*

Although the two-sided upwards or downwards exit problems are of interest in their own right and do have applications in many areas, e.g., dividend problems in risk theory (see [17]), the corresponding one-sided exit problem(s) have received a great deal of interest in the literature and have many applications in ruin theory (see [11, 17] and [38] among others). One such quantity, which is used in the final section to derive the so-called bankruptcy probability, can be obtained by taking the limit as $a \to \infty$ in Theorem 9 from which we obtain the following corollary.

Corollary 19. *Then for all $x \geq 0$,*

$$
\mathbb{E}_x \left[\prod_{n=1}^{\infty} q(X_n) \mathbf{1}_{(\tau_{-1}^{-} = \infty)} \right] = a_{\mathcal{W}^{-1}(\infty)} \mathcal{W}_q(x), \tag{2.2.18}
$$

where $a_{\mathcal{W}^{-1}(\infty)} = \lim_{a \to \infty} \mathcal{W}_q(a)^{-1}$.

Proof. To prove the result, it is sufficient to show the existence and finiteness of the limit. First note that both $\mathcal{W}_q(a)$ (hence also $\mathcal{W}_q(a)^{-1}$) is monotone in a. Moreover, $\mathcal{W}_q(a)^{-1}$ is bounded since, by definition

$$\mathcal{W}_q(a)^{-1} = p_0\mathbb{E}\left[\prod_{n=1}^{\tau_a^+} q(X_n)\mathbf{1}_{(\tau_a^+<\tau_{-1}^-)}\right] \leqslant p_0\mathbb{P}(\tau_a^+ < \tau_{-1}^-),$$

which completes the proof. $\square$

It is worth pointing out that each of the results presented so far are more general than they may first appear and contain, by a suitable choice of the 'q-killing' function, other well known transforms from the literature. One such example is the generalized version for the transform of the undershoot (deficit) below the lower level and is given in the following proposition.

Proposition 20. *For* $\xi \in (0,1]$*, let*

$$q(x) = \begin{cases} \widetilde{q}(x), & x \geq 0, \\ \xi^{-x}, & x < 0, \end{cases}$$

where $\widetilde{q} : \mathbb{N} \to (0,1]$*. Then, it follows that for* $x \leq a$*,*

$$\mathcal{B}(x,a) := \mathbb{E}_x\Big(\prod_{i=1}^{\tau_{-1}^- -1} \widetilde{q}(X_i)\xi^{-X_{\tau_{-1}^-}}\mathbf{1}_{(\tau_{-1}^-<\tau_a^+)}\Big) = \widetilde{\mathcal{Z}}_q(x,\xi) - \frac{\mathcal{W}_q(x)}{\mathcal{W}_q(a)}\widetilde{\mathcal{Z}}_q(a,\xi),$$

where $\widetilde{\mathcal{Z}}_q$ *satisfies the following recursive equation*

$$\widetilde{\mathcal{Z}}_q(x,\xi) = \sum_{k=-1}^{x} p_{k+1}q(x-k)\widetilde{\mathcal{Z}}_q(x-k,\xi) + \sum_{k=x+1}^{\infty} p_{k+1}\xi^{-(x-k)}.$$

In particular, if we further let $\widetilde{q}(x) = \nu$ for all $x \geqslant 0$, then we note that $\nu\mathcal{B}(x,a)$ reduces to the joint transform of the time, and deficit, below 0 as seen in (Eq. (12) and Remark 18 in [5]). It is worth highlighting that this method is not possible in the classical setting of [5], where the authors must first determine the more complicated joint transform and can then recover the simpler two-sided exit identity as a special case.

2.2.3 Resolvents

In this subsection, we establish identities for resolvent of an q-killed upwards skip-free random walk, which can be used to determine the distribution the level of a q-killed random walk right before the exiting some interval. We point out that in [5], the resolvent measure is obtained based on Proposition 3.2 in [30], where a combinatorial approach is used whilst in [22] the law of running infima (known by Wiener-Hopf factorisation) is considered. In this study, we employ first-step analysis to derive semi-explicit expressions for the q-killed resolvent measure, given in the next theorem. Before introducing the theorem, we give the following remark which is needed for the derivation of the resolvent measure.

Remark 21. *In fact, the recursion in Eq. (2.2.7) can be extended to include the case* $u = z - 1$, *such that for all* $u \geqslant z - 1$,

$$\mathcal{W}_q(u, z) = \sum_{k=-1}^{u-z} p_{k+1} q(u-k)\mathcal{W}_q(u-k, z) - q(z)\mathbf{1}_{(u-z=-1)}. \tag{2.2.19}$$

Theorem 22 (Resolvent)**.** *For* $x, y \in [0, a]$, *resolvent of the process killed on exiting* $\{0, \dots a-1\}$ *is given by*

$$\begin{aligned}\mathcal{U}_q(x, y) &:= \sum_{n=0}^{\infty} \mathbb{E}_x \Big[\prod_{i=1}^{n} q(X_i) \mathbf{1}_{(X_n = y,\, n < \tau_{-1}^- \wedge \tau_a^+)} \Big] \\ &= q(y+1)^{-1} \left(\frac{\mathcal{W}_q(a, y+1)\mathcal{W}_q(x)}{\mathcal{W}_q(a)} - \mathcal{W}_q(x, y+1) \right),\end{aligned} \tag{2.2.20}$$

where $\mathcal{W}_q(\cdot, \cdot)$ *is given in the Eq. (2.2.7).*

Proof. By conditioning on the first period of time, we note that for $x \in [0, a-1]$ the resolvent measure $\mathcal{U}_q$ satisfies the following recursive equation

$$\begin{aligned}\mathcal{U}_q(x, y) &:= \sum_{n=0}^{\infty} \mathbb{E}_x \Big[\prod_{i=1}^{n} q(X_i) \mathbf{1}_{(X_n = y,\, n < \tau_{-1}^- \wedge \tau_a^+)} \Big] \\ &= \mathbf{1}_{(X_0 = y)} + \sum_{n=1}^{\infty} \mathbb{E}_x \Big[\prod_{i=1}^{n} q(X_i) \mathbf{1}_{(X_n = y,\, n < \tau_{-1}^- \wedge \tau_a^+)} \Big] \\ &= \sum_{k=0}^{x+1} \mathbb{P}_x(C_1 = k) q(x+1-k) \sum_{n=1}^{\infty} \mathbb{E}_{x+1-k} \Big[\prod_{i=1}^{n-1} q(X_i) \mathbf{1}_{(X_{n-1} = y,\, n-1 < \tau_{-1}^- \wedge \tau_a^+)} \Big] \\ &\quad + \mathbf{1}_{(x=y)}\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=0}^{x+1} p_k q(x+1-k)\mathcal{U}_q(x+1-k,y) + \mathbf{1}_{(x=y)} \\
&= \sum_{k=-1}^{x} p_{k+1} q(x-k)\mathcal{U}_q(x-k,y) + \mathbf{1}_{(x=y)}. \qquad (2.2.21)
\end{aligned}
$$

On the other hand, from Eqs. (2.2.2) – (2.2.19), we note that for some constant c_a, it follows that

$$
\begin{aligned}
&c_a\mathcal{W}_q(x) - q(y+1)^{-1}\mathcal{W}_q(x,y+1) \\
&= \mathbf{1}_{(x=y)} + \sum_{k=-1}^{x} p_{k+1} q(x-k)\big(c_a\mathcal{W}_q(x-k) - q(y+1)^{-1}\mathcal{W}_q(x-k,y+1)\big),
\end{aligned}
$$

since $\sum_{k=x-y}^{x} q(x-k)p_{k+1}\mathcal{W}_q(x-k,y+1) = 0$ and thus, satisfies the same recursive equation as $\mathcal{U}_q$. In particular, we have that

$$\mathcal{U}_q(x,y) = c_a\mathcal{W}_q(x) - q(y+1)^{-1}\mathcal{W}_q(x,y+1),$$

when

$$c_a = \frac{q(y+1)^{-1}\mathcal{W}_q(a,y+1)}{\mathcal{W}_q(a)},$$

due to the boundary condition $\mathcal{U}_q(a,y) = 0$, which completes the proof. ☐

Remark 23. *Note that when $q(\cdot) = \nu$, Theorem 22 reduces to (no q-killed) resolvent of X killed on exiting $\{0,\dots,a-1\}$, i.e.,*

$$U_\nu(x,y) = \nu^{-1}\left(\frac{W_\nu(x)}{W_\nu(a)}W_\nu(a-1-y) - W_\nu(x-y-1)\right),$$

which is the same equation as Eq. (2.1.10) (see also Remark 15 in [5]).

2.2.4 Exit Times for Reflections

In this subsection we derive exit identities for reflected q-killed upwards skip-free random walk. We should point out, as in the case of spectrally negative Levý process (see for example [33]), these identities can be derived by means of martingale properties of scale functions. However, in this study, we will demonstrate how the 'q-killing' function can be used to develop a probabilistic argument in terms of the exit identities, given in Theorem 9 and Theorem 15. We also remark out that the random walk reflected from below has numerous applications in actuarial science, particularly in risk models with capital injections (see for instance [7]).

Let us define upwards skip-free random walk reflected at zero and $a > 0$, respectively by

$$Y_n = X_n - I_n, \quad \text{and} \quad \widetilde{Y}_n = X_n - S_n, \tag{2.2.22}$$

where $I_n := \inf_{0 \leq k \leq n}(X_k \wedge 0)$ and $S_n := \sup_{0 \leq k \leq n}(X_k \vee a) - a$. The first passage times for the reflected processes is then denoted by

$$\widehat{\tau}_a^+ = \inf\{n \geq 0 : Y_n \geq a\}, \text{ and } \widetilde{\tau}_{-1}^- = \inf\{n \geq 0 : \widetilde{Y}_n \leq -1\}. \tag{2.2.23}$$

Theorem 24. *For $x \leq a$, we have*

$$\widehat{\mathcal{C}}(x,a) := \mathbb{E}_x\Big[\prod_{i=1}^{\widehat{\tau}_a^+} q(Y_i)\mathbf{1}_{(\widehat{\tau}_a^+ < \infty)}\Big] = \frac{\mathcal{Z}_q(x)}{\mathcal{Z}_q(a)}, \tag{2.2.24}$$

where $\mathcal{Z}_q(\cdot)$ is given in Eq. (2.2.3) with $q(x) \equiv q(0)$ for all $x \leq 0$.

Proof. For the reflected process, we have the following two scenarios; either the process exits from above before being reflected (at zero) or the process reflects from below (which is equivalent to the non-reflected process down-crossing zero) before reaching level $a \in \mathbb{N}$. As such, if we consider a specific q-killing function which takes general values $q(x)$ for $x \geqslant 0$ but constant and equal to $q(0)$ otherwise, i.e. $q(x) \equiv q(0)$ for all $x < 0$, it follows that

$$\widehat{\mathcal{C}}(x,a) = \mathcal{A}(x,a) + \mathcal{B}(x,a)\widehat{\mathcal{C}}(0,a), \tag{2.2.25}$$

where $\mathcal{A}(\cdot,a)$ and $\mathcal{B}(\cdot,a)$ have the specific q-killing function defined above and thus, it only remains to calculate the constant $\widehat{\mathcal{C}}(0,a)$. Substituting $x = 0$ in the Eq. (2.2.25) gives

$$(1 - \mathcal{B}(0,a))\widehat{\mathcal{C}}(0,a) = \mathcal{A}(0,a),$$

and thus, by using Eq. (2.2.4) and Eq. (2.2.9), we get that

$$\begin{aligned}
\widehat{\mathcal{C}}(x,a) &= \mathcal{A}(x,a) + \mathcal{B}(x,a)\frac{\mathcal{A}(0,a)}{1 - \mathcal{B}(0,a)} \\
&= \frac{\mathcal{W}_q(x)}{\mathcal{W}_q(a)} + \Big(\mathcal{Z}_q(x) - \frac{\mathcal{W}_q(x)}{\mathcal{W}_q(a)}\mathcal{Z}_q(a)\Big) \times \Big(\frac{\mathcal{W}_q(0)/\mathcal{W}_q(a)}{1 - \Big(1 - \frac{\mathcal{W}_q(0)}{\mathcal{W}_q(a)}\mathcal{Z}_q(a)\Big)}\Big)
\end{aligned}$$

$$= \frac{\mathcal{W}_q(x)}{\mathcal{W}_q(a)} + \Big(\mathcal{Z}_q(x) - \frac{\mathcal{W}_q(x)}{\mathcal{W}_q(a)}\mathcal{Z}_q(a)\Big)\frac{1}{\mathcal{Z}_q(a)} = \frac{Z_q(x)}{Z_q(a)},$$

with $q(x) \equiv q(0)$ for $x \leqslant 0$. $\square$

Next, using similar arguments given in Theorem 22, we also derive the corresponding resolvent for this reflected random walk.

Theorem 25. *For $x, y \in [0, a]$, resolvent of the q-killed upwards skip-free random walk reflected at 0 and killed on exiting $\{0, \dots, a-1\}$ is given by*

$$\begin{aligned}\mathcal{L}_q(x,y) &:= \sum_{n=0}^{\infty} \mathbb{E}_x\big[\prod_{i=1}^{n} q(Y_i)\mathbf{1}_{(Y_n=y,\, n<\hat{\tau}_a^+)}\big] \\ &= q(y+1)^{-1}\Big(\frac{\mathcal{Z}_q(x)}{\mathcal{Z}_q(a)}\mathcal{W}_q(a, y+1) - \mathcal{W}_q(x, y+1)\Big),\end{aligned}$$

where $\mathcal{Z}_q(\cdot)$ and $\mathcal{W}_q(\cdot,\cdot)$ are defined in Eq. (2.2.3) and Eq. (2.2.7) respectively with $q(x) \equiv q(0)$ for all $x \leq 0$.

Proof. In this proof we follow the same steps with the proof of Theorem 22. By conditioning on the first period of time, we obtain a recursive equation as follows

$$\begin{aligned}\mathcal{L}_q(x,y) &= \sum_{n=0}^{\infty} \mathbb{E}_x\big[\prod_{i=1}^{n} q(Y_i)\mathbf{1}_{(Y_n=y,\, n<\hat{\tau}_a^+)}\big] \\ &= \mathbf{1}_{(Y_0=y)} + \sum_{n=1}^{\infty} \mathbb{E}_x\big[\prod_{i=1}^{n} q(Y_i)\mathbf{1}_{(Y_n=y,\, n<\hat{\tau}_a^+)}\big].\end{aligned}$$

Since the process reflected at zero, whenever it exits from 0, it is evaluated as the process takes zero value. Therefore, we have

$$\begin{aligned}&\mathcal{L}_q(x,y) \\ &\quad= \mathbf{1}_{(x=y)} + \sum_{k=0}^{x+1} \mathbb{P}(C_1 = k)q(x+1-k)\sum_{n=1}^{\infty}\mathbb{E}_{x+1-k}\Big[\prod_{i=1}^{n-1} q(Y_i)\mathbf{1}_{(Y_{n-1}=y,\, n-1<\hat{\tau}_a^+)}\Big] \\ &\qquad + \sum_{k=x+2}^{\infty} \mathbb{P}(C_1 = k)q(0)\sum_{n=1}^{\infty}\mathbb{E}_0\Big[\prod_{i=1}^{n-1} q(Y_i)\mathbf{1}_{(Y_{n-1}=y,\, n-1<\hat{\tau}_a^+)}\Big] \\ &\quad= \mathbf{1}_{(x=y)} + \sum_{k=0}^{x+1} p_k q(x+1-k)\mathcal{L}_q(x+1-k, y) + \sum_{k=x+2}^{\infty} p_k q(0)\mathcal{L}_q(0,y) \\ &\quad= \mathbf{1}_{(x=y)} + \sum_{k=-1}^{x} p_{k+1} q(x-k)\mathcal{L}_q(x-k, y) + \sum_{k=x+1}^{\infty} p_{k+1} q(0)\mathcal{L}_q(0,y). \qquad (2.2.26)\end{aligned}$$

Moreover, since $q(x) \equiv q(0)$ for $x \leq 0$, Eq. (2.2.3) becomes

$$\mathcal{Z}_q(x) = \sum_{k=-1}^{x} p_{k+1} q(x-k)\mathcal{Z}_q(x-k) + q(0)\sum_{k=x+1}^{\infty} p_{k+1}, \tag{2.2.27}$$

which, along with Eq. (2.2.19), yields that for some constant c_L, we have

$$\begin{aligned}
&c_L \mathcal{Z}_q(x) - q(y+1)^{-1}\mathcal{W}_q(x, y+1) \\
&\quad = c_L\Big(\sum_{k=-1}^{x} p_{k+1} q(x-k)\mathcal{Z}_q(x-k) + q(0)\sum_{k=x+1}^{\infty} p_{k+1}\Big) \\
&\qquad - q(y+1)^{-1}\sum_{k=-1}^{x-y-1} p_{k+1} q(x-k)\mathcal{W}_q(x-k, y+1) + \mathbf{1}_{(x=y)} \\
&\quad = \mathbf{1}_{(x=y)} + c_L q(0)\sum_{k=x+1}^{\infty} p_{k+1} \\
&\qquad + \sum_{k=-1}^{x} p_{k+1} q(x-k)\big[c_L \mathcal{Z}_q(x-k) - q(y+1)^{-1}\mathcal{W}_q(x-k, y+1)\big],
\end{aligned}$$

since $\sum_{k=x-y}^{x} q(x-k)p_{k+1}\mathcal{W}_q(x-k, y+1) = 0$. Hence, after noting that

$$c_L \mathcal{Z}_q(0) - q(y+1)^{-1}\mathcal{W}_q(0, y+1) = c_L,$$

since $\mathcal{Z}_q(0) = 1$ and $\mathcal{W}_q(0, y+1) = 0$, for $y \in [0, a)$, the expression on the right-hand side of the above equation satisfies the same recursion as $\mathcal{L}_q$. Finally, using the fact that $\mathcal{L}_q(a, y) = 0$, we have that

$$\mathcal{L}_q(x, y) = c_L \mathcal{Z}_q(x) - q(y+1)^{-1}\mathcal{W}_q(x, y+1),$$

with

$$c_L = \frac{q(y+1)^{-1}\mathcal{W}_q(a, y+1)}{\mathcal{Z}_q(a)},$$

which proves the theorem. □

To complete this section, in the following theorem we find one-sided exit problem for q-killed upwards skip-free random walk reflected at $a > 0$.

Theorem 26. *For $x \leq a$, we have*

$$\widetilde{\mathcal{C}}(x, a) := \mathbb{E}_x\Big[\prod_{i=1}^{\widetilde{\tau}^-_{-1}} q(\widetilde{Y}_i)\mathbf{1}_{(\widetilde{\tau}^-_{-1} < \infty)}\Big] = \mathcal{Z}_q(x) - \mathcal{W}_q(x)\Gamma_q(a),$$

with

$$\Gamma_q(a) = \frac{\mathcal{Z}_q(a+1)}{\mathcal{W}_q(a+1)} + \frac{1}{\mathcal{W}_q(a+1) - \mathcal{W}_q(a)}\Big(\frac{\mathcal{W}_q(a)\mathcal{Z}_q(a+1)}{\mathcal{W}_q(a+1)} - \mathcal{Z}_q(a)\Big),$$

where $\mathcal{W}_q$ and $\mathcal{Z}_q$ are given in Eqs. (2.2.2) – (2.2.3) respectively with $q(x) \equiv q(a)$ for $x \geq a+1$.

Proof. Using similar arguments as in Theorem 24, we consider a specific q-killing function of the form $q(x)$ for all $x \leq a$ and $q(x) \equiv q(a)$ for all $x \geq a+1$. Moreover, we have the following two scenarios; either the process exits from below before reaching level $a+1$ or it exits from above (hits level $a+1$) and is reflected at the level a. Therefore, we have

$$\widetilde{\mathcal{C}}(x,a) = \mathcal{B}(x,a+1) + \mathcal{A}(x,a+1)\widetilde{\mathcal{C}}(a,a), \tag{2.2.28}$$

where in $\mathcal{B}(\cdot,a+1)$ and $\mathcal{A}(\cdot,a+1)$ are understood to have the specific q-killing function defined above. Then, it only remains to compute $\widetilde{\mathcal{C}}(a,a)$. Substituting $x = a$ in the Eq. (2.2.28) gives

$$\widetilde{\mathcal{C}}(a,a) = \frac{\mathcal{B}(a,a+1)}{1 - \mathcal{A}(a,a+1)},$$

and thus, Eq. (2.2.28) becomes

$$\begin{aligned}
\widetilde{\mathcal{C}}(x,a) &= \mathcal{B}(x,a+1) + \mathcal{A}(x,a+1)\frac{\mathcal{B}(a,a+1)}{1 - \mathcal{A}(a,a+1)} \\
&= \mathcal{Z}_q(x) - \frac{\mathcal{W}_q(x)}{\mathcal{W}_q(a+1)}\mathcal{Z}_q(a+1) \\
&\quad + \frac{\mathcal{W}_q(x)}{\mathcal{W}_q(a+1)} \frac{\mathcal{Z}_q(a) - \frac{\mathcal{W}_q(a)}{\mathcal{W}_q(a+1)}\mathcal{Z}_q(a+1)}{1 - \frac{\mathcal{W}_q(a)}{\mathcal{W}_q(a+1)}} \\
&= \mathcal{Z}_q(x) - \frac{\mathcal{W}_q(x)}{\mathcal{W}_q(a+1)}\mathcal{Z}_q(a+1) + \frac{\mathcal{W}_q(x)}{\mathcal{W}_q(a+1)} \\
&\quad \times \frac{\mathcal{Z}_q(a)\mathcal{W}_q(a+1) - \mathcal{Z}_q(a+1)\mathcal{W}_q(a)}{\mathcal{W}_q(a+1) - \mathcal{W}_q(a)} \\
&= \mathcal{Z}_q(x) + \mathcal{W}_q(x)\frac{\mathcal{Z}_q(a)}{\mathcal{W}_q(a+1) - \mathcal{W}_q(a)} \\
&\quad - \frac{\mathcal{Z}_q(a+1)}{\mathcal{W}_q(a+1)}\mathcal{W}_q(x)\Big(1 + \frac{\mathcal{W}_q(a)}{\mathcal{W}_q(a+1) - \mathcal{W}_q(a)}\Big).
\end{aligned}$$

□

2.2.5 One-Sided Exit Upwards and Its Resolvent

In this subsection, we introduce the third scale function, namely $\mathcal{H}_q$, which satisfies the following recursive equation

$$\mathcal{H}_q(x) = \sum_{k=-1}^{\infty} p_{k+1} q(x-k)\mathcal{H}_q(x-k), \tag{2.2.29}$$

and plays a fundamental role in the one-sided upwards exit problem and the corresponding potential measure, which are given in the following theorem.

Proposition 27 (One-sided upwards exit)**.** *For $x \le a$, it follows that*

$$\mathbb{E}_x\left[\prod_{n=1}^{\tau_a^+} q(X_n)\mathbf{1}_{(\tau_a^+<\infty)}\right] = \frac{\mathcal{H}_q(x)}{\mathcal{H}_q(a)}, \tag{2.2.30}$$

where $\mathcal{H}_q$ satisfies Eq. (2.2.29). Moreover, for $x, y \le a$, the corresponding resolvent is given by

$$\begin{aligned}\Xi_q(x,y) &:= \sum_{n=0}^{\infty} \mathbb{E}_x\left[\prod_{i=1}^{n} q(X_i)\mathbf{1}_{(X_n=y,\, n<\tau_a^+)}\right]\\ &= q(y+1)^{-1}\left(\frac{\mathcal{H}_q(x)}{\mathcal{H}_q(a)}\mathcal{W}_q(a,y+1) - \mathcal{W}_q(x,y+1)\right).\end{aligned} \tag{2.2.31}$$

Proof. The results are a direct consequence of Corollary 11 (with $y = a$) and Theorem 22, respectively, after taking the limits as $z \to -\infty$ and defining $\mathcal{H}_q(x) := \lim_{z\to-\infty} \mathcal{W}_q(x,z)$. ☐

Unfortunately, due to the infinite summation in Eq. (2.2.29), it is not possible to compute $\mathcal{H}_q$ recursively in the general case, unlike for $\mathcal{W}_q$ and $\mathcal{Z}_q$. In the classical case, where $q(x) = \nu$ for all $x \in \mathbb{Z}$, such that the killing rate becomes independent of the level, the recursion reduces to

$$\mathcal{H}_q(x) = \nu \sum_{k=-1}^{\infty} p_{k+1}\mathcal{H}_q(x-k).$$

Assuming that $\mathcal{H}_q$ takes the form $\mathcal{H}_q(x) = \varphi_\nu^{-x}$ for some constant value $\varphi_\nu \in \mathbb{R}^+$, it is easy to see that φ_ν is the solution to Lundberg's equation (see [16])

$$\varphi_\nu = \nu\widetilde{p}(\varphi_\nu).$$

We then obtain the classical one-sided upwards exit result, i.e. $\mathbb{E}_x\left[\nu^{\tau_a^+}\mathbf{1}_{(\tau_a^+<\infty)}\right] = \varphi_\nu^{a-x}$ given in Eq. (2.1.3) (see also [5] and references therein).

We have already seen in Proposition 20, how the generality of the 'q-killing' allows us to retrieve different quantities by appropriate choices of the killing function itself. We end this section by presenting another example of this property.

One solution to dealing with the infinite summation in Eq. (2.2.29) is to choose $q(\cdot)$ such that $q(x) = 0$ for $x < z$, with $z < a$. In this case, each $\mathcal{H}_q(x)$ can be written as a factor of $\mathcal{H}_q(z)$ and thus, is sufficient to determine Eq. (2.2.30). However, the reader may notice that this is equivalent to the two-sided upwards problem, i.e. $\mathcal{H}_q(x) = \mathcal{W}_q(x, z)$. That is, in the q-killing model, it is possible to recover the two-sided exit problems (since a similar arguments holds for the one and two-sided downwards exit as well) from the corresponding one-sided problems. This is not possible in the classic case. In fact, with the above observation in mind, it is possible to think of the classical model as a model with level dependent killing. That is, has killing probability $1 - \nu \in [0, 1)$ between the barriers and probability one above or below the barrier depending on the exit problem itself. In the final section, we will consider an example of a specific q-killing function which corresponds to the so-called bankruptcy model within the risk theory literature.

2.2.6 Example: Omega-model in Ruin Theory

As an example, we derive an explicit expression for the probability of bankruptcy for the discrete risk model. It is well known within actuarial science, in the discrete time setting one could model the surplus of an insurance company as an upwards skip-free random walk, often called as compound binomial risk model.Within the classical risk theory, the process stops when the event of ruin (first time the process becomes negative) occurs. As an extension to this, the process, in the so-called as omega risk model, the surplus process is allowed to carry on whilst negative until the event of bankruptcy occurs (see [18], for the continuous time setting). Bankruptcy may occur in one of the two ways: it may happen in the red zone, i.e., when the surplus process in $[-d, 0)$ and there is a level dependent bankruptcy probability at each period of time or it may occur if the surplus process falls below some fixed level $-d < 0$. Then, the bankruptcy probability is defined as

$$\psi(x) = 1 - \mathbb{E}_x\left[\prod_{n=1}^{\infty} q(X_n)\mathbf{1}_{(\tau_{-d-1}^{-}=\infty)}\right] = 1 - a_{\mathcal{W}^{-1}(\infty,-d)}\mathcal{W}_q(x,-d),$$

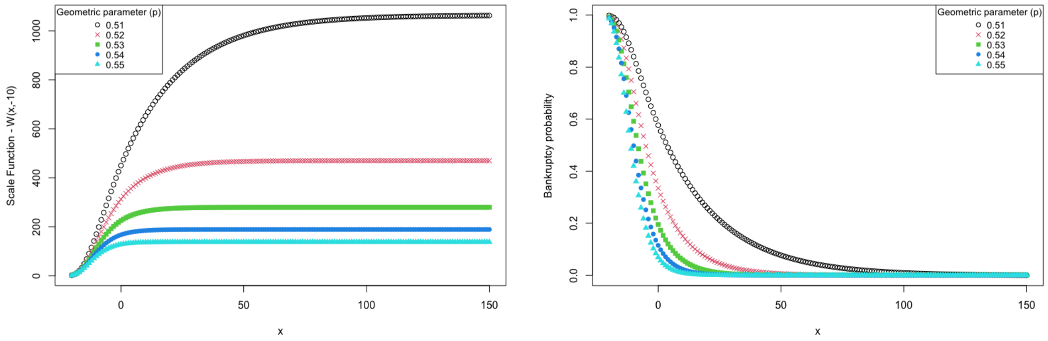
where $a_{\mathcal{W}^{-1}(\infty,-d)} = \lim_{a\to\infty} \mathcal{W}_q(a,-d)^{-1}$.

On order to model the bankruptcy probability described above, we take the specific $q(\cdot)$ function for bankruptcy, namely bankruptcy function, for $\gamma_0, \gamma_1 \in (0,1]$

$$q(x) = 1 - \gamma_0\gamma_1^{(x+d)}\mathbf{1}_{\{x\in[-d,0)\}},$$

which is typically decreasing function of x and equals to one on the positive half line so that bankruptcy does not occur when the surplus is positive. In this case, the recursive equation given in Eq. (2.2.7) becomes

$$\begin{aligned}\mathcal{W}_q(x,-d) = \sum_{k=-1}^{x+d} p_{k+1}\mathcal{W}_q(x-k,-d)& \\ - \gamma_0 \sum_{k=-1}^{x+d} p_{k+1}\gamma_1^{x+d-k}\mathbf{1}_{\{x-k\in[-d,0)\}}\mathcal{W}_q(x-k,-d).&\end{aligned}$$



(a) $\mathcal{W}_q(x,-10)$ scale function.

(b) Probability of bankruptcy

Figure 2.2: Bankruptcy example.

In order to demonstrate the behavior of $\mathcal{W}_q(\cdot,-d)$ scale function and associated bankruptcy probability, let us consider a specific example with $d = 10$, $\gamma_0 = 0.5$,

$\gamma_1 = 0.7$. We will also assume that the jump size distribution, p_k, is geometrically distributed with varying success parameter. In order to keep in line with the risk theory literature, we will only consider success probabilities greater than 0.5 to ensure a positive asymptotic drift of the random walk (net profit condition), see for example [16] among others. For the bankruptcy probability, the limit $a_{\mathcal{W}^{-1}(\infty,-10)}$ has been approximated by using a sufficiently large value of a, see Figure 2.2.

2.2.7 Numerical Illustrations and Examples

In this subsection, we provide an implementation of fluctuation identities, specifically two-sided exit identities. Firstly, we conduct some simulation studies. Then, we solve numerical examples by utilizing the explicit solutions of these fluctuation identities derived in Section 2.2.1 and Section 2.2.2. We numerically investigate the impact of the distribution of the claim sizes and its parameter, as well as q-killing function and its parameter.

For simulation and numerical studies, we determine our parameters as follows.

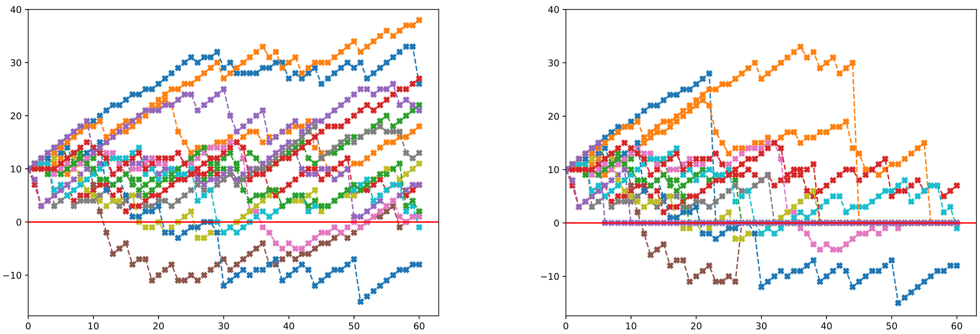
- For p.m.f $p_k = \mathbb{P}(C_1 = k)$ for $k \in \mathbb{N}_0$, we assume that IID random variables $\{C_i\}_{i\in\mathbb{N}} \in \mathbb{N}_0$ are distributed as geometric with parameter p. Note that in actuarial applications C_i's denotes claims so that p_k becomes the claim size distribution. Also, it is well-known that geometric claim distribution is the most commonly used distribution in discrete time modeling.

- For q-killing, we choose the killing function $q(x) = \nu^x$ where $\nu \in (0, 1]$. As we mentioned earlier, this function can be interpreted as a killing parameter depending on the present value of the upwards skip-free random walk. For example, in the economical aspect q function can be seen as a discrete discounting, whereas in the mathematical aspect, it is a generator of the probability generating function (z-transform), which changes according to the current level of the process.

2.2.7.1 Simulation of the q-killed Upwards Skip-Free Random Walk

As previously mentioned, the term "q-killing" refers to the level-dependent killing probability for an upwards skip-free random walk under some specific choices. In this section, we simulate an upwards skip-free random walk versus q-killed random walk. Simulation of the q-killed random walk has been done by determining the killing probability $(q(X_1), q(X_2), \ldots)$ for each level of the process $(X_1, X_2, \ldots)$, then generating geometric random variates based on these probabilities that we have found and comparing them with time.

In Figure 2.3, we demonstrate the behavior of the q-killed upwards skip-free random walk and a not killed upwards skip-free random walk. According to the parameter choice mentioned before, we generate these random walks with the same seed to provide convenient comparison. When we compare Figure 2.3a and Figure 2.3b, we immediately observe that not every simulated path reaches the terminal time; some of them are killed before reaching terminal time ($T = 60$) in Figure 2.3b while the simulated paths of upwards skip-free random walk given in Figure 2.3a, they continue their lifetime till reaching terminal time. For illustration, let us turn our focus to orange paths in Figure 2.3a. One of them goes to the dummy state, which means it cannot be observable after approximately time 45, while the other one goes to the dummy state around time 55.



(a) (b)

Figure 2.3: Upwards Skip-Free Random Walk and Killed Upwards Skip-Free Random Walk.

2.2.7.2 Numerical Results

This section provides numerical examples for the two-sided exit upwards and downwards problem derived in Theorem 9 and Theorem 15. In order to find numerical values of the fluctuation identities, we need to evaluate recursively equations of $\mathcal{W}_q(x)$ and $\mathcal{Z}_q(x)$ for $x \in \mathbb{N}_0$ given in the Eqs. (2.2.2) – (2.2.3) respectively. In this subsection, we first investigate the behavior of q-killed scale functions in order to gain a better understanding of exit problems that we shall examine later in this section.

We use the same parameter that we introduced in the previous part of this subsection. Now, we analyze our numerical results under different values of p and ν.

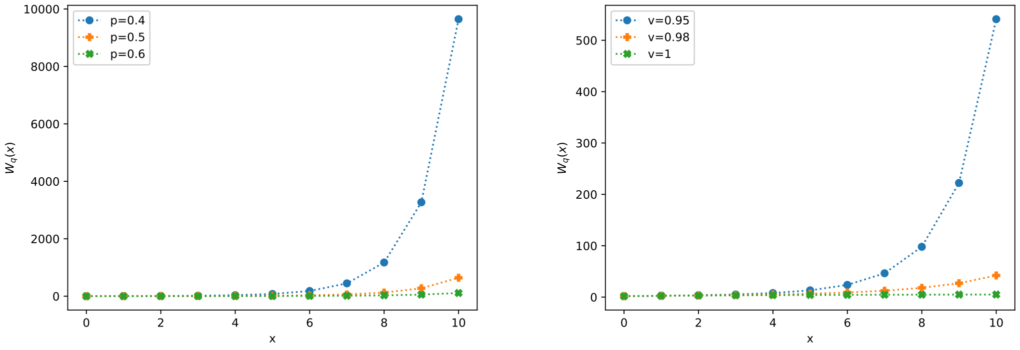
Scale Functions

In order to comprehend the behavior of two-sided exit problems under different parameters of p and ν, we first evaluate the behavior of scale functions $\mathcal{W}_q(x)$ and $\mathcal{Z}_q(x)$ under these parameters that we choose.

In Figure 2.4a, we demonstrate the behavior of $\mathcal{W}_q(x)$ when we take different parameters of geometric distribution (i.e. $p = 0.4, 0.5, 0.6$) when $\nu = 0.96$ whereas in Figure 2.4b, when we take different parameters for the killing function $q(x) = \nu^x$ (i.e. $\nu = 0.95, 0.98, 1$) when $p = 0.55$. As noted in Section 2.2, the scale functions are recursive equations with some initial values, which are unnecessary to find two-sided exit problems. However, for consistency with the literature (see [5]), we set $\mathcal{W}_q(0) = 1/p_0$.

As seen in Table 2.1 (also in Figure 2.4a) when the probability of getting claim $(1-p)$ is increased the values of scale function are increasing. Moreover in Table 2.2 (also in Figure 2.4b) when the value of ν is increased the values of scale function are decreasing. We remark that when $\nu = 1$, the q-killed scale function turns out to be the function given in [5] (see Eq. (2.1.6) in Section 2).

Furthermore, we show the effect of the geometric distribution parameter p and killing parameter ν for $\mathcal{Z}_q(x)$ in Figure 2.5. Similar to the example for $\mathcal{W}_q(x)$, we take the different parameters for the geometric distribution ($p = 0.4, 0.5, 0.6$) with $\nu = 0.96$



(a)

(b)

Figure 2.4: $\mathcal{W}_q(x)$ for different choice of p and ν.

Table 2.1: Different values of $\mathcal{W}_q(x)$ with different p.

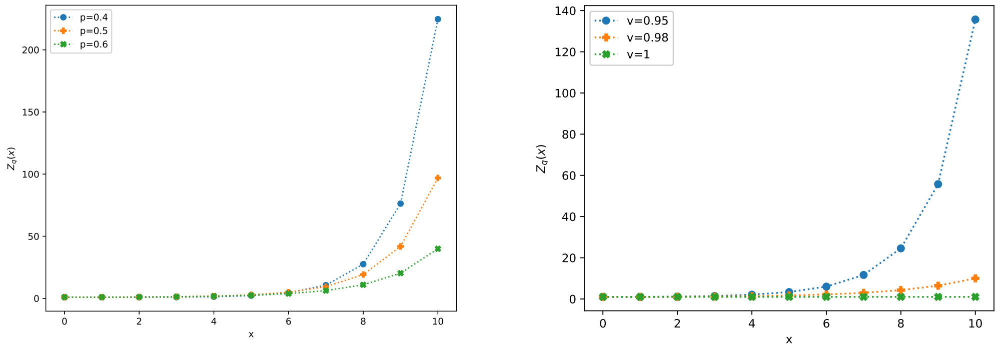
	$\mathcal{W}_q(0)$	$\mathcal{W}_q(1)$	$\mathcal{W}_q(2)$	$\mathcal{W}_q(3)$	$\mathcal{W}_q(4)$	$\mathcal{W}_q(5)$	$\mathcal{W}_q(6)$	$\mathcal{W}_q(7)$	$\mathcal{W}_q(8)$	$\mathcal{W}_q(9)$	$\mathcal{W}_q(10)$
$p = 0.4$	2.5	4.9479	9.3531	18.040	36.582	78.976	182.13	448.30	1174.9	3270.2	9646.4
$p = 0.5$	2	3.125	4.6115	6.8926	10.8007	18.0395	32.2938	61.9447	126.9709	277.2387	643.0286
$p = 0.6$	1.6667	2.1991	2.7713	3.5635	4.8174	6.9334	10.6599	17.4918	30.5611	56.7106	111.5221

Table 2.2: Different values of $\mathcal{W}_q(x)$ with different $q(x) = \nu^x$.

	$\mathcal{W}_q(0)$	$\mathcal{W}_q(1)$	$\mathcal{W}_q(2)$	$\mathcal{W}_q(3)$	$\mathcal{W}_q(4)$	$\mathcal{W}_q(5)$	$\mathcal{W}_q(6)$	$\mathcal{W}_q(7)$	$\mathcal{W}_q(8)$	$\mathcal{W}_q(9)$	$\mathcal{W}_q(10)$
$\nu = 0.95$	1.8182	2.6185	3.627	5.1927	7.9481	13.1852	23.7659	46.4286	97.933	222.2502	541.0794
$\nu = 0.98$	1.8182	2.5384	3.2566	4.0844	5.1625	6.687	8.9569	12.4568	18.0077	27.0458	42.1513
$\nu = 1$	1.8182	2.4876	3.0353	3.4834	3.8501	4.1501	4.3955	4.5963	4.7606	4.8951	5.0051

and different parameters for the killing function as ($\nu = 0.95, 0.98, 1$) with $p = 0.55$. Similar to $\mathcal{W}_q(x)$, choice of the initial value of the q-killed scale function leans on the literature so that we choose $\mathcal{Z}_q(0) = 1$.

Not surprisingly, as seen in Table 2.3 (also in Figure 2.5a), the values of $\mathcal{Z}_q(x)$ are increasing when the probability of getting claim $(1-p)$ is increased, while in Table 2.4 (also in Figure 2.5b), the values of $\mathcal{Z}_q(x)$ decrease when ν is increased. As expected, the values of $\mathcal{Z}_q(x)$ is always equal to one when $\nu = 1$ since the q-killed scale function turns out to be the function of classical discrete scale function given in [5] when $\nu = 1$



(a) (b)

Figure 2.5: $\mathcal{Z}_q(x)$ for different choice of p and ν.

is always equal to one (see Eq. (2.1.9) in Section 2).

Table 2.3: Different values of $\mathcal{Z}_q(x)$ with different p.

	$\mathcal{Z}_q(0)$	$\mathcal{Z}_q(1)$	$\mathcal{Z}_q(2)$	$\mathcal{Z}_q(3)$	$\mathcal{Z}_q(4)$	$\mathcal{Z}_q(5)$	$\mathcal{Z}_q(6)$	$\mathcal{Z}_q(7)$	$\mathcal{Z}_q(8)$	$\mathcal{Z}_q(9)$	$\mathcal{Z}_q(10)$
$p = 0.4$	1	0.9375	0.9155	0.9975	1.3193	2.2101	4.5304	10.6606	27.5244	76.2707	224.7037
$p = 0.5$	1	0.9964	1.0772	1.3089	1.8139	2.8439	4.9491	9.3875	19.1658	41.7945	96.9012
$p = 0.6$	1	1.0218	1.1245	1.3484	1.7634	2.5019	3.8253	6.2646	10.9384	20.2939	39.9062

Table 2.4: Different values of $\mathcal{Z}_q(x)$ with different $q(x) = \nu^x$.

	$\mathcal{Z}_q(0)$	$\mathcal{Z}_q(1)$	$\mathcal{Z}_q(2)$	$\mathcal{Z}_q(3)$	$\mathcal{Z}_q(4)$	$\mathcal{Z}_q(5)$	$\mathcal{Z}_q(6)$	$\mathcal{Z}_q(7)$	$\mathcal{Z}_q(8)$	$\mathcal{Z}_q(9)$	$\mathcal{Z}_q(10)$
$\nu = 0.95$	1	1.0139	1.136	1.4415	2.0767	3.3554	5.9878	11.6586	24.5672	55.7379	135.6877
$\nu = 0.98$	1	1.0062	1.053	1.1595	1.3515	1.669	2.1774	2.9872	4.29	6.4239	9.999
$\nu = 1$	1	1	1	1	1	1	1	1	1	1	1

Moreover, if we compare Table 2.1 and Table 2.2 with Table 2.3 and Table 2.4 respectively, we immediately observe that increases in the values of $\mathcal{W}_q(x)$ are more rapid than the increases in the values of $\mathcal{Z}_q(x)$. At first glance, it might seem strange since in the calculation of the $\mathcal{Z}_q(x)$ we additionally consider an infinite sum. However, if we magnify on the recursive equations given in Eqs. (2.2.2) – (2.2.3), we notice that these equations are actually backward recursive equations so that this infinite sum comes into account as a difference. In other words, the calculation is done by considering the following forward recursive equation

$$\mathcal{Z}_q(x+1) = \frac{(1-p_1 q(x))\mathcal{Z}_q(x) - \sum_{k=1}^{x} p_{k+1} q(x-k)\mathcal{Z}_q(x-k)}{p_0 q(x+1)} - \frac{\sum_{k=x+1}^{\infty} p_{k+1} q(x-k)}{p_0 q(x+1)},$$

while for $\mathcal{W}_q(x)$, we consider

$$\mathcal{W}_q(x+1) = \frac{(1-p_1 q(x))\mathcal{W}_q(x) - \sum_{k=1}^{x} p_{k+1} q(x-k)\mathcal{W}_q(x-k)}{p_0 q(x+1)}.$$

This clearly explains why $\mathcal{Z}_q(x)$ increases less than $\mathcal{W}_q(x)$.

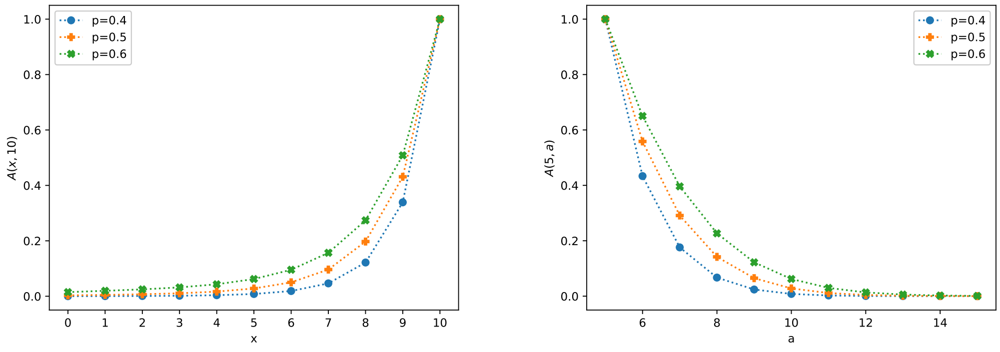
Two-Sided Exit Upwards

After exemplifying the scale functions, we move on to examine the behaviors of two-sided exit upwards when we change the parameter for geometric distribution and q-killing function. In the actuarial context, two-sided exit upwards problem is used for the calculations of, for instance, the probability of paying dividends to shareholders and related identities.

In Figure 2.6a, we show $\mathcal{A}(x, a)$ when the level above is fixed as $a = 10$ while in Figure 2.6b when initial value of the process is fixed as $x = 5$ under different parameters of geometric distribution i.e. $p = 0.4, 0.5, 0.6$.

As depicted in Table 2.5 and Table 2.6 (also in Figure 2.6a and Figure 2.6b respectively), decrease in getting claim $(1-p)$ results in increase in the two-sided exit upwards problem in both cases. This result can be interpreted as if we have less claim, then the probability of reaching level a before deficit occurs is higher so that the probability of paying dividends to the shareholders is higher, for instance.

Furthermore, from Table 2.5 (or Figure 2.6a), we examine that the increase of the initial value of x with fixed $a = 10$ increases the probability of reaching level a for a q-killed upwards skip-free process. On the other hand, in Table 2.6 (or Figure 2.6b), with fixed initial value $x = 5$ when the level a (barrier) increases unsurprisingly, the probability of reaching level a decreases for such processes.



(a)

(b)

Figure 2.6: $\mathcal{A}(x,a)$ for different choice of p.

Table 2.5: Different values of $\mathcal{A}(x,10)$ with different p.

	$\mathcal{A}(0,10)$	$\mathcal{A}(1,10)$	$\mathcal{A}(2,10)$	$\mathcal{A}(3,10)$	$\mathcal{A}(4,10)$	$\mathcal{A}(5,10)$	$\mathcal{A}(6,10)$	$\mathcal{A}(7,10)$	$\mathcal{A}(8,10)$	$\mathcal{A}(9,10)$	$\mathcal{A}(10,10)$
$p=0.4$	2.5916×10^{-4}	5.1293×10^{-4}	9.6959×10^{-4}	1.8702×10^{-3}	3.7923×10^{-3}	8.1871×10^{-3}	1.8881×10^{-2}	4.6473×10^{-2}	1.2180×10^{-1}	3.3901×10^{-1}	1
$p=0.5$	3.1103×10^{-3}	4.8598×10^{-3}	7.1716×10^{-3}	1.0719×10^{-2}	1.6797×10^{-2}	2.8054×10^{-2}	5.0221×10^{-2}	9.6333×10^{-2}	1.9746×10^{-1}	4.3115×10^{-1}	1
$p=0.6$	1.4945×10^{-2}	1.9719×10^{-2}	2.4850×10^{-2}	3.1953×10^{-2}	4.3197×10^{-2}	6.2171×10^{-2}	9.5586×10^{-2}	1.5685×10^{-1}	2.7404×10^{-1}	5.0851×10^{-1}	1

Table 2.6: Different values of $\mathcal{A}(5,a)$ with different p.

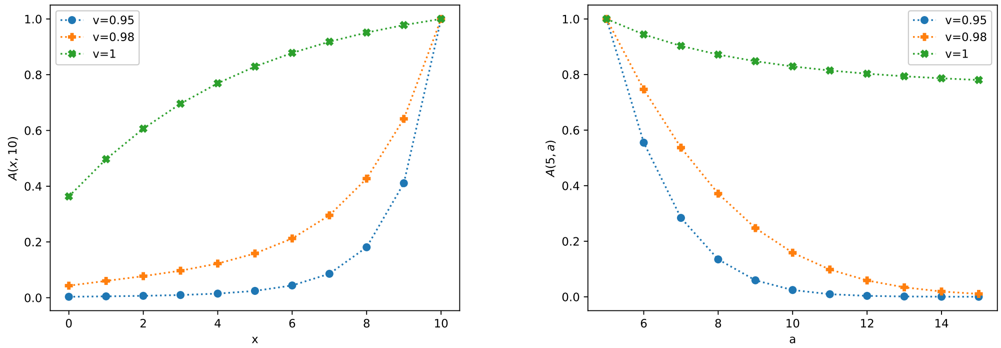
	$\mathcal{A}(5,5)$	$\mathcal{A}(5,6)$	$\mathcal{A}(5,7)$	$\mathcal{A}(5,8)$	$\mathcal{A}(5,9)$	$\mathcal{A}(5,10)$	$\mathcal{A}(5,11)$	$\mathcal{A}(5,12)$	$\mathcal{A}(5,13)$	$\mathcal{A}(5,14)$	$\mathcal{A}(5,15)$
$p=0.4$	1	4.3361×10^{-1}	1.7617×10^{-1}	6.7221×10^{-2}	2.4150×10^{-2}	8.1871×10^{-3}	2.6238×10^{-3}	7.9615×10^{-4}	2.2901×10^{-4}	6.2517×10^{-5}	1.6211×10^{-5}
$p=0.5$	1	5.5860×10^{-1}	2.9122×10^{-1}	1.4208×10^{-1}	6.5068×10^{-2}	2.8054×10^{-2}	1.1413×10^{-2}	4.3892×10^{-3}	1.5982×10^{-3}	5.5166×10^{-4}	1.8071×10^{-4}
$p=0.6$	1	6.5042×10^{-1}	3.9638×10^{-1}	2.2687×10^{-1}	1.2226×10^{-1}	6.2171×10^{-2}	2.9887×10^{-2}	1.3603×10^{-2}	5.8694×10^{-3}	2.4034×10^{-3}	9.3481×10^{-4}

We also analyze the impacts of the parameter of $q(x)$ on the two-sided exit upwards problem. As seen in Table 2.7 and Table 2.8 (also in Figure 2.7a and Figure 2.7b respectively), decrease in the killing parameter that is a decrease in $(1-q(x))$ leads to the higher p.g.f. of reaching level a for q-killed upwards skip-free random walk. This result is also logical since the process is killed after a longer time, it has a higher chance to reach level a.

Also note that in Figure 2.7a, it might be awkward to see that $\mathcal{A}(x,10)$ when $\nu=1$ behaves differently than the other ones. One possible reason for that is the increase in the scale function $\mathcal{W}_q(x)$ is slower than the other ones (see Table 2.2).

One last remark about the two-sided exit upwards problem in that when $\nu=1$,

$\mathcal{A}(x,a)$ turns out to be the fluctuation identity given in Eq. (2.1.4).



(a)

(b)

Figure 2.7: $\mathcal{A}(x,a)$ for different choice of ν.

Table 2.7: Different values of $\mathcal{A}(x,10)$ with different ν.

	$\mathcal{A}(0,10)$	$\mathcal{A}(1,10)$	$\mathcal{A}(2,10)$	$\mathcal{A}(3,10)$	$\mathcal{A}(4,10)$	$\mathcal{A}(5,10)$	$\mathcal{A}(6,10)$	$\mathcal{A}(7,10)$	$\mathcal{A}(8,10)$	$\mathcal{A}(9,10)$	$\mathcal{A}(10,10)$
$\nu = 0.95$	0.0034	0.0048	0.0067	0.0096	0.0147	0.0244	0.0439	0.0858	0.181	0.4108	1
$\nu = 0.98$	0.0431	0.0602	0.0773	0.0969	0.1225	0.1586	0.2125	0.2955	0.4272	0.6416	1
$\nu = 1$	0.3633	0.497	0.6064	0.696	0.7692	0.8292	0.8782	0.9183	0.9512	0.978	1

Table 2.8: Different values of $\mathcal{A}(5,a)$ with different ν.

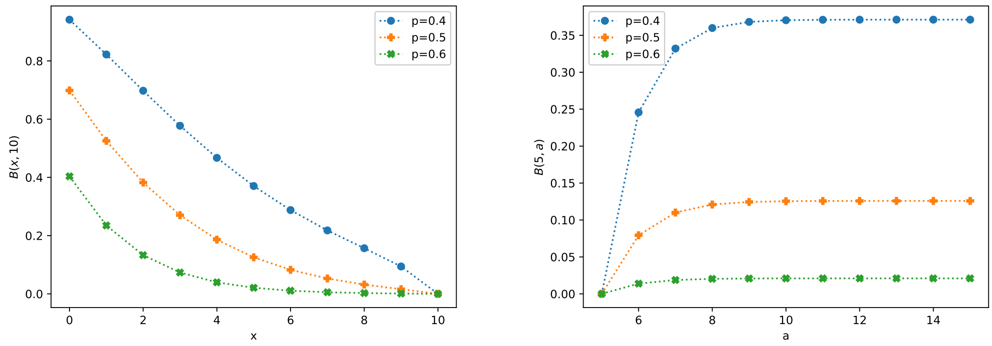
	$\mathcal{A}(5,5)$	$\mathcal{A}(5,6)$	$\mathcal{A}(5,7)$	$\mathcal{A}(5,8)$	$\mathcal{A}(5,9)$	$\mathcal{A}(5,10)$	$\mathcal{A}(5,11)$	$\mathcal{A}(5,12)$	$\mathcal{A}(5,13)$	$\mathcal{A}(5,14)$	$\mathcal{A}(5,15)$
$\nu = 0.95$	1	5.5479×10^{-1}	2.8399×10^{-1}	1.3463×10^{-1}	5.9326×10^{-2}	2.4368×10^{-2}	9.3520×10^{-3}	3.35969×10^{-3}	1.1314×10^{-3}	3.5769×10^{-4}	1.0626×10^{-4}
$\nu = 0.98$	1	7.4658×10^{-1}	5.3682×10^{-1}	3.7134×10^{-1}	2.4725×10^{-1}	1.5864×10^{-1}	9.8229×10^{-2}	5.8771×10^{-2}	3.4016×10^{-2}	1.9066×10^{-2}	1.0357×10^{-2}
$\nu = 1$	1	9.4416×10^{-1}	9.0291×10^{-1}	8.7175×10^{-1}	8.4781×10^{-1}	8.2918×10^{-1}	8.1453×10^{-1}	8.0293×10^{-1}	7.9368×10^{-1}	7.8627×10^{-1}	7.8030×10^{-1}

Two-Sided Exit Downwards

In this part, we investigate the other important exit identity, i.e., two-sided exit downwards. In this case, the results will be the p.g.f. of crossing the lower level before exiting the level a for a q-killed upwards skip-free random walk. In actuarial and financial context, this identity is used for calculating probability of ruin/bankruptcy. Since we have $q(x)$ function, finding the values of two-sided exit downwards gives information about the probability of bankruptcy, which allows the process to go below zero; in that case the occurrence of bankruptcy is weighted with the function $q(x)$. An example of calculating the probability of bankruptcy can be found in Section 2.2.6.

Assuming the same parameter choice as the previous part, we analyze the behavior of the two-sided exit downwards problem with some figures and tables, organized under different parameters of p and ν.

In Figure 2.8a, we demonstrate $\mathcal{B}(x, a)$ when we have a fixed level above, i.e., $a = 10$ while in Figure 2.8b when we choose the fixed initial value of the process $x = 5$ under different parameters of geometric distribution (i.e. $p = 0.4, 0.5, 0.6$). As seen in Table 2.9 - Table 2.10[1] (also in Figure 2.8a and Figure 2.8b respectively), two-sided exit downwards values drop when the probability of having claim $(1 - p)$ is decreased. This expected result can translated into real-life example as follows: if we have less claim, the p.g.f. of reaching lower level -1 before reaching level a is less probable. Another comment is about the increase in level a. In Figure 2.8b and also can be understood from Table 2.10, when the level a increases, $\mathcal{B}(x, a)$ increases in every situation (getting more claim or not). The reason behind that result is when the upper barrier increases, there is more possibility of the process to go to level -1 till reaching this barrier.



(a)

(b)

Figure 2.8: $\mathcal{B}(x, a)$ for different choice of p.

Moreover, we analyze the effects of the parameters of $q(x)$ on the two-sided exit

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[1] In Table 2.10, the values might seem equal, but it is not the case when we take more than four decimal digits, these values are actually slightly different.

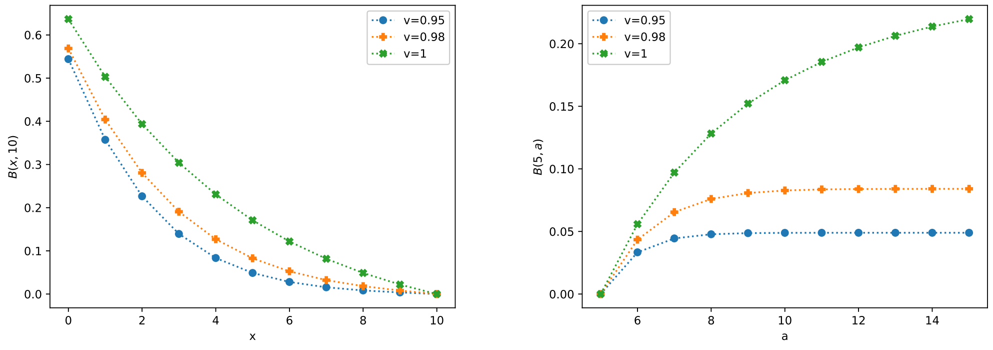
Table 2.9: Different values of $\mathcal{B}(x,10)$ with different p.

	$\mathcal{B}(0,10)$	$\mathcal{B}(1,10)$	$\mathcal{B}(2,10)$	$\mathcal{B}(3,10)$	$\mathcal{B}(4,10)$	$\mathcal{B}(5,10)$	$\mathcal{B}(6,10)$	$\mathcal{B}(7,10)$	$\mathcal{B}(8,10)$	$\mathcal{B}(9,10)$	$\mathcal{B}(10,10)$
$p=0.4$	0.9418	0.8222	0.6977	0.5773	0.4672	0.3704	0.2877	0.2179	0.1568	0.0941	0
$p=0.5$	0.6986	0.5255	0.3823	0.2702	0.1863	0.1254	0.0826	0.0528	0.0319	0.016	0
$p=0.6$	0.4036	0.2349	0.1329	0.0733	0.0396	0.0209	0.0108	0.0055	0.0026	0.001	0

Table 2.10: Different values of $\mathcal{B}(5,a)$ with different p.

	$\mathcal{B}(5,5)$	$\mathcal{B}(5,6)$	$\mathcal{B}(5,7)$	$\mathcal{B}(5,8)$	$\mathcal{B}(5,9)$	$\mathcal{B}(5,10)$	$\mathcal{B}(5,11)$	$\mathcal{B}(5,12)$	$\mathcal{B}(5,13)$	$\mathcal{B}(5,14)$	$\mathcal{B}(5,15)$
$p=0.4$	0	0.2456	0.332	0.3599	0.3681	0.3704	0.371	0.3711	0.3712	0.3712	0.3712
$p=0.5$	0	0.0793	0.1101	0.1209	0.1244	0.1254	0.1257	0.1258	0.1258	0.1258	0.1258
$p=0.6$	0	0.0139	0.0187	0.0203	0.0208	0.0209	0.0209	0.0209	0.0209	0.0209	0.0209

downwards problem. As clearly seen in Table 2.11 and Table 2.12 (also in Figure 2.9a and Figure 2.9b respectively), when the killing parameter is decreased (that is, decrease in $(1-q(x))$) probability of reaching level below before reaching level a is higher. One explanation for this result might be related to there is a higher possibility of reaching level -1 if the process is killed after a longer time.



(a) (b)

Figure 2.9: $\mathcal{B}(x,a)$ for different choice of ν.

With the same reasoning as the previous figure/table for $\mathcal{B}(5,a)$, Figure 2.9b (also in Table 2.12) reveals that increase in the level above a causes the increase in the two-sided exit downwards problem. Also, we note that when $\nu=1$, $\mathcal{B}(x,a)$ turns out to be the fluctuation identity of upwards skip-free random walk (non-killed version) given in Eq. (2.1.8).

Table 2.11: Different values of $\mathcal{B}(x, 10)$ with different ν.

	$\mathcal{B}(0,10)$	$\mathcal{B}(1,10)$	$\mathcal{B}(2,10)$	$\mathcal{B}(3,10)$	$\mathcal{B}(4,10)$	$\mathcal{B}(5,10)$	$\mathcal{B}(6,10)$	$\mathcal{B}(7,10)$	$\mathcal{B}(8,10)$	$\mathcal{B}(9,10)$	$\mathcal{B}(10,10)$
$\nu = 0.95$	0.5441	0.3572	0.2264	0.1393	0.0835	0.0489	0.028	0.0157	0.0083	0.0038	0
$\nu = 0.98$	0.5687	0.4041	0.2805	0.1906	0.1269	0.0828	0.0526	0.0322	0.0183	0.0082	0
$\nu = 1$	0.6367	0.503	0.3936	0.304	0.2308	0.1708	0.1218	0.0817	0.0488	0.022	0

Table 2.12: Different values of $\mathcal{B}(5, a)$ with different ν.

	$\mathcal{B}(5,5)$	$\mathcal{B}(5,6)$	$\mathcal{B}(5,7)$	$\mathcal{B}(5,8)$	$\mathcal{B}(5,9)$	$\mathcal{B}(5,10)$	$\mathcal{B}(5,11)$	$\mathcal{B}(5,12)$	$\mathcal{B}(5,13)$	$\mathcal{B}(5,14)$	$\mathcal{B}(5,15)$
$\nu = 0.95$	0	0.0334	0.0445	0.0478	0.0487	0.0489	0.049	0.049	0.049	0.049	0.049
$\nu = 0.98$	0	0.0435	0.0655	0.076	0.0807	0.0828	0.0836	0.0839	0.084	0.0841	0.0841
$\nu = 1$	0	0.0558	0.0971	0.1283	0.1522	0.1708	0.1855	0.1971	0.2063	0.2137	0.2197

CHAPTER 3

FLUCTUATION THEORY FOR DIFFERENT TYPES OF SPECTRALLY NEGATIVE LÉVY PROCESSES

In this chapter, starting with the characterization of Lévy processes, step by step, we built an overview of the level-dependent Lévy processes studied in [13]. We first introduce basic concepts and notations for spectrally negative Lévy processes (SNLP) and a suite of exit identities of these processes with respect to (so-called) scale functions. Then, we review the fluctuation theory of refracted and reflected refracted spectrally negative Lévy processes analyzed in [25] and [31], respectively. Particularly, we concentrate on the theory of multi-refracted SNLP and its generalization, which is well-studied in [13]. In all theorems of this section, we observe that scale functions are employed predominantly to explain exit problems and their resolvents semi-explicitly. For a wide range of reviews about utilizing the scale function in different contexts, we refer to [3].

In the following sections of this chapter, we present our study related to reflected level-dependent SNLP and omega-killed (reflected) level-dependent SNLP. In the first part of our study, we analyze potential measures of reflected level-dependent SNLP by adopting the theory of reflected refracted SNLP to the multi-refracted case. We discuss firstly the existence of SDE of the reflected multi-refracted SNLP. Then, we derive the potential measure and one-sided exit problem of the reflected multi-refracted SNLP for the bounded variation case, and by using a strong approximation of the bounded variation case, we prove for the unbounded variation case. In order to derive the same identities for the reflected level-dependent SNLP, we approximate a sequence of reflected multi-refracted SNLP to level-dependent as a limiting argument which demonstrates the existence and uniqueness of reflected level-dependent SNLP.

In the second part of our study, we study the fluctuation theory of ω-killed level-dependent SNLP. Additionally, we present the results of the reflected case, i.e., the potential measure and one-sided exit problem for the reflected ω-killed level-dependent SNLP. We finalize this study with an essential example of finding the probability of bankruptcy if the surplus process is modeled by ω-killed level-dependent SNLP. The primary method throughout all proofs in this part leans on the Poisson observation method in [26, 29, 28] and classical fluctuation identities.

3.1 Spectrally Negative Lévy Processes

This subsection presents some introduction related to fluctuation theory for spectrally negative Lévy processes. We start with the definition and fundemental properties of Lévy processes. Then, we turn our attention to scale functions (or q-scale function) which are auxiliary functions to solve boundary crossing problems, potential measures and related path decomposition. We note that the aforementioned problems do not have explicit solutions, and thus the scale function provides nice semi-explicit expressions. That is the reason why applied probability models heavily rely on these functions.

In probability theory, Lévy process, named after the French mathematician Paul Lévy, is a stochastic process representing the motion of a point whose successive displacements are random, and independent and identically distributed (IID) over different time intervals of the same length. The most well-known examples of Lévy processes are the Brownian motion, the Poisson and the compound Poisson processes or Cramér-Lundberg process in the context of actuarial literature, the Gaussian process, the stable processes, and many others. Hence, as a result of the mathematical tractability and the preeminent advantages in modeling, Lévy processes are proven to play a central role in several fields of science, such as physics, in the study of turbulence and laser cooling; in quantum field theory; in engineering, for the study of networks, queues, and dams; in economics, for continuous time-series models; in the actuarial science, for the calculation of insurance and reinsurance risk; and in mathematical finance, for option pricing. A comprehensive overview of several applications of Lévy processes can be found in [8, 15, 24, 35].

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t : t \geq 0\}, \mathbb{P})$ be a filtered probability space satisfying usual assumptions with complete filtration $(\mathcal{F}_t)_{t\geq 0}$. Consider a real-valued stochastic process $X = \{X_t,\ t \geq 0\}$, the law of that is denoted by $\mathbb{P}_x$ such that $X_0 = x$ and its associated expectation by $\mathbb{E}_x$. Also, for convenience, when $x = 0$, we always write $\mathbb{P}$ and $\mathbb{E}$.

Definition 28 (Lévy processes [24])**.** *A stochastic process $X = \{X_t\}_{t\geq 0}$ on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is said to be* Lévy process*, if it satisfies the following conditions*

i. *The paths of X_t are $\mathbb{P}$-almost surely right continuous with left limits, namely* cádlág *process,*

ii. $\mathbb{P}(X_0 = 0) = 1$,

iii. *$X_t - X_s$ is equal in distribution for $0 \leq s \leq t$ to X_{t-s}* (stationary increments),

iv. *$X_t - X_s$ is independent of $(X_u)_{u<s}$ for $0 \leq s \leq t$* (independent increments).

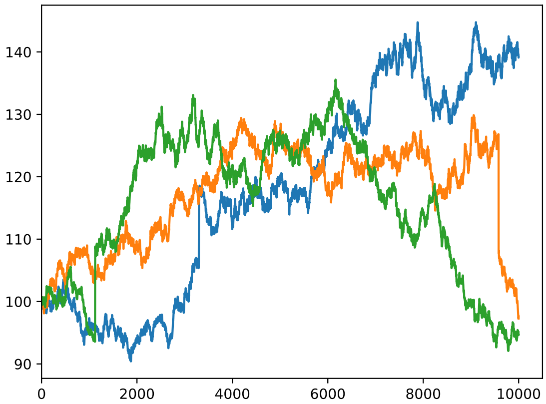


Figure 3.1: Simulated sample paths of a Lévy process.

Theorem 29 (Lévy-Khintchine formula [24])**.** *Under law of $\mathbb{P}_x$, Lévy process X is infinitely divisible if and only if there exits a triplet (μ, σ, ν), where $\mu \in \mathbb{R}$, $\sigma \geq 0$ and ν is a measure concentrated on $\mathbb{R} \setminus \{0\}$ satisfying $\int_{\mathbb{R}}(1 \wedge |x|^2)\nu(\mathrm{d}x) < \infty$ such that*

$$\mathbb{E}[e^{i\theta X_t}] = e^{-t\Psi(\theta)}, \quad \textit{for all} \quad t \geq 0,$$

where

$$\Psi(\theta) = -i\theta\mu + \frac{1}{2}\sigma^2\theta^2 + \int_{\mathbb{R}} (1 - e^{i\theta x} + i\theta x \mathbf{1}_{(|x|<1)})\nu(\mathrm{d}x). \tag{3.1.1}$$

Note that the triple (μ, σ, ν) is called *characteristic triplet* which is uniquely defined for X and $\Psi(\theta)$ is called *characteristic exponent* or *Lévy exponent*. Most of the time it is easy to work with *Laplace exponent* which is defined by

$$\psi(\theta) = \log \mathbb{E}[e^{\theta X_1}] = -\Psi(-i\theta). \tag{3.1.2}$$

Through this work, it is assumed that X has no positive jumps. In this case X is called as *spectrally negative Lévy process* (SNLP). Mathematically, $X = \{X_t\}_{t\geq 0}$ is a *spectrally negative* if the measure ν is carried on $(-\infty, 0)$ i.e. $\nu(0,\infty) = 0$. Since such processes have no positive jumps, its Laplace exponent always exists for all $\theta \geq 0$ and is given by

$$\psi(\theta) = \mu\theta + \frac{\theta^2\sigma^2}{2} + \int_{(-\infty,0)} (e^{\theta x} - 1 - \theta x \mathbf{1}_{\{x>-1\}})\nu(\mathrm{d}x),$$

which is continuous, strictly convex function and tends to infinity as $\theta \to \infty$ that is $\lim_{\theta\to\infty} \psi(\theta) = \infty$ and $\psi(0) = 0$. Thus, there exists a function $\Phi(\theta) : [0,\infty) \to [0,\infty)$ defined by

$$\Phi(q) = \sup\{\theta \geq 0 : \psi(\theta) = q\}, \quad \text{(right inverse of } \psi\text{)},$$

such that $\psi(\Phi(q)) = q$ for $q \geq 0$. We note that $\Phi(q)$ exists and can be explicitly calculated for the Brownian motion case, the compound Poisson case (which is called Lundberg's equation within the actuarial literature), to mention a few.

Due to having jumps in only one direction, spectrally negative Lévy processes offer a significant advantage for many calculations. That is one of the reasons why they are massively used in actuarial and mathematical finance applications.

Next, we introduce the so-called scale functions, which plays a vital role for developing fluctuation identities of spectrally negative Lévy processes, see [8, 15, 19, 22, 23, 24].

Definition 30 ($W^{(q)}$-scale function [24]). *For any $q \geq 0$, the q-scale function $W^{(q)} : \mathbb{R} \to [0,\infty)$ as a unique right continuous function whose Laplace transform satisfies*

$$\int_0^\infty e^{-\theta x} W^{(q)}(x)\mathrm{d}x = \frac{1}{\psi(\theta) - q}, \quad \theta > \Phi(q), \tag{3.1.3}$$

with $W^{(q)}(0) := \lim_{x \downarrow 0} W^{(q)}(x)$.

The function is defined above is positive, monotone, strictly increasing, and strictly convex for $x \geq 0$. It can be extended to the whole real line by setting $W^{(q)}(x) = 0$ for $x < 0$. Further, for $q > 0$, q-scale function can be regarded as the scale function of the original process X killed at independent exponential times with parameter q.

Next, we present adjoint scale function which is closely related to $W^{(q)}(x)$.

Definition 31 ($Z^{(q)}$-scale function [24])**.** *For* $q \geq 0$*, we define* $Z^{(q)} : \mathbb{R} \to [1, \infty)$ *by*

$$Z^{(q)}(x) = 1 + q \int_0^x W^{(q)}(y) \mathrm{d}y, \tag{3.1.4}$$

with $Z^{(q)}(x) = 1$ *for* $x \leq 0$*, where* $W^{(q)}(x)$ *is given in Eq. (3.1.3).*

Notice that $Z^{(q)}$ function inherits some properties of $W^{(q)}(x)$; in particularly, it is increasing, continuous and strictly convex on $(0, \infty)$.

Note that we denote $W(x) := W^{(0)}(x)$ and $Z(x) := Z^{(0)}(x)$ for $q = 0$. For further analytical properties of both scale functions, one can see [Chap. 8 in [24] or as a summary, see [19]].

It is unclear from the aforementioned definitions of the scale function alone why classical applied probability models rely on such functions. Later we shall see that scale functions appear to solve boundary-crossing problems and related path decompositions. An immediate example of a boundary-crossing identity is the so-called first passage problems and their resolvents, which are given by the following theorems. Before stating, we define the first passage times above and below a level x respectively by

$$\tau_x^+ = \inf\{t > 0 : X_t > x\}, \quad \text{and} \quad \tau_x^- = \inf\{t > 0 : X_t < x\}, \tag{3.1.5}$$

with $\inf \emptyset = \infty$.

Theorem 32 (One and two-sided exit [24])**.** *For* $0 \leq x \leq a$ *and* $q \geq 0$

$$\mathbb{E}_x \left[e^{-q\tau_a^+} \mathbf{1}_{(\tau_a^+ < \tau_0^-)} \right] = \frac{W^{(q)}(x)}{W^{(q)}(a)}, \tag{3.1.6}$$

and

$$\mathbb{E}_x\left[e^{-q\tau_0^-}\mathbf{1}_{(\tau_0^-<\tau_a^+)}\right] = Z^{(q)}(x) - \frac{W^{(q)}(x)}{W^{(q)}(a)}Z^{(q)}(a), \tag{3.1.7}$$

where $W^{(q)}(x)$ *and* $Z^{(q)}(x)$ *are given in Eqs. (3.1.3)-(3.1.4) respectively. Also, we have the one-sided exit problems as follows*

$$\mathbb{E}_x\left[e^{-q\tau_a^+}\mathbf{1}_{(\tau_a^+<\infty)}\right] = e^{-\Phi(q)(a-x)},$$

and for $a \to \infty$ *in Eq. (3.1.7)*

$$\mathbb{E}_x\left[e^{-q\tau_0^-}\mathbf{1}_{(\tau_0^-<\infty)}\right] = Z^{(q)}(x) - \frac{q}{\Phi(q)}W^{(q)}(x),$$

where $q/\Phi(q)$ *is a limit of* $Z^{(q)}(a)/W^{(q)}(a)$ *as* $a \to \infty$.

Next, we provide one of the important analytical properties of scale functions which is asymptotic behavior at ∞.

Lemma 33 (Asymptotic behavior of scale function [22])**.** *For* $q \geq 0$*, we have*

$$\lim_{x\to\infty} e^{-\Phi(q)x}W^{(q)}(x) = \frac{1}{\psi'(\Phi(q))} = \Phi'(q),$$

where $\psi'(\Phi(q)) = \Phi'(q)^{-1}$ *since* $\psi(\Phi(q)) = q$ *and taking derivative with respect to* q*. Also, we have the following limiting argument*

$$\lim_{x\to\infty} Z^{(q)}(x)/W^{(q)}(x) = \frac{q}{\Phi(q)},$$

where the RHS of the above equation is understood as $\lim_{q\downarrow 0} q/\Phi(q) = 0 \vee (1/\psi'(0^+))$ *when* $q = 0$.

Exit problems given above express the (discounted) probabilities of exiting in the interval $[0, a]$ above and below in terms of scale functions, which originally roots back to Takács (1966) in [36]. With the next theorem, we also provide semi-explicit expressions for the expected occupation measure of X killed on exiting an interval $[0, a]$. Such occupation measures are known as *resolvents* and play a crucial role in establishing information about the level of the process.

Theorem 34 (Resolvents [24])**.** *For* $0 \leq x \leq a$ *and* $q \geq 0$*, the resolvent of the process* X *killed on exiting the interval* $[0, a]$ *is defined by*

$$U^{(q)}(a, x, \mathrm{d}y) := \mathbb{E}_x\left[\int_0^\infty e^{-qt}\mathbf{1}_{(X_t\in[0,a],\, t<\tau_0^-\wedge\tau_a^+)}\mathrm{d}t\right]$$

$$
\begin{aligned}
&= \int_0^\infty e^{-qt}\mathbb{P}_x(X_t \in \mathrm{d}y,\ t < \tau_0^- \wedge \tau_a^+)\mathrm{d}t \\
&= \int_{[0,a]} \Big(\frac{W^{(q)}(x)W^{(q)}(a-y)}{W^{(q)}(a)} - W^{(q)}(x-y)\Big)\mathrm{d}y, \qquad (3.1.8)
\end{aligned}
$$

while the resolvent of the process X killed on exiting $[0,\infty)$ is defined by

$$
\begin{aligned}
R^{(q)}(x,\mathrm{d}y) &:= \int_0^\infty e^{-qt}\mathbb{P}_x(X_t \in \mathrm{d}y,\ t < \tau_0^-)\mathrm{d}t \\
&= \int_0^\infty \Big(e^{-\Phi(q)y}W^{(q)}(x) - W^{(q)}(x-y)\Big)\mathrm{d}y. \qquad (3.1.9)
\end{aligned}
$$

Notice that Eq. (3.1.9) is the limiting case of Eq. (3.1.8) as $a \to \infty$.

3.2 Refracted Spectrally Negative Lévy Processes

Refracted spectrally negative Lévy process is described as a Lévy process whose dynamics alternate by subtracting a fixed linear drift whenever the spectrally negative Lévy process is above a positive pre-specified level. In Figure 3.2, we demonstrate a sample path of a particular case of spectrally negative Lévy process with bounded variation which is refracted case when it exceeds the level b.

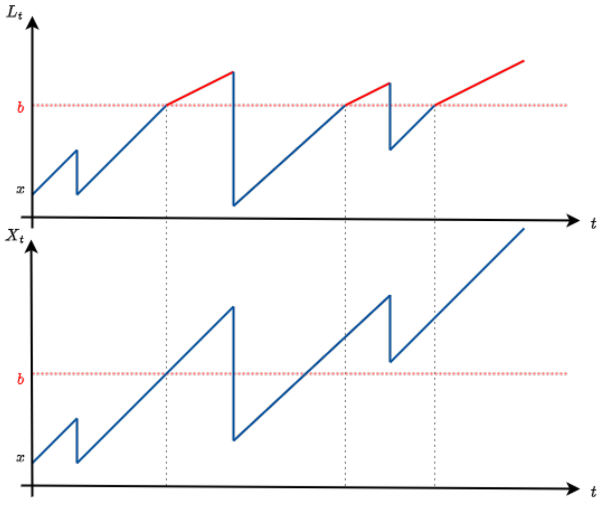


Figure 3.2: Sample path of a refracted process of bounded variation.

More formally, refracted Lévy process L_t satisfies the stochastic differential equation

(SDE) of the form

$$\mathrm{d}L_t = \mathrm{d}X_t - \delta \mathbf{1}_{\{L_t > b\}}\mathrm{d}t, \quad t \geq 0,$$

or equivalently,

$$L_t = X_t - \delta \int_0^t \mathbf{1}_{\{L_s > b\}}\mathrm{d}s, \quad t \geq 0, \tag{3.2.1}$$

where $\delta > 0$ is a linear drift rate. It is well-known that when X has paths of bounded variation if and only if $\sigma = 0$ and $\int_{(-1,0)} x\nu(\mathrm{d}x) < \infty$. In this case X is written as

$$X_t = ct - S_t, \quad t \geq 0, \tag{3.2.2}$$

where $c := \mu - \int_{(-1,0)} x\nu(\mathrm{d}x) > 0$ is a constant and $S = \{S_t : t \geq 0\}$ is a pure jump compound Poisson subordinator.

As alluded before and seen in the Figure 3.2, in the setting of the Lévy insurance risk model instead of constant linear drift (or premia in the insurance context) is modified depending upon the current process level. In other words, it can be thought as an aggregate insurance risk process when dividends are paid out at a rate δ whenever it exceeds the level b.

Futher, it is important to mention that above the level b, refracted process evolves as a *drift-changed process* which is defined as

$$X_1(t) := X(t) - \delta t, \quad t \geq 0,$$

then, the associated scale functions $W_1^{(q)}$ satisfies the following Laplace transform

$$\int_0^\infty e^{-\theta x} W_1^{(q)}(x)\mathrm{d}x = \frac{1}{\psi_{X_1}(\theta) - q}, \quad \theta > \varphi_1(q), \tag{3.2.3}$$

where $\psi_{X_1}(\theta) := \psi(\theta) - \delta\theta$ for $\theta \geq 0$ is the Laplace exponent for $X_1(t)$ and

$$\varphi_1(q) := \sup\{\theta \geq 0 : \psi_{X_1}(\theta) = q\},$$

Then, the adjoint scale function $Z_1^{(q)}$ is defined as

$$Z_1^{(q)}(x) = 1 + q \int_0^x W_1^{(q)}(y)\mathrm{d}y. \tag{3.2.4}$$

Whilst studying the process in the form of Eq. (3.2.1), the first issue one confronts is whether a solution to SDE exits or not. For the mathematical convenience, it is imposed that when X has paths of bounded variation, $\delta < c$.

Theorem 35 (Existence and Uniqueness [25])**.** *For a fixed $X_0 = x \in \mathbb{R}$, there exists unique strong solution to Eq. (3.2.1).*

Remark 36. *The way to prove the above theorem is initially to show that in the bounded variation case, Eq. (3.2.1) has a unique pathwise solution, (see Chp.10.4 in [24] or [25]). Then, using the fact that any spectrally negative Lévy process with unbounded variation paths, X, there exists a sequence of bounded variation spectrally negative Lévy processes, $X^{(n)}$ such that for each $t > 0$,*

$$\lim_{n\uparrow\infty} \sup_{s\in[0,t]} |X_s^{(n)} - X_s| = 0, \quad \textit{a.s.},$$

see (p.210, in [8] and Def. 11 of [25]), it can be established, by means of convergence, that a strong solution exists in the unbounded variation case, i.e.

$$\lim_{n\uparrow\infty} \sup_{s\in[0,t]} |L_s^{(n)} - L_s^{(\infty)}| = 0, \quad \textit{a.s.},$$

where $L^{(n)}$ is the sequence of pathwise solutions associated with bounded variation case. Note that the associated process $L^{(\infty)}$ holds that $\mathbb{P}_x(L_t^{(\infty)} = a) = 0$ for Lebesgue almost every $t \geq 0$.

Another observation from the construction of the unique pathwise solution is that L adapted to natural filtration $(\mathcal{F}_t)_{t\geq 0}$ of X. This indicates that L is a strong Markov process. For detailed proof, please see [24, 25].

3.3 Reflected-Refracted Spectrally Negative Lévy Processes

In this subsection, considering a combination of refracted and reflected process, we present several fluctuation identities studied in [34]. Mental picture of the reflected-refracted process is as follows: given a spectrally negative Lévy process and two bounds, the path of the process is reflected at the lower boundary, whereas a linear drift at a constant rate δ is subtracted from the increments of the process whenever the path of the process is above the upper boundary $b > 0$, see Figure 3.3.

The reflected-refracted Lévy process is given in [31] as a decomposition of reflected and refracted as follows

$$V_t := X_t + R_t - \delta \int_0^t \mathbf{1}_{\{V_s > b\}} \mathrm{d}s, \tag{3.3.1}$$

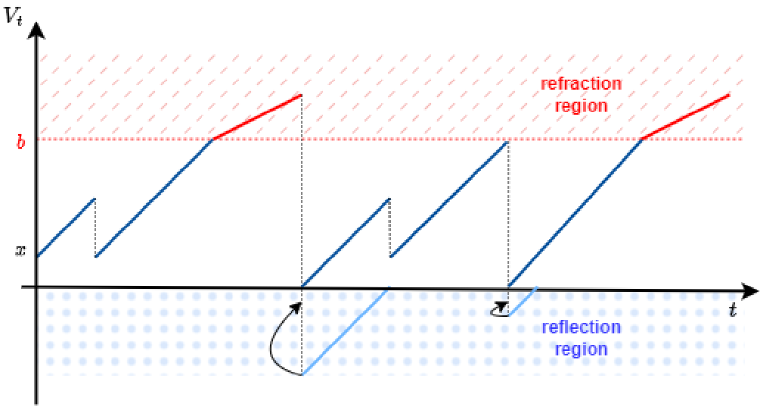


Figure 3.3: Sample path of a reflected-refracted Lévy process of bounded variation.

where $R_t := (\sup_{0\leq s\leq t}(-V_s) \vee 0) = (-\inf_{0\leq s\leq t}(V_s)) \vee 0$, which pushes the process upward whenever it attempts to go below the level 0.

Before the exit identities of such processes, it is necessary to show that reflected-refracted Lévy process V given in Eq. (3.3.1) associated with the X of unbounded variation can be approximated by those of bounded variation. In the following proposition, we present the strong approximation which also assures the existence of the solution of Eq. (3.3.1).

Proposition 37 (Strong Approximation of V_t [31])**.** *Assume that X with paths of unbounded variation and $(X^{(n)})_{n\geq 1}$ (of bounded variation) is a strongly approximating sequence for X. Moreover, define V and $V^{(n)}$ as the reflected-refracted process associated with X and $X^{(n)}$, respectively. Then, $V^{(n)}$ is a strongly approximating sequence of V.*

Now, since we are interested in the fluctuation theory of V, let us define the first passage times for reflected-refracted SNLP for $a > 0$ as follows

$$T_a^+ = \inf\{t > 0 : V_t > a\} \quad \text{and} \quad T_a^- = \inf\{t > 0 : V_t < a\}.$$

Then, the resolvent measure killed on exiting $[0, a]$ for a reflected-refracted SNLP process is given by the following theorem.

Theorem 38 (Resolvent of reflected-refracted SNLP [34])**.** *For $q \geq 0, x \leq a$ and a*

Borel set $\mathfrak{B}$ *on* $[0, a]$*,*

$$\mathbb{E}_x\Big[\int_0^{T_a^+} e^{-qt} 1_{\{V_t \in \mathfrak{B}\}} \mathrm{d}t\Big] = \int_{\mathfrak{B}} \Big(w^{(q)}(a, z) \frac{z^{(q)}(x)}{z^{(q)}(a)} - w^{(q)}(x, z) \Big) \mathrm{d}z,$$

where, for all $0 \leq z \leq a$*,*

$$\begin{aligned} w^{(q)}(x, z) =& \mathbf{1}_{\{0<z<b\}} \big(W^{(q)}(x - z) + \delta \int_b^x W_1^{(q)}(x - y) W^{(q)\prime}(y - z) \mathrm{d}y \big) \\ & + \mathbf{1}_{\{b<z<x\}} W_1^{(q)}(x - z), \end{aligned} \tag{3.3.2}$$

and

$$z^{(q)}(x) = Z^{(q)}(x) + q\delta \int_b^x {W_1}^{(q)}(x - y) W^{(q)}(y) \mathrm{d}y, \quad x \in \mathbb{R}, \quad q \geq 0, \tag{3.3.3}$$

where $W_1^{(q)}(x)$ *is the scale function of the drift changed process defined in Eq. (3.2.3).*

Using the above resolvent identity, one can derive the one-sided exit problem for reflected-refracted process as given below.

Theorem 39 (One-Sided Exit [34])**.** *For any* $q \geq 0$ *and* $x \leq a$*, we have*

$$\mathbb{E}_x\big[e^{-qT_a^+} \mathbf{1}_{(T_a^+ < \infty)}\big] = \frac{z^{(q)}(x)}{z^{(q)}(a)},$$

where $z^{(q)}(x)$ *is given in Eq. (3.3.3)*

3.4 Multi-Refracted Spectrally Negative Lévy Processes

An extension of a refracted process is given in Eq. (3.2.1) is introduced by Czarna et.al. (2019) [13] as a multi-refracted spectrally negative Lévy process by revising $\phi(x) = \delta \mathbf{1}_{(x>b)}$ in Eq. (3.2.1) with the refraction of multiple levels with the following some non-decreasing function

$$\phi_k(x) = \sum_{j=1}^{k} \delta_j \mathbf{1}_{(x>b_j)}. \tag{3.4.1}$$

For $k \geq 1$, we set $0 < \delta_1, \ldots, \delta_k$, and $-\infty < b_1 < \cdots < b_k < \infty$. Then, considering the function given in Eq. (3.4.1), the corresponding SDE is defined as

$$\mathrm{d}U_k(t) = \mathrm{d}X(t) - \sum_{i=1}^{k} \delta_i \mathbf{1}_{(U_k(t)>b_i)} \mathrm{d}t, \tag{3.4.2}$$

and the behavior of a multi-refracted SNLP is given in Figure 3.4.

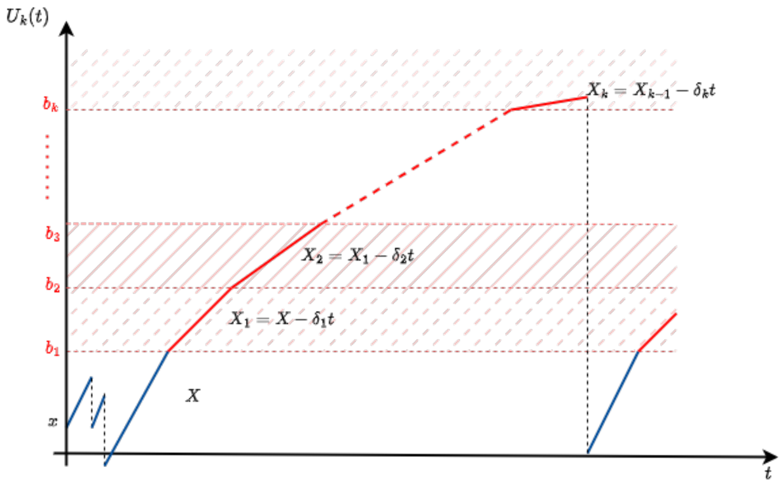


Figure 3.4: Sample path of a multi-refracted Lévy process of bounded variation.

Theorem 40 (Existence and Uniqueness [13])**.** *There exist a unique strong solution $U_k(t)$ to the SDE given Eq. (3.4.2) for $k \geq 1$, $0 < \delta_1, \ldots, \delta_k$, and $-\infty < b_1 < \cdots < b_k < \infty$.*

We remark that under the above assumptions of U_k, the process U_k is a strong Markov process. When X has paths of bounded variation, it is assumed that

$$0 < \delta_1 + \cdots + \delta_k < c := \mu - \int_{(-1,0)} x\nu(\mathrm{d}x). \tag{3.4.3}$$

As mentioned previously, the special case $k = 1$ is already studied in [25] as refracted Lévy process. The crucial observation for the SDE given in Eq. (3.4.2) is that for any $0 \leq j \leq k$ in each refraction level interval $(b_j, b_{j+1}]$, where $b_0 := -\infty$ and $b_{k+1} := \infty$, the process $U_k(t)$ evolves as $X_j := \{t \geq 0 : X(t) - \sum_{i=1}^{j} \delta_i t\}$ which is the *drift changed spectrally negative Lévy process* that is not the negative subordinator, since (3.4.3) is assumed. Then, the Laplace exponent of X_j on $[0, \infty)$ is

$$\theta \mapsto \psi_j(\theta) := \psi(\theta) - (\delta_1 + \cdots + \delta_j)\theta,$$

with right-inverse

$$\varphi_j(q) = \sup\{\theta \geq 0 : \ \psi_j(\theta) = q\}.$$

Furthermore, for all $0 \leq j \leq k$, X_j has the same Lévy measure and diffusion coefficient with X in Section 3.1. One can notice the recursive relationship between the process X_j and X_{j+1} easily, that is $X_{j+1} = \{t \geq 0 : X_j(t) - \delta_{j+1} t\}$, see Figure 3.4. These observations allow us to introduce the scale functions of drift-changed process $X_k(t)$, i.e. $W_k^{(q)}$ and $Z_k^{(q)}$ for any $k \geq 0$.

For $q \geq 0$, the q-scale function $W_k^{(q)}$ of the process satisfies the following Laplace transform

$$\int_0^{\infty} e^{-\theta x} W_k^{(q)}(x) \mathrm{d}x = \frac{1}{\psi_k(\theta) - q}, \quad \text{for} \quad \theta > \varphi_k(q). \tag{3.4.4}$$

We also define

$$Z_k^{(q)}(x) = 1 + q \int_0^x W_k^{(q)}(y) \mathrm{d}y, \quad x \in \mathbb{R}. \tag{3.4.5}$$

Notice that when $k = 0$, $W_k^{(q)}$ and $Z_k^{(q)}$ turns out to be the scale functions for spectrally Lévy process X which is defined Eq. (3.1.3) and Eq. (3.1.4), respectively. Also, note that when $k = 1$, U_k reduces to the refracted process given in Section 3.2, and thus the scale functions for drift changed process are the same with Eq. (3.2.3) and Eq. (3.2.4), respectively.

For $a \in \mathbb{R}$ and $k \geq 1$, the following first-passage stopping times are defined

$$\begin{aligned} \tau_k^{a,-} &:= \inf\{t > 0 : X_k(t) < a\} \quad \text{and} \quad \tau_k^{a,+} := \inf\{t > 0 : X_k(t) \geq a\}, \\ \kappa_k^{a,-} &:= \inf\{t > 0 : U_k(t) < a\} \quad \text{and} \quad \kappa_k^{a,+} := \inf\{t > 0 : U_k(t) \geq a\}. \end{aligned}$$

Theorem 41 (Resolvent measure for drift changed process [13]). *For* $b_k \leq x \leq a$ *and* $q \geq 0$, *the density of a resolvent measure of the process* X_k *(drift changed process) killed on exiting the interval* $[b_k, a]$ *is defined by*

$$\mathbb{E}_x\Big[\int_0^{\tau_k^{b_k,-} \wedge \tau_k^{a,+}} e^{-qt} \mathbf{1}_{(X_k(t) \in dy)} \mathrm{d}t\Big] = \Big(\frac{W_k^{(q)}(x - b_k) W_k^{(q)}(a - y)}{W_k^{(q)}(a - b_k)} - W_k^{(q)}(x - y)\Big) \mathrm{d}y.$$

Further, the potential measure and exit identities of U_k in Eq. (3.4.2) are based on the multi-refracted scale functions $\mathbb{W}_k^{(q)}$ and $\mathbb{Z}_k^{(q)}$ are defined below.

Definition 42 (Multi-refracted scale functions [13]). *For* $x \in \mathbb{R}$, *and* $0 < b_1 < \cdots < b_k \leq a$ *we define the two families of functions* $\{\mathbb{W}_k^{(q)}(x),\ x \in \mathbb{R}\}$ *and* $\{\mathbb{Z}_k^{(q)}(x),\ x \in \mathbb{R}\}$ *as the solution to the following integral equations,*

$$\begin{aligned} \mathbb{W}_k^{(q)}(x) &= \mathbb{W}_{k-1}^{(q)}(x) + \delta_k \int_{b_k}^x W_k^{(q)}(x - y) \mathbb{W}_{k-1}^{(q)\prime}(y) \mathrm{d}y, \\ \mathbb{Z}_k^{(q)}(x) &= \mathbb{Z}_{k-1}^{(q)}(x) + \delta_k \int_{b_k}^x W_k^{(q)}(x - y) \mathbb{Z}_{k-1}^{(q)\prime}(y) \mathrm{d}y, \end{aligned} \tag{3.4.6}$$

with initial functions $\mathbb{W}_0^{(q)}(x) = W^{(q)}(x)$ *and* $\mathbb{Z}_0^{(q)}(x) = Z^{(q)}(x)$*, respectively and* $W_k^{(q)}(\cdot)$ *is given in Eq. (3.4.4).*

Further, for $x, d \in \mathbb{R}$*, and* $d < b_1 < \cdots < b_k \leq a$*, we define more general scale function as follows*

$$\mathbb{W}_k^{(q)}(x; d) = \mathbb{W}_{k-1}^{(q)}(x; d) + \delta_k \int_{b_k}^{x} W_k^{(q)}(x-y) \mathbb{W}_{k-1}^{(q)\prime}(y; d) \mathrm{d}y, \tag{3.4.7}$$

with $\mathbb{W}_0^{(q)}(x; d) = W^{(q)}(x; d)$.

Proposition 43 (Proposition 17 in [13]). *Assume that* $\phi_k(x) = \sum_{i=1}^{k} \delta_i \mathbf{1}_{(x>b_i)}$ *and* $k, q \geq 0$*. Then for* $x, d \in \mathbb{R}$ *and* $d < b_1 < \cdots < b_k$*, the multi-refracted scale functions given in Eq. (3.4.7) and Eq. (3.4.6) are unique solutions to the following integral equations respectively*

$$\mathbb{W}_k^{(q)}(x; d) = W^{(q)}(x-d) + \int_{d}^{x} W^{(q)}(x-y) \phi_k(y) \mathbb{W}_k^{(q)\prime}(y; d) \mathrm{d}y, \tag{3.4.8}$$

$$\mathbb{Z}_k^{(q)}(x) = Z^{(q)}(x) + \int_{d}^{x} W^{(q)}(x-y) \phi_k(y) \mathbb{Z}_k^{(q)\prime}(y) \mathrm{d}y. \tag{3.4.9}$$

For more details about the above proposition, we refer to Proposition 17 of [13].

For the analytical properties of the multi-refracted scale functions and the fluctuation identities of multi-refracted process in terms of scale functions defined above, we refer to (Sec 2.3 in [13] and Theorem 5 and Theorem 7 in [13]), respectively. Now, we give some useful definition and lemma, which are essential to prove the resolvent measure of multi-refracted Lévy process reflected its infimum in Section 3.6.1.

Definition 44 (Definition 4 in [13]). *For any* $0 \leq k, -\infty =: b_0 < b_1 < \cdots < b_k < b_{k+1} := \infty$*, and* $y \in \mathbb{R}$*, define*

$$\Xi_{\phi_k}(y) := 1 - W^{(q)}(0)\phi_k(y).$$

For the unbounded variation case, by the fact that $W^{(q)}(0) = 0$*, we get* $\Xi_{\phi_k}(y) = 1$ *for all* $y \in \mathbb{R}$*. On the other hand, in the bounded variation case for* $y \in (b_i, b_{i+1}]$ *and* $i \leq k-1$*, we have that*

$$\Xi_{\phi_k}(y) = 1 - W^{(q)}(0) \sum_{j=1}^{i} \delta_j = \prod_{j=1}^{i} \left(1 - \delta_j W_{j-1}^{(q)}(0)\right),$$

and similarly for $y > b_k$*, we obtain* $\Xi_{\phi_k}(y) = \prod_{j=1}^{k} \left(1 - \delta_j W_{j-1}^{(q)}(0)\right)$.

Lemma 45 (Lemma 14 in [13]). *For any* $q, k \geq 0, x \in \mathbb{R}, b_0 = -\infty$, *and* $b_k = \max_{1 \leq j \leq k} b_j < y$

$$\mathbb{W}_k^{(q)}(x; y) = \Xi_{\phi_k}(y) W_k^{(q)}(x - y).$$

We conclude this section with the following remark. Note that the identities for $\mathbb{W}_k^{(q)}$ and $\mathbb{Z}_k^{(q)}$ given in this remark will be used in Section 3.6.1 to simplify some expressions and help to avoid the use of Lévy measure.

Remark 46 (Remark 34 in [13]). *For* $0 \leq x \leq a$ *and* $d < b_1 < \cdots < b_k < a$

$$\begin{aligned}\int_0^{a-b_k} \int_y^\infty & \mathbb{W}_{k-1}^{(q)}(b_k + y - \theta; d) \\ & \times \left[\frac{W_k^{(q)}(x - b_k) W_k^{(q)}(a - b_k - y)}{W_k^{(q)}(a - b_k)} - W_k^{(q)}(x - b_k - y) \right] \nu(\mathrm{d}\theta)\mathrm{d}y \\ &= -\frac{W_k^{(q)}(x - b_k)}{W_k^{(q)}(a - b_k)} \Big(\mathbb{W}_{k-1}^{(q)}(a; d) + \delta_k \int_{b_k}^a W_k^{(q)}(a - y) \mathbb{W}_{k-1}^{(q)\prime}(y; d)\mathrm{d}y \Big) \\ &\quad + \mathbb{W}_{k-1}^{(q)}(x; d) + \delta_k \int_{b_k}^x W_k^{(q)}(x - y) \mathbb{W}_{k-1}^{(q)\prime}(y; d)\mathrm{d}y. \end{aligned} \tag{3.4.10}$$

Similarly, we have

$$\begin{aligned}\int_0^{a-b_k} \int_y^\infty & \mathbb{Z}_{k-1}^{(q)}(b_k + y - \theta) \Big[\frac{W_k^{(q)}(x - b_k) W_k^{(q)}(a - b_k - y)}{W_k^{(q)}(a - b_k)} - W_k^{(q)}(x - b_k - y) \Big] \\ & \times \nu(\mathrm{d}\theta)\mathrm{d}y \\ &= -\frac{W_k^{(q)}(x - b_k)}{W_k^{(q)}(a - b_k)} \Big(\mathbb{Z}_{k-1}^{(q)}(a) + \delta_k \int_{b_k}^a W_k^{(q)}(a - y) \mathbb{Z}_{k-1}^{(q)\prime}(y)\mathrm{d}y \Big) \\ &\quad + \mathbb{Z}_{k-1}^{(q)}(x) + \delta_k \int_{b_k}^x W_k^{(q)}(x - y) \mathbb{Z}_{k-1}^{(q)\prime}(y)\mathrm{d}y. \end{aligned} \tag{3.4.11}$$

Proof. The proof of Eq. (3.4.10) can be found in [13]. We prove only Eq. (3.4.11). Using the definition of $\mathbb{Z}_{k-1}^{(q)}(x)$ given in [p. 5445 in [13]] that is

$$\mathbb{Z}_{k-1}^{(q)}(x) := 1 + q \sum_{i=0}^{k-1} \int_{b_i}^{b_{i+1}} \frac{\mathbb{W}_{k-1}^{(q)}(x; y)}{\Xi_{\phi_i}(y)} \mathrm{d}y, \quad \text{where} \quad b_0 = -\infty, \quad b_k = x. \tag{3.4.12}$$

Then, the LHS of Eq. (3.4.11) can be written as

$$\int_0^{a-b_k} \int_y^\infty \Big(1 + q \sum_{i=0}^{k-1} \int_{b_i}^{b_{i+1}} \frac{\mathbb{W}_{k-1}^{(q)}(b_k + y - \theta; z)}{\Xi_{\phi_i}(z)} \mathrm{d}z \Big)$$

$$
\begin{aligned}
&\times\Big[\frac{W_k^{(q)}\left(x-b_k\right)W_k^{(q)}\left(a-b_k-y\right)}{W_k^{(q)}\left(a-b_k\right)}-W_k^{(q)}\left(x-b_k-y\right)\Big]\nu(\mathrm{d}\theta)\mathrm{d}y\\
=&\int_0^\infty\int_y^\infty\Big[\frac{W_k^{(q)}\left(x-b_k\right)W_k^{(q)}\left(a-b_k-y\right)}{W_k^{(q)}\left(a-b_k\right)}-W_k^{(q)}\left(x-b_k-y\right)\Big]\nu(\mathrm{d}\theta)\mathrm{d}y\\
&+q\int_0^\infty\int_y^\infty\sum_{i=0}^{k-1}\Big(\int_{b_i}^{b_{i+1}}\frac{\mathbb{W}_{k-1}^{(q)}(b_k+y-\theta;z)}{\Xi_{\phi_i}(z)}\mathrm{d}z\Big)\\
&\times\Big[\frac{W_k^{(q)}(x-b_k)W_k^{(q)}(a-b_k-y)}{W_k^{(q)}\left(a-b_k\right)}-W_k^{(q)}\left(x-b_k-y\right)\Big]\nu(\mathrm{d}\theta)\mathrm{d}y\\
=&\int_0^\infty\int_y^\infty\Big[\frac{W_k^{(q)}\left(x-b_k\right)W_k^{(q)}\left(a-b_k-y\right)}{W_k^{(q)}\left(a-b_k\right)}-W_k^{(q)}\left(x-b_k-y\right)\Big]\nu(\mathrm{d}\theta)\mathrm{d}y\\
&+q\sum_{i=0}^{k-1}\int_{b_i}^{b_{i+1}}\frac{1}{\Xi_{\phi_i}(z)}\Big[\int_0^\infty\int_y^\infty\mathbb{W}_{k-1}^{(q)}(b_k+y-\theta;z)\\
&\times\Big[\frac{W_k^{(q)}\left(x-b_k\right)W_k^{(q)}\left(a-b_k-y\right)}{W_k^{(q)}\left(a-b_k\right)}-W_k^{(q)}\left(x-b_k-y\right)\Big]\nu(\mathrm{d}\theta)\mathrm{d}y\Big]\mathrm{d}z.
\end{aligned}
$$

By using Eq. (3.4.10), we have

$$
\begin{aligned}
&\int_0^{a-b_k}\int_y^\infty\Big(1+q\sum_{i=0}^{k-1}\int_{b_i}^{b_{i+1}}\frac{\mathbb{W}_{k-1}^{(q)}(b_k+y-\theta;z)}{\Xi_{\phi_i}(z)}\mathrm{d}z\Big)\\
&\qquad\times\Big[\frac{W_k^{(q)}\left(x-b_k\right)W_k^{(q)}\left(a-b_k-y\right)}{W_k^{(q)}\left(a-b_k\right)}-W_k^{(q)}\left(x-b_k-y\right)\Big]\nu(\mathrm{d}\theta)\mathrm{d}y\\
&\quad=\int_0^\infty\int_y^\infty\Big[\frac{W_k^{(q)}\left(x-b_k\right)W_k^{(q)}\left(a-b_k-y\right)}{W_k^{(q)}\left(a-b_k\right)}-W_k^{(q)}\left(x-b_k-y\right)\Big]\nu(\mathrm{d}\theta)\mathrm{d}y\\
&\qquad+q\sum_{i=0}^{k-1}\int_{b_i}^{b_{i+1}}\frac{1}{\Xi_{\phi_i}(z)}\Big(-\frac{W_k^{(q)}(x-b_k)}{W_k^{(q)}(a-b_k)}\mathbb{W}_k^{(q)}(a;z)+\mathbb{W}_k^{(q)}(x;z)\Big)\mathrm{d}z\\
&\quad=Z_k^{(q)}(x-b_k)-\frac{W_k^{(q)}(x-b_k)}{W_k^{(q)}(a-b_k)}Z_k^{(q)}(a-b_k)\\
&\qquad+q\sum_{i=0}^{k-1}\int_{b_i}^{b_{i+1}}\frac{1}{\Xi_{\phi_i}(z)}\Big(-\frac{W_k^{(q)}(x-b_k)}{W_k^{(q)}(a-b_k)}\mathbb{W}_k^{(q)}(a;z)+\mathbb{W}_k^{(q)}(x;z)\Big)\mathrm{d}z,
\end{aligned}
$$

where in the last line follows by using the identity

$$
\begin{aligned}
&\int_0^\infty\int_y^\infty\Big[\frac{W_k^{(q)}\left(x-b_k\right)W_k^{(q)}\left(a-b_k-y\right)}{W_k^{(q)}\left(a-b_k\right)}-W_k^{(q)}\left(x-b_k-y\right)\Big]\nu(\mathrm{d}\theta)\mathrm{d}y\\
&\qquad=\mathbb{E}_x\Big[e^{-q\tau_k^{b_k,-}}\mathbf{1}_{(\tau_k^{b_k,-}<\tau_k^{a,+})}\Big]=Z_k^{(q)}(x-b_k)-\frac{W_k^{(q)}(x-b_k)}{W_k^{(q)}(a-b_k)}Z_k^{(q)}(a-b_k).
\end{aligned}
$$

Further using definition of $Z_k^{(q)}$ given in Eq. (3.4.5), we obtain that

$$
\begin{aligned}
&\int_0^{a-b_k}\int_y^{\infty} \mathbb{Z}_{k-1}^{(q)}(b_k+y-\theta)\\
&\qquad \times \Big[\frac{W_k^{(q)}(x-b_k)\,W_k^{(q)}(a-b_k-y)}{W_k^{(q)}(a-b_k)} - W_k^{(q)}(x-b_k-y)\Big]\nu(\mathrm{d}\theta)\mathrm{d}y\\
&\quad = 1+q\int_0^{x-b_k} W_k^{(q)}(x-b_k-z)\mathrm{d}z\\
&\qquad - \frac{W_k^{(q)}(x-b_k)}{W_k^{(q)}(a-b_k)}\big(1+q\int_0^{a-b_k} W_k^{(q)}(a-b_k-z)\mathrm{d}z\big)\\
&\qquad + q\sum_{i=0}^{k-1}\int_{b_i}^{b_{i+1}} \frac{1}{\Xi_{\phi_i}(z)}\Big(-\frac{W_k^{(q)}(x-b_k)}{W_k^{(q)}(a-b_k)}\mathbb{W}_k^{(q)}(a;z)+\mathbb{W}_k^{(q)}(x;z)\Big)\mathrm{d}z\\
&\quad = 1+q\int_{b_k}^{x} W_k^{(q)}(x-b_k-z)\mathrm{d}z\\
&\qquad - \frac{W_k^{(q)}(x-b_k)}{W_k^{(q)}(a-b_k)}\big(1+q\int_{b_k}^{a} W_k^{(q)}(a-b_k-z)\mathrm{d}z\big)\\
&\qquad + q\sum_{i=0}^{k-1}\int_{b_i}^{b_{i+1}} \frac{1}{\Xi_{\phi_i}(z)}\Big(-\frac{W_k^{(q)}(x-b_k)}{W_k^{(q)}(a-b_k)}\mathbb{W}_k^{(q)}(a;z)+\mathbb{W}_k^{(q)}(x;z)\Big)\mathrm{d}z\\
&\quad = 1+q\sum_{i=0}^{k}\int_{b_i}^{b_{i+1}} \frac{1}{\Xi_{\phi_i}(z)}\mathbb{W}_k^{(q)}(x;z)\mathrm{d}z\\
&\qquad - \frac{W_k^{(q)}(x-b_k)}{W_k^{(q)}(a-b_k)}\Big(1+q\sum_{i=0}^{k}\int_{b_i}^{b_{i+1}} \frac{1}{\Xi_{\phi_i}(z)}\mathbb{W}_k^{(q)}(a;z)\mathrm{d}z\Big) \quad \text{(by Lemma 45)}\\
&\quad = \mathbb{Z}_k^{(q)}(x) - \frac{W_k^{(q)}(x-b_k)}{W_k^{(q)}(a-b_k)}\mathbb{Z}_k^{(q)}(a).
\end{aligned}
$$

Then, recalling the definition of $\mathbb{Z}_k^{(q)}(x)$ in the Eq. (3.4.6), the result follows. ☐

3.5 Level-Dependent Spectrally Negative Lévy Processes

A generalization of the refracted Lévy process or multi-refracted process is so-called as *level-dependent Lévy process* $U = \{U_t,\ t \geq 0\}$, which is the solution of the following SDE

$$\mathrm{d}U_t = \mathrm{d}X_t - \phi(U_t)\mathrm{d}t, \tag{3.5.1}$$

where X_t is a spectrally negative Lévy process. A special case is when $X_t := ct - S_t$ with $c > 0$ is a constant and $S = \{S_t, t \geq 0\}$ is a pure jump subordinator and in the bounded variation case, Eq. (3.5.1) can be written as

$$\mathrm{d}U_t = c\mathrm{d}t - \mathrm{d}S_t - \phi(U_t)\mathrm{d}t = -\mathrm{d}S_t + p(U_t)\mathrm{d}t,$$

with $p(U_t) = ct - \phi(U_t)$, which corresponds to a risk model with the premium rate function depending on the current level of the process, see Chapter VII in [2].

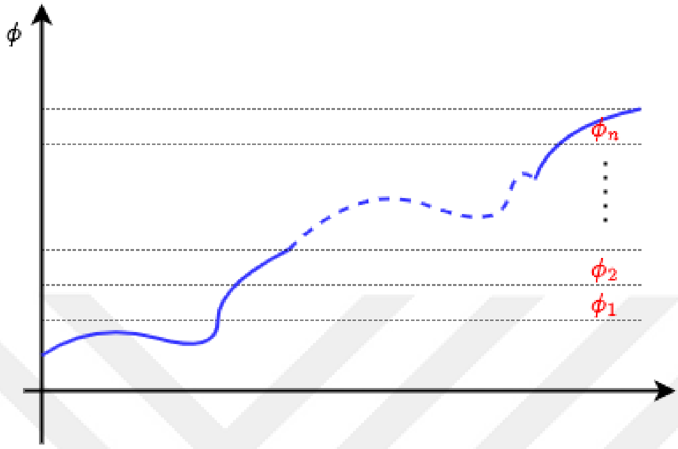


Figure 3.5: Level-dependent SNLP is a generalization of the multi-refracted SNLP.

Fluctuation theory of level-dependent Lévy processes is studied in [13] as a parallel theory of refracted processes in [25]. In fact, firstly the results of refracted process is extended by changing $\phi(x) = \delta\mathbf{1}_{(x>b)}$ where $\delta > 0$ to the refraction of multiple levels with the function given in Eq. (3.4.1) . Then, these results are generalized to the processes with general ϕ, which is non-decreasing and locally Lipschitz continuous. Fluctuation identities of that process, namely level-dependent spectrally negative Lévy process, are developed by approximating ϕ by an approximating sequence of multi-refracted rate functions ϕ_n, see Figure 3.5. In the limit, a level-dependent process U and the corresponding scale functions are obtained in the study of [13]. In this subsection, we give the fundamental aspects related to level-dependent spectrally negative Lévy process introduced in [13] rigorously.

The following assumption is imposed throughout the remainder of this study.

Assumption 47. *The non-decreasing function ϕ is either locally Lipschitz continuous or in the form of Eq. (3.4.1). Moreover, $\phi(x) = 0$ for $x \leq d'$, where $d' \in \mathbb{R}$ is fixed and for the bounded variation case, it is assumed that $\phi(x) < c$.*

Under Assumption 47, a unique strong solution to Eq. (3.5.1) exists and it is given in the following theorem.

Theorem 48 (Solution of SDE [13])**.** *Suppose that locally Lipschitz continuous rate function ϕ satisfies Assumption 47. Then, there exits a unique strong solution U to the SDE given in Eq. (3.5.1) with general rate function ϕ. Further, the sequence $(U_n)_{n\geq 1}$ defined by the approximating sequence of rate functions $(\phi_n)_{n\geq 1}$ converges uniformly to U with the corresponding rate function ϕ a.s. on the compact time intervals.*

Remark 49 (Breakdown of SDE solution [13])**.** *For the multi-refracted case i.e. when ϕ is in the form of Eq. (3.4.1), it is shown that multi-refracted SDE admits a unique solution in the strong sense (see Theorem 1 and Lemma 2 in [13]). Existence and uniqueness of a solution to Eq. (3.5.1) with a general rate function ϕ that satisfies Assumption 47 is proved by defining an approximating sequence of m_n-multi-refracted Lévy process and taking limit of this sequence. In order to this, the general rate function ϕ is approximated by a sequence $(\phi_n)_{n\geq 1}$ that has the following properties for $x \in \mathbb{R}$,*

- *(a)* $\lim_{n\to\infty} \phi_n = \phi$ *uniformly on the compact sets,*
- *(b)* $\phi_1(x) \leq \phi_2(x) \leq \cdots \leq \phi(x)$, *(non-decreasing)*
- *(c) For each* $n \geq 1$, *the rate function* $\phi_n(x) = \sum_{j=1}^{m_n} \delta_j^n \mathbf{1}_{(x>b_j^n)}$ *for some* $m_n \in \mathbb{N}$, $0 < b_1^n < \cdots < b_{m_n}^n$ *and* $\delta_j^n > 0$ *for each* $j = 1, \ldots, m_n$.

Details of the proof for both the multi-refracted case and the level-dependent case can be found in [13].

Remark 50. *We highlight that in [13], ϕ_n is not precisely the same rate function given in the Section 3.4 since according to the convention in the Section 3.4, n refers to the number of barriers. Here, the sequence of general rate function ϕ_n is an m_n-multi-refracted rate function. From now on, we choose to use ϕ_n notation for the clarity. The construction of specific sequence of $(\phi_n)_{n\geq 1}$, which satisfies the condition in the remark above, as follows. A grid is chosen for each $n \geq 1$ such that $\Pi^n = \{b_l^n = l2^{-n} : l = 1, \ldots, m_n = n2^n\}$ and set $\delta_j^n = \phi(b_j^n) - \phi(b_{j-1}^n)$, with $b_0^n = 0$.*

Then, the approximating sequence of the rate function ϕ is defined as

$$\phi_n(x) = \sum_{j=1}^{m_n} \delta_j^n \mathbf{1}_{\{x>b_j^n\}} \quad \textit{for } \ n \geq 1 \ \textit{ and } x \in \mathbb{R}.$$

For any $n \geq 1$, there exists a unique strong solution $(U_n)_{n\geq 1}$ with rate function $(\phi_n)_{n\geq 1}$ to Eq. (3.5.1) by means of uniform convergence on a compact sets of a sequence of m_n-multi-refracted Lévy processes to the solution U with corresponding rate function..

As indicated before, the fluctuation theory of the level-dependent Lévy processes is established in terms of level-dependent scale functions which are introduced in the following definition.

Definition 51 (Level-dependent scale functions [13])**.** *For $x \in \mathbb{R}$, we define the two families of functions $\{\mathbb{W}^{(q)}(x), \ x \in \mathbb{R}\}$ and $\{\mathbb{Z}^{(q)}(x), \ x \in \mathbb{R}\}$ as the solution to the following integral equations, provided that $\mathbb{W}^{(q)}$ and $\mathbb{Z}^{(q)}$ are almost everywhere differentiable*

$$\mathbb{W}^{(q)}(x) = W^{(q)}(x) + \int_0^x W^{(q)}(x-y)\phi(y)\mathbb{W}^{(q)\prime}(y)\mathrm{d}y, \tag{3.5.2}$$

$$\mathbb{Z}^{(q)}(x) = Z^{(q)}(x) + \int_0^x W^{(q)}(x-y)\phi(y)\mathbb{Z}^{(q)\prime}(y)\mathrm{d}y, \tag{3.5.3}$$

with the boundary conditions $\mathbb{W}^{(q)}(0) = W^{(q)}(0)$ and $\mathbb{Z}^{(q)}(0) = 1$ where $W^{(q)}(\cdot)$ and $Z^{(q)}(\cdot)$ are given in Eq. (3.1.3) and Eq. (3.1.4), respectively.

We note that the general derivative of $W^{(q)}$ is not defined for all $x \in \mathbb{R}$. Hence, $W^{(q)\prime}$ is understood as the right-derivative, which always exists (see Lemma 2.3 in [22]). Also, if the derivative of $\mathbb{W}^{(q)}$ does not exist, $\mathbb{W}^{(q)\prime}$ is also understood to be right-derivative. Then, for any $x \geq 0$, the derivative of level-dependent scale functions, $\mathbb{W}^{(q)\prime}$ and $\mathbb{Z}^{(q)\prime}$ are solutions to following the Voltera-type equations (for more details, see Eqs. (41) – (42) of [13])

$$\mathbb{W}^{(q)\prime}(x) = \Xi_\phi(x)^{-1} W^{(q)\prime}(x_+) + \Xi_\phi(x)^{-1} \int_0^x W^{(q)\prime}(x-y)\phi(y)\mathbb{W}^{(q)\prime}(y)\mathrm{d}y, \tag{3.5.4}$$

$$\mathbb{Z}^{(q)\prime}(x) = \Xi_\phi(x)^{-1} q W^{(q)}(x) + \Xi_\phi(x)^{-1} \int_0^x W^{(q)\prime}(x-y)\phi(y)\mathbb{Z}^{(q)\prime}(y)\mathrm{d}y, \tag{3.5.5}$$

with the initial conditions $\mathbb{W}^{(q)}(0) = W^{(q)}(0)$ and $\mathbb{Z}^{(q)}(0) = 1$, and for any $x \geq 0$, $\Xi_\phi(x)$ is strictly positive function by Assumption 47 and defined as

$$\Xi_\phi(x) := 1 - W^{(q)}(0)\phi(x), \tag{3.5.6}$$

where

$$W^{(q)}(0) = \begin{cases} 0 & \text{if } X \text{ has unbounded variation,} \\ 1/c & \text{if } X \text{ has bounded variation,} \end{cases}$$

with $c := \mu - \int_{x\in(-1,0)} x\nu(\mathrm{d}x) > 0$.

The above equations with the initial conditions define uniquely $\mathbb{W}^{(q)}$ and $\mathbb{Z}^{(q)}$, respectively. Further, the solution of the integral equations in Eqs. (3.5.2) – (3.5.3) can be obtained by considering the lemma below.

Lemma 52 (Lemma 22 in [13]). *Assume that $\mathbb{W}^{(q)}$ and $\mathbb{Z}^{(q)}$ are a.e. differentiable. Then, $\mathbb{W}^{(q)}$ and $\mathbb{Z}^{(q)}$ are the solutions to Eqs. (3.5.2) – (3.5.3) respectively, if and only if $\mathbb{W}^{(q)\prime}$ and $\mathbb{Z}^{(q)\prime}$ are solutions to the Eqs. (3.5.4) – (3.5.5) with the boundary conditions $\mathbb{W}^{(q)}(0) = W^{(q)}(0)$ and $\mathbb{Z}^{(q)}(0) = Z^{(q)}(0)$, respectively.*

Based on the lemma above, in [13], it is shown that that the integral equations in Eqs. (3.5.4) – (3.5.5) have a unique solution. In more details from [13], we note that Eqs. (3.5.4) – (3.5.5) are the second kind of Volterra equation of the form, for $x \geq d$

$$u(x;d) = g(x;d) + \int_d^x K(x,y)u(y;d)\mathrm{d}y, \tag{3.5.7}$$

where $K(x,y) = \Xi_\phi(x)^{-1}\phi(y)W^{(q)\prime}((x-y)+)$.

Notice that setting $g(x;d) = \Xi_\phi(x)^{-1}W^{(q)\prime}(x+)$ and $g(x;d) = \Xi_\phi(x)^{-1}qW^{(q)}(x)$ and $d = 0$, we recover immediately Eqs. (3.5.4) – (3.5.5), respectively. Note that although the forms of $K(\cdot,\cdot)$ and $g(\cdot,\cdot)$ given in the previous sentence do not depend on d, the notation with d has been adopted in Eq. (3.5.7) in order to be used universally and match the Volterra eqution both for $\mathbb{W}^{(q)}(x)$ and $\mathbb{W}^{(q)}(x;d)$, similarly for $\mathbb{Z}^{(q)}(x)$ and $\mathbb{Z}^{(q)}(x;d)$ that we will define next. Finally, using successive approximation method and obtaining the Neumann series, one can show that the solution of the second kind of Volterra equation in Eq. (3.5.7) exists and unique as it is given in the following lemma (for the full proof, see p. 5425 – 5428 in [13]).

Lemma 53 (Existence to the solution of level-dependent scale functions [13])**.** *There exists a unique differentiable* u *that solves the integral equation in Eq. (3.5.7) of the form of*

$$u(x;d) = g(x;d) + \int_d^x K^*(x,y)g(y;d)\mathrm{d}y,$$

where $K^*(x,y) = \sum_{n\geq 1} K^{(n)}(x,y)$ *(which is convergent if there exist a convergent majorant function) with* $K^{(1)} = K$ *and* $K^{(n+1)}(x,y) = \int_y^x K^{(n)}(x,s)K(s,y)\mathrm{d}s$*.*

Now, we define the general version of level-dependent scale functions Eqs. (3.5.2) – (3.5.3), given by, for $x > d$,

$$\mathbb{W}^{(q)}(x;d) = W^{(q)}(x-d) + \int_d^x W^{(q)}(x-y)\phi(y)\mathbb{W}^{(q)\prime}(y;d)\mathrm{d}y, \tag{3.5.8}$$

$$\mathbb{Z}^{(q)}(x;d) = Z^{(q)}(x-d) + \int_d^x W^{(q)}(x-y)\phi(y)\mathbb{Z}^{(q)\prime}(y;d)\mathrm{d}y, \tag{3.5.9}$$

with the boundary conditions $\mathbb{W}^{(q)}(d;d) = W^{(q)}(0)$ and $\mathbb{Z}^{(q)}(d;d) = 1$. Notice that $\mathbb{W}^{(q)}(x) = \mathbb{W}^{(q)}(x;0)$ and $\mathbb{Z}^{(q)}(x) = \mathbb{Z}^{(q)}(x;0)$. In the same line of logic with Eqs. (3.5.4) – (3.5.5), it has been shown in [13], that $\mathbb{W}^{(q)\prime}(x;d)$ and $\mathbb{Z}^{(q)\prime}(x;d)$ satisfy the second kind of Volterra equations

$$\mathbb{W}^{(q)\prime}(x;d) = \Xi_\phi(x)^{-1}\big[W^{(q)\prime}((x-d)_+) + \int_d^x W^{(q)\prime}(x-y)\phi(y)\mathbb{W}^{(q)\prime}(y;d)\mathrm{d}y\big],$$

$$\mathbb{Z}^{(q)\prime}(x;d) = \Xi_\phi(x)^{-1}\big[qW^{(q)}(x-d) + \int_d^x W^{(q)\prime}(x-y)\phi(y)\mathbb{Z}^{(q)\prime}(y;d)\mathrm{d}y\big],$$

respectively.

Since the above equations reduce to Eqs. (3.5.8) – (3.5.9) (after setting non-negative function $g(x;d) = \Xi_\phi(x)^{-1}W^{(q)\prime}((x-d)+)$ and $g(x;d) = \Xi_\phi(x)^{-1}qW^{(q)}(x-d)$, respectively) using Lemma 53, we guarantee that Eqs. (3.5.8) – (3.5.9) have a unique solution.

Finally following remark offers an observation in regards to the scale functions in the case where approximated sequence is used. Also, this remark will be used in Section 3.6.2.

Remark 54. *Recall that* $(U_n)_{n\geq 1}$ *is defined by an approximating sequence of* $(\phi_n)_{n\geq 1}$ *for* ϕ *(see [13]). Then, for each* $n \geq 1$*, the scale functions* $\mathbb{W}_n^{(q)}$ *and* $\mathbb{Z}_n^{(q)}$ *associated with the level-dependent Lévy process* $(U_n)_{n\geq 1}$ *satisfy Eqs. (3.4.8) – (3.4.9), respectively. Note that the scale functions* $\mathbb{W}_n^{(q)}$ *and* $\mathbb{Z}_n^{(q)}$ *converges to level-dependent* *scale functions* $\mathbb{W}^{(q)}$ *and* $\mathbb{Z}^{(q)}$ *given in Eqs. (3.5.2) – (3.5.3), because of the fact that* $\lim_{n\to\infty}\mathbb{W}_n^{(q)\prime}(x;y) = \mathbb{W}^{(q)\prime}(x;y)$ *and* $\lim_{n\to\infty}\mathbb{Z}_n^{(q)\prime}(x) = \mathbb{Z}^{(q)\prime}(x)$, *where* $\mathbb{W}^{(q)\prime}$ *and* $\mathbb{Z}^{(q)\prime}$ *are the unique solutions to Eqs. (3.5.4) – (3.5.5), respectively. For more details, see Lemma 22 and Theorem 27 in [13].*

We are now ready to provide the closed-form expressions for the fluctuation identities of level-dependent Lévy process with general rate function ϕ. Let us define the first passage stopping times for the level-dependent process, given by

$$\kappa_x^+ := \inf\{t > 0 : U_t \geq x\} \quad \text{and} \quad \kappa_x^- := \inf\{t > 0 : U_t < x\}. \tag{3.5.10}$$

Theorem 55 (Two-sided exit problem of level-dependent [13]). *For* $0 \leq x \leq a$ *and* $q \geq 0$, *we have*

$$\mathbb{E}_x\Big[e^{-q\kappa_a^+}\mathbf{1}_{(\kappa_a^+<\kappa_0^-)}\Big] = \frac{\mathbb{W}^{(q)}(x)}{\mathbb{W}^{(q)}(a)}, \tag{3.5.11}$$

$$\mathbb{E}_x\Big[e^{-q\kappa_0^-}\mathbf{1}_{(\kappa_0^-<\kappa_a^+)}\Big] = \mathbb{Z}^{(q)}(x) - \frac{\mathbb{W}^{(q)}(x)}{\mathbb{W}^{(q)}(a)}\mathbb{Z}^{(q)}(a), \tag{3.5.12}$$

where $\mathbb{W}^{(q)}(x)$ *and* $\mathbb{Z}^{(q)}(x)$ *are given in Eq. (3.5.2) and Eq. (3.5.3) respectively.*

Theorem 56 (One-sided exit problem of level-dependent [13]). *It is also worth remarking that by taking limit of Eq. (3.5.12) as* $a \to \infty$, *we obtain one-sided exit problem of level-dependent Lévy processes provided that* $C^{(q)} := \lim_{a\to\infty}\frac{\mathbb{Z}^{(q)}(a)}{\mathbb{W}^{(q)}(a)}$ *exists and finite, it is given by*

$$\mathbb{E}_x\Big[e^{-q\kappa_0^-}\mathbf{1}_{(\kappa_0^-<\infty)}\Big] = \mathbb{Z}^{(q)}(x) - C^{(q)}\mathbb{W}^{(q)}(x),$$

and also we have, for $x \leq a$

$$\mathbb{E}_x\Big[e^{-q\kappa_a^+}\mathbf{1}_{(\kappa_a^+<\infty)}\Big] = \frac{u^{(q)}(x)}{u^{(q)}(a)},$$

where $u^{(q)}(x)$ *is given by*

$$u^{(q)}(x) = e^{\Phi(q)x} + \int_{d'}^{x} W^{(q)}(x-y)\phi(y)u^{(q)\prime}(y)\mathrm{d}y, \quad x \in \mathbb{R}, \tag{3.5.13}$$

with $u^{(q)\prime}$ *is the unique solution to*

$$u^{(q)\prime}(x) = \Xi_\phi(x)^{-1}\Phi(q)e^{\Phi(q)x} + \Xi_\phi(x)^{-1}\int_{d'}^{x} W^{(q)\prime}(x-y)\phi(y)u^{(q)\prime}(y)\mathrm{d}y, \quad x \in \mathbb{R}.$$

Hence, by defining $u^{(q)}(x) = 1 + \int_{d'}^{x} u^{(q)\prime}(y)\mathrm{d}y$ *and proceeding like in Lemma 52, the unique solution to* $u^{(q)}$ *given in Eq. (3.5.13) is obtained. For more details, see Lemma 28 in [13].*

In the following theorem, we present semi-explicit expressions for the resolvent measure of the level-dependent Lévy process U_t.

Theorem 57 (Resolvents for level-dependent [13])**.** *Fix a Borel set $\mathfrak{B} \subseteq \mathbb{R}$. Then,*

(i) For $q \geq 0$ and $d \leq x \leq a$, the resolvent of level-dependent Lévy process U_t killed on exiting $[d, a]$ is given by

$$\mathbb{E}_x\Big[\int_0^{\kappa_a^+ \wedge \kappa_d^-} e^{-qt}\mathbf{1}_{(U(t)\in\mathfrak{B})}dt\Big] = \int_{\mathfrak{B}\cap(d,a)} \Xi_\phi(y)^{-1}\Big(\frac{\mathbb{W}^{(q)}(x;d)}{\mathbb{W}^{(q)}(a;d)}\mathbb{W}^{(q)}(a;y) - \mathbb{W}^{(q)}(x;y)\Big)\mathrm{d}y, \quad (3.5.14)$$

where $\mathbb{W}^{(q)}(x;y)$ is solution to Eq. (3.5.8) with $\mathbb{W}^{(q)}(y;y) = W^{(q)}(0)$.

(ii) The resolvent for level-dependent Lévy process killed on exiting an interval $[d, \infty)$ given for $q > 0$ and $x \geq 0$ with assuming that there exists $c^{(q)}(y; d) > 0$ such that $\mathbb{W}^{(q)}(x; d)c^{(q)}(y; d) - \mathbb{W}^{(q)}(x; y) \geq 0$,

$$\mathbb{E}_x\Big[\int_0^{\kappa_d^-} e^{-qt}\mathbf{1}_{(U(t)\in\mathfrak{B})}\mathrm{d}t\Big] = \int_{\mathfrak{B}\cap(0,\infty)} \Xi_\phi(y)^{-1}\Big(c^{(q)}(y;d)\mathbb{W}^{(q)}(x;d) - \mathbb{W}^{(q)}(x;y)\Big)\mathrm{d}y, \quad (3.5.15)$$

where $c^{(q)}(y; d) := \lim_{a\to\infty} \mathbb{W}^{(q)}(a; y)/\mathbb{W}^{(q)}(a; d)$.

(iii) The resolvent of level-dependent Lévy process killed on exiting $(-\infty, a]$ is given for $q \geq 0$ and $x \leq a$,

$$\mathbb{E}_x\Big[\int_0^{\kappa_a^+} e^{-qt}\mathbf{1}_{(U(t)\in\mathfrak{B})}\mathrm{d}t\Big] = \int_{\mathfrak{B}\cap(-\infty,a)} \Xi_\phi(y)^{-1}\left(\frac{u^{(q)}(x)}{u^{(q)}(a)}\mathbb{W}^{(q)}(a;y) - \mathbb{W}^{(q)}(x;y)\right)\mathrm{d}y, \quad (3.5.16)$$

where $u^{(q)}(x) := \lim_{d\to\infty} \frac{\mathbb{W}^{(q)}(x;d)}{W^{(q)}(-d)}$ is solution to Eq. (3.5.13) for any $x \in \mathbb{R}$.

3.6 Potential Measures of Reflected Level-Dependent Spectrally Negative Lévy Processes

In this section, we study *reflected level-dependent spectrally negative Lévy processes*, which can be seen as a combination of the level-dependent and reflected Lévy processes. As seen in the previous section, the level-dependent Lévy process solves the stochastic differential equation (SDE)

$$\mathrm{d}U(t) = \mathrm{d}X(t) - \phi(U(t))\mathrm{d}t,$$

where X is a spectrally negative Lévy process and ϕ is a non-decreasing and locally Lipschitz continuous which is called as *general rate function*, with special case the multi-refracted Lévy processes where $\phi_k(x) = \sum_{j=1}^{k} \delta_j \mathbf{1}_{\{x \geq b_j\}}$. In this study, we extend the theory given in [31] first to reflected multi-refracted spectrally negative Lévy processes with some non-decreasing function ϕ_k and then to the case of a general non-decreasing continuously differentiable ϕ function. The dynamics of reflected multi-refracted spectrally negative Lévy processes can be understood as follows: given lower and upper boundaries $(b_j)_{1 \leq j \leq k}$, we reflect the path of the process at the lower boundary while whenever it is above upper boundaries multi-refraction occurs in which the process behaves like a drift changed process. Along the way, we develop some fluctuation formulas, such as potential measure and one-sided exit problem for reflected multi-refracted spectrally negative Lévy processes, after proving the existence and uniqueness of the constructed SDE. Then, we propose another process with non-decreasing and locally Lipschitz continuous general function ϕ, so-called as reflected level-dependent spectrally negative Lévy process by extending the methodology in [13]. We approximate a general rate function ϕ by a sequence of some multi-refracted rate functions $(\phi_n)_{n \geq 0}$, which defines a sequence of reflected multi-refracted Lévy processes so that we prove existence and uniqueness of the reflected level-level dependent SNLP by a limiting argument. After that, we derive the potential measures and one-sided exit problem for the reflected level-dependent spectrally negative Lévy processes in terms of scale functions defined in Section 3.5. It is known from [13] that the sequences of multi-refracted scale functions converge to the level-dependent scale functions, which implies that we can obtain the fluctuation identities of reflected level-dependent process as the limit of corresponding identities for the sequence of reflected multi-refracted Lévy process. The results we prove in this section do not exist in the current literature.

3.6.1 Reflected Multi-refracted Spectrally Negative Lévy Processes

Let us start with the definition of SDE of the multi-refracted spectrally negative Lévy process reflected at its infimum, given by

$$\mathrm{d}V_k(t) = \mathrm{d}X(t) + \mathrm{d}R_k(t) - \sum_{i=1}^{k} \delta_i \mathbf{1}_{(V_k(t)>b_i)}\mathrm{d}t, \tag{3.6.1}$$

or equivalently,

$$V_k(t) = X(t) + R_k(t) - \sum_{i=1}^{k} \delta_i \int_0^t \mathbf{1}_{(V_k(s)>b_i)}\mathrm{d}s,$$

where

$$R_k(t) = \sup_{0\leq s\leq t} (-V_k(s)) \vee 0 = (-\inf_{0\leq s\leq t}(V_k(s))) \vee 0,$$

modifies the process by pushing upwards when it attempts to go below zero while the summation in the above equation i.e. $\sum_{i=1}^{k} \delta_i \int_0^t \mathbf{1}_{(V_k(s)>b_i)}$ adds multi-refraction property and is a right-continuous process. Also, note that we use equivalently the notation $V_0(t) = X_0(t) = X(t)$.

As we mentioned before, similar to [13] and [31], we first obtain the fluctuation identities for the case that X is of bounded variation and then we extend these results for the case that X is of unbounded variation via strong approximation. Hence, we show that for $V_k(t)$ given in Eq. (3.6.1), a strong approximation exists between the bounded and the unbounded variation case. This also guarantees the existence of the solution for Eq. (3.6.1). To do this, we first need the following lemma.

Lemma 58. *For $k \geq 1$ and fixed $t > 0$, let $(x_s)_{0\leq s\leq t}$ and $(\tilde{x}_s)_{0\leq s\leq t}$ be the paths of two different Lévy processes such that*

$$\sup_{0\leq s\leq t} |x_s - \tilde{x}_s| < \varepsilon, \quad \textit{for some } \varepsilon > 0.$$

Also, fix $z_k, \tilde{z}_k \in \mathbb{R}$ and $0 \leq t_0 < t$.

(i) *Define the reflected paths $y_s(z_k, t_0)$ and $\tilde{y}_s(\tilde{z}_k, t_0)$ on $[t_0, t]$ of the shifted paths*

$$z_k + (x_s - x_{t_0}) \quad \textit{and} \quad \tilde{z}_k + (\tilde{x}_s - \tilde{x}_{t_0}),$$

respectively. In other words, for all $t_0 \leq s \leq t$, let

$$y_s(z_k, t_0) := z_k + (x_s - x_{t_0}) + \big(-\inf_{t_0\leq u\leq s} [z_k + (x_u - x_{t_0})]\big) \vee 0,$$

$$\tilde{y}_s(\tilde{z}_k, t_0) := \tilde{z}_k + (\tilde{x}_s - \tilde{x}_{t_0}) + \big(-\inf_{t_0 \le u \le s} [\tilde{z}_k + (\tilde{x}_u - \tilde{x}_{t_0})]\big) \vee 0.$$

Then, we have

$$\sup_{t_0 \le s \le t} |y_s(z_k, t_0) - \tilde{y}_s(\tilde{z}_k, t_0)| < 2|z_k - \tilde{z}_k| + 4\varepsilon. \tag{3.6.2}$$

(ii) *Similarly, we define the multi-refracted paths* $a_s(z_{k-1}, t_0)$ *and* $\tilde{a}_s(\tilde{z}_{k-1}, t_0)$ *for all* $t_0 \le s \le t$ *that solves*

$$a_s(z_{k-1}, t_0) := z_{k-1} + (x_s - x_{t_0}) - \sum_{i=1}^{k} \delta_i \int_{t_0}^{s} \mathbf{1}_{\{a_u(z_{k-1}, t_0) > b_i\}} \mathrm{d}u,$$
$$\tilde{a}_s(\tilde{z}_{k-1}, t_0) := \tilde{z}_{k-1} + (\tilde{x}_s - \tilde{x}_{t_0}) - \sum_{i=1}^{k} \delta_i \int_{t_0}^{s} \mathbf{1}_{\{\tilde{a}_u(\tilde{z}_{k-1}, t_0) > b_i\}} \mathrm{d}u.$$

Hence,

$$\sup_{t_0 \le s \le t} |a_s(z_{k-1}, t_0) - \tilde{a}_s(\tilde{z}_{k-1}, t_0)| < 2|z_{k-1} - \tilde{z}_{k-1}| + 4\varepsilon. \tag{3.6.3}$$

Proof. We prove this lemma in a similar way of [31].

(i) Take $z = z_k$ and $\tilde{z} = \tilde{z}_k$ for $k \ge 1$ in Lemma A.1 of [31]. Then, the result follows.

(ii) Take $z = z_{k-1}$ and $\tilde{z} = \tilde{z}_{k-1}$ for $k \ge 1$ in Lemma A.1 of [31]. Also, note that from [13] since

$$\sup_{0 \le s \le t} |\Delta^{(n,m)} U_k(s) - \Delta^{(n,m)} X(s)| \le \eta,$$

where $\Delta^{(n,m)} U := U_k^{(n)}(s) - U_k^{(m)}(s)$ and $|\Delta^{(n,m)} X| < \eta$, it holds that

$$\sup_{0 \le s \le t} |U_k^{(n)}(s) - U_k^{(m)}(s)| < 2\eta,$$

where $U_k^{(n)}(t)$ is a sequence of the multi-refracted pathwise solution to

$$\mathrm{d}U_k^{(n)}(t) = \mathrm{d}X^{(n)}(t) - \sum_{i=1}^{k} \delta_i \int_0^t \mathbf{1}_{\{U_k^{(n)}(s) > b_i\}} \mathrm{d}s.$$

Then, using exactly the same steps in [31], the result follows.

$\square$

Now, in order to use the strong approximation, recall that for any spectrally negative Lévy process of unbounded variation paths, there exists a strongly approximating sequence with bounded variation, as it is shown in the definition below. The labeling of the result below as definition is made in order to keep the same labeling with the literature.

Definition 59 (Definition 11 in [25] and p. 210 in [8])**.** *For any spectrally negative Lévy process with unbounded variation paths, X, there exists a sequence of bounded variation $X^{(n)}$ such that*

$$\lim_{n\to\infty} \sup_{0\le s\le t} |X^{(n)}(s) - X(s)| = 0 \ \textit{a.s.} \quad \textit{for each} \ t > 0.$$

Furthermore, when $X^{(n)}$ is given in the form of Eq. (3.2.2) the drift coefficient tends to infinity as $n \to \infty$. This fact implies later that for all n sufficiently large $X^{(n)}$ satisfies the condition of $0 < \delta_1 + \cdots + \delta_k < c := \mu - \int_{(-1,0)} x\nu(\mathrm{d}x)$ instinctively. Such a sequence is called as strongly approximating sequence for X.

In the following proposition, we give the similar representation for reflected multi-refracted Lévy processes.

Proposition 60. *Assume that X is of unbounded variation and $\{X^{(n)}\}_{n\ge1}$ is a strongly approximating sequence of bounded variation. In addition, we define V_k and $V_k^{(n)}$ as the reflected multi-refracted processes associated with X and $X^{(n)}$, respectively. Then, $V_k^{(n)}$ is the strongly approximating sequence of V_k.*

Proof. For $k \ge 1$, $0 < \delta_1, \delta_2, \dots, \delta_k$ and $-\infty < b_1 < b_2 < \cdots < b_k < \infty$, we shall prove that there exists a strong approximating sequence for reflected multi-refracted Lévy process by extending the arguments of Proposition 2.1 in [31].

With $\beta_k := b_k/2 > 0$ for $k \ge 1$, we define a sequence of increasing random times $(\underline{\tau}_{1,k}, \overline{\tau}_1, \underline{\tau}_{2,k}, \overline{\tau}_2, \dots, \underline{\tau}_{\nu,k}, \overline{\tau}_\nu)$ as follows.

$$\begin{aligned}
&\underline{\tau}_{1,k} := \inf\{s > 0 : V_{k-1}(s) > \beta_k\}, \quad k = 1, 2, \dots,\\
&\sigma_{1,k} := \inf\{s > \underline{\tau}_{1,k} : V_k(s) = 0\}\\
&\qquad \text{with } \ \sigma_{1,0} := \inf\{s > \underline{\tau}_{1,1} : V_k(s) = 0\}, \quad k = 0, 1, 2, \dots,\\
&\widetilde{\sigma}_1 := \inf\{k \ge 0 : \sigma_{1,k}\} = \inf\{\sigma_{1,0}, \sigma_{1,1}, \dots, \sigma_{1,k}\},
\end{aligned}$$

$$\overline{\tau}_{1,k} := \sup\{s < \widetilde{\sigma}_1 : V_{k-1}(s) > \beta_k\}, \quad k = 1, 2, \dots,$$

$$\overline{\tau}_1 := \sup\{k \geq 0 : \overline{\tau}_{1,k}\} = \sup\{\overline{\tau}_{1,1}, \overline{\tau}_{1,2}, \dots, \overline{\tau}_{1,k}\},$$

and for all $\nu \geq 2$,

$$\underline{\tau}_{\nu,k} := \inf\{s >: \widetilde{\sigma}_{\nu-1} : V_{k-1}(s) > \beta_k\}, \quad k = 1, 2, \dots,$$

$$\sigma_{\nu,k} := \inf\{s > \underline{\tau}_{\nu,k} : V_k(s) = 0\}$$

$$\text{with } \sigma_{\nu,0} := \inf\{s > \underline{\tau}_{\nu,1} : V_k(s) = 0\}, \quad k = 0, 1, 2, \dots,$$

$$\widetilde{\sigma}_\nu := \inf\{k \geq 0 : \sigma_{\nu,k}\} = \inf\{\sigma_{\nu,0}, \sigma_{\nu,1}, \dots, \sigma_{\nu,k}\},$$

$$\overline{\tau}_{\nu,k} := \sup\{s < \widetilde{\sigma}_\nu : V_{k-1}(s) > \beta_k\}, \quad k = 1, 2, \dots,$$

$$\overline{\tau}_\nu := \sup\{k \geq 0 : \overline{\tau}_{\nu,k}\} = \sup\{\overline{\tau}_{\nu,1}, \overline{\tau}_{\nu,2}, \dots, \overline{\tau}_{\nu,k}\},$$

for convenience we also let $\overline{\tau}_0 := 0$.

Based on the above stopping times, we can break down the paths of the process as the sequence of multi-refracted paths and reflected paths, see Figure 3.6.

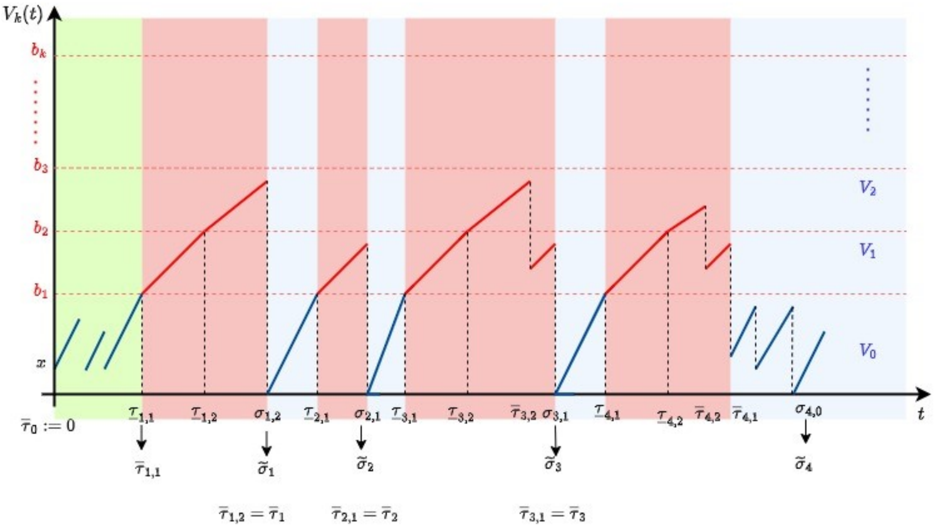


Figure 3.6: An example of the sequences of multi-refracted and reflected paths for the bounded variation case.

Let N_1 and N_2 denote the number of times switching to multi-refracted paths and reflected paths until $t > 0$, respectively and are defined as follows

$$N_1 := \sup\{\nu \geq 0 : \underline{\tau}_{\nu,1} < t\} \text{ and } N_2 := \sup\{\nu \geq 0 : \overline{\tau}_\nu < t\}.$$

Thus,

$$N = 1 + N_1 + N_2,$$

is the total number of times switching has occurred until time $t > 0$ plus one to account for the period until the first refraction.

Further, we define

$$\underline{\beta}_k := \min_{1 \leq \nu \leq N_1} \inf_{s \in [\underline{\tau}_{\nu,1}, \overline{\tau}_\nu \wedge t)} V_k(s), \text{ and } \underline{\beta} = \min\{k \geq 0 : \underline{\beta}_k\},$$

where $\underline{\beta} > 0$.

Based on Definition 59, in order to prove the strong approximation for $V_k(t)$, it suffices to show that there exists a finite C and $\underline{n} \in \mathbb{N}$ such that

$$\sup_{0 \leq s \leq t} |V_k^{(n)}(s) - V_k(s)| \leq C \sup_{0 \leq s \leq t} |X^{(n)}(s) - X(s)|, \quad n \geq \underline{n}, \tag{3.6.4}$$

where in this proof we choose $\underline{n}$ large enough so that

$$\sup_{m \geq \underline{n}} \sup_{0 \leq s \leq t} |X^{(m)}(s) - X(s)| < [4(2^N - 1)]^{-1} \underline{\beta}, \tag{3.6.5}$$

or equivalently,

$$[4(2^N - 1)] \sup_{m \geq \underline{n}} \sup_{0 \leq s \leq t} |X^{(m)}(s) - X(s)| < \underline{\beta}, \tag{3.6.6}$$

which we will see in later that, for all $n \geq \underline{n}$, this bound confirms that $\underline{\tau}_{\nu,1}$ and $\overline{\tau}_\nu$ can act as switching times for both $V_k(t)$ and $V_k^{(n)}(t)$ i.e. on each interval $[\underline{\tau}_{\nu,1}, \overline{\tau}_\nu]$ and $[\overline{\tau}_\nu, \underline{\tau}_{\nu+1,1}]$ are multi-refracted paths and reflected paths for both $V_k(t)$ and $V_k^{(n)}(t)$, respectively.

Let us fix $n > \underline{n}$ and $\varepsilon := \sup_{0 \leq s \leq t} |X^{(n)}(s) - X(s)|$.

Now, defining a sequence (this is not a stochastic process) $(\eta_\nu)_{0 \leq \nu \leq N}$ such that $\eta_0 = 0$ (the same sequence defined in the proof of Proposition 2.1 in [31]) and noticing that $\eta_{\nu+1} = 2\eta_\nu + 4\varepsilon$ is a sequence that gives

$$\eta_\nu = 4(2^N - 1)\varepsilon,$$

and by Eq. (3.6.5), we have

$$4(2^N - 1)\varepsilon < \underline{\beta} < b_1/2.$$

Then, we shall show that the followings for reflected and multi-refracted paths of the process

$$\sup_{\overline{\tau}_\nu \le s \le \underline{\tau}_{\nu+1,1} \wedge t} |V_k(s) - V_k^{(n)}(s)| \le \eta_{2\nu+1}, \text{ a.s. } 0 \le \nu \le N_2, \tag{3.6.7}$$

$$\sup_{\underline{\tau}_{\nu,1} \le s \le \overline{\tau}_\nu \wedge t} |V_k(s) - V_k^{(n)}(s)| \le \eta_{2\nu}, \text{ a.s. } 0 \le \nu \le N_1, \tag{3.6.8}$$

and hence Eq. (3.6.4) holds true with $C = 4(2^N - 1)$. Thus, the results above lead us to prove the following claims.

Claim 1 (Reflected Paths): Fix $\nu \ge 0$. Assume that $\overline{\tau}_\nu < t$ (or $\nu \le N_2$) and $\tilde{\eta} := 2\eta + 4\varepsilon < \underline{\beta} < b_1/2$. Also, suppose that

$$|R_k(\overline{\tau}_{\nu}-) - R_k^{(n)}(\overline{\tau}_{\nu}-)| = |V_k(\overline{\tau}_{\nu}-) + \Delta X(\overline{\tau}_\nu) - (V_k^{(n)}(\overline{\tau}_{\nu}-) + \Delta X^{(n)}(\overline{\tau}_\nu))| \le \eta.$$

Then, we have

$$|V_k(s) - V_k^{(n)}(s)| \le \tilde{\eta}, \quad \text{for} \quad \overline{\tau}_\nu \le s \le \underline{\tau}_{\nu+1,1} \wedge t.$$

Proof. Consider the reflected paths on $[\overline{\tau}_\nu, \underline{\tau}_{\nu+1,1} \wedge t]$. For every sequence ν such that a reflection from any $k \ge 1$, we have that

$$\begin{aligned} Y_k(s) := {} & V_k(\overline{\tau}_{\nu}-) + \Delta X(\overline{\tau}_\nu) + (X(s) - X(\overline{\tau}_\nu)) \\ & + \left[-\inf_{\overline{\tau}_\nu \le u \le s} \left(V_k(\overline{\tau}_{\nu}-) + \Delta X(\overline{\tau}_\nu) + (X(s) - X(\overline{\tau}_\nu))\right)\right] \vee 0, \end{aligned}$$

and

$$\begin{aligned} Y_k^{(n)}(s) := {} & V_k^{(n)}(\overline{\tau}_{\nu}-) + \Delta X^{(n)}(\overline{\tau}_\nu) + (X^{(n)}(s) - X^{(n)}(\overline{\tau}_\nu)) \\ & + \left[-\inf_{\overline{\tau}_\nu \le u \le s} \left(V_k^{(n)}(\overline{\tau}_{\nu}-) + \Delta X^{(n)}(\overline{\tau}_\nu) + (X^{(n)}(s) - X^{(n)}(\overline{\tau}_\nu))\right)\right] \vee 0. \end{aligned}$$

Applying Lemma 58(i) for $z_k = V_k(\overline{\tau}_{\nu}-) + \Delta X(\overline{\tau}_\nu)$ and $\tilde{z}_k = V_k^{(n)}(\overline{\tau}_{\nu}-) + \Delta X^{(n)}(\overline{\tau}_\nu)$ and $t_0 = \overline{\tau}_\nu$, we obtain that

$$\begin{aligned} |Y_k(s) - Y_k^{(n)}(s)| & < 2\left|V_k(\overline{\tau}_{\nu}-) + \Delta X(\overline{\tau}_\nu) - (V_k^{(n)}(\overline{\tau}_{\nu}-) + \Delta X^{(n)}(\overline{\tau}_\nu))\right| + 4\varepsilon \\ & \le 2\eta + 4\varepsilon = \tilde{\eta} < \underline{\beta}, \end{aligned}$$

for all $s \in [\overline{\tau}_\nu, \underline{\tau}_{\nu+1,1} \wedge t]$. Since there is no refraction (or multi-refraction) on this interval for $V_k(s)$, we can conclude that there is no refraction for $V_k^{(n)}(s)$ as well. Therefore, $V_k(s)$ and $V_k^{(n)}(s)$ coincide with defined $Y_k(s)$ and $Y_k^{(n)}(s)$ above. Thus, Eq. (3.6.7) holds true where $\tilde{\eta}$ has been replaced by $\eta_{2\nu+1}$ to take into account of all reflected path sequences. □

Claim 2 (Multi-refracted Paths): Fix $\nu \geq 0$ and $k \geq 1$. Assume that $\underline{\tau}_{\nu,1} < t$ (or $\nu \leq N_2$) and $\tilde{\eta} := 2\eta + 4\varepsilon < \underline{\beta} < b_1/2$. For the multi-refracted paths we have

$$|V_{k-1}(\underline{\tau}_{\nu,k}) - V_{k-1}^{(n)}(\underline{\tau}_{\nu,k})| \leq \eta.$$

Then,

$$|V_k(s) - V_k^{(n)}(s)| \leq \tilde{\eta}, \quad \text{for} \quad \underline{\tau}_{\nu,1} \leq s \leq \overline{\tau}_\nu \wedge t,$$

and

$$|V_k(\overline{\tau}_{\nu}-) + \Delta X(\overline{\tau}_\nu) - (V_k^{(n)}(\overline{\tau}_{\nu}-) + \Delta X^{(n)}(\overline{\tau}_\nu))| \leq \tilde{\eta}, \quad \text{if } \overline{\tau}_\nu < t.$$

Proof. Consider the multi-refracted paths A_k and $A_k^{(n)}$ on $[\underline{\tau}_{\nu,1}, \overline{\tau}_\nu \wedge t]$ that solve following equations

$$\begin{aligned} A_k(s) &:= V_0(\underline{\tau}_{\nu,1}) + (X(s) - X(\underline{\tau}_{\nu,1})) - \sum_{i=1}^{k} \delta_i \int_{\underline{\tau}_{\nu,1}}^{s} \mathbf{1}_{\{A_k(u) > b_i\}} \mathrm{d}u \\ &= V_{k-1}(\underline{\tau}_{\nu,k}) + (X(s) - X(\underline{\tau}_{\nu,k})) - \sum_{i=1}^{k} \delta_i \int_{\underline{\tau}_{\nu,k}}^{s} \mathbf{1}_{\{A_k(u) > b_i\}} \mathrm{d}u, \end{aligned}$$

and,

$$\begin{aligned} A_k^{(n)}(s) &:= V_0^{(n)}(\underline{\tau}_{\nu,1}) + (X^{(n)}(s) - X^{(n)}(\underline{\tau}_{\nu,1})) - \sum_{i=1}^{k} \delta_i \int_{\underline{\tau}_{\nu,1}}^{s} \mathbf{1}_{\{A_k^{(n)}(u) > b_i\}} \mathrm{d}u \\ &= V_{k-1}^{(n)}(\underline{\tau}_{\nu,k}) + (X^{(n)}(s) - X^{(n)}(\underline{\tau}_{\nu,k})) - \sum_{i=1}^{k} \delta_i \int_{\underline{\tau}_{\nu,k}}^{s} \mathbf{1}_{\{A_k^{(n)}(u) > b_i\}} \mathrm{d}u, \end{aligned}$$

where the last line of the both above equations are obtained by using the same arguments in the proof of Theorem 1 of [13]. In that case, multi-refracted paths are considered as refracted paths of each $k \geq 1$ for all $\underline{\tau}_{\nu,k} \leq s \leq \overline{\tau}_\nu \wedge t$.

Applying Lemma 58(ii) for $z_{k-1} = V_{k-1}(\underline{\tau}_{\nu,k})$ and $\tilde{z}_{k-1} = V_{k-1}^{(n)}(\underline{\tau}_{\nu,k})$ and $t_0 = \underline{\tau}_{\nu,k}$ for any $k \geq 1$ gives that

$$\begin{aligned} |A_k(s) - A_k^{(n)}(s)| &< 2|V_{k-1}(\underline{\tau}_{\nu,k}) - V_{k-1}^{(n)}(\underline{\tau}_{\nu,k})| + 4\varepsilon \\ &\leq 2\eta + 4\varepsilon = \tilde{\eta} < \underline{\beta}, \end{aligned}$$

for all $\underline{\tau}_{\nu,1} \leq s \leq \overline{\tau}_\nu \wedge t$. Since there is no reflection for both V_k and $V_k^{(n)}$ on that interval and we have $A_k(s) = V_k(s^-) + \Delta X(s)$ and $A_k^{(n)}(s) = V_k^{(n)}(s^-) + \Delta X^{(n)}(s)$ for all $\underline{\tau}_{\nu,1} \leq s \leq \overline{\tau}_\nu \wedge t$. Thus, we obtain the Eq. (3.6.8), i.e.

$$\sup_{\underline{\tau}_{\nu,1} \leq s \leq \overline{\tau}_\nu \wedge t} |V_k(s) - V_k^{(n)}(s)| \leq \eta_{2\nu},$$

where $\eta_{2\nu}$ is replaced by $\tilde{\eta}$. Moreover, the second statement of this proof is obvious since for any $k \geq 1$, the multi-refracted paths might jumps to reflection region if $\overline{\tau}_\nu < t$ which is proven in the Claim 1. $\square$

Now, we will show Eqs. (3.6.7) – (3.6.8) by using mathematical induction. We start with $\nu = 0$ by Claim 1, it is obvious that

$$|V_k(0^-) + \Delta X(0) - (V_k(0^-) + \Delta X(0))| = |x - x| = \eta_0 = 0,$$

since for convenience we take $\overline{\tau}_0 := 0$. Then, applying repeatedly Claim 1 and Claim 2 one after other for N times, Eqs. (3.6.7) – (3.6.8) hold true.

Finally, repeating the same reasoning given in Lemma 4.1 of [31] (using the fact that we will see later the resolvent in Theorem 61 for bounded variation case has a density), we verify that for all driving processes X of unbounded variation when x is fixed we have for $1 \leq j \leq k$ then

$$\mathbb{P}_x(V_k^{(\infty)}(t) = b_j) = 0 \text{ for almost every } t \geq 0, \tag{3.6.9}$$

which leads us to obtain that

$$\begin{aligned}\lim_{n\to\infty} V_k^{(n)}(t) &= \lim_{n\to\infty}\Big(X^{(n)}(t) + R_k^{(n)}(t) - \sum_{i=1}^{k}\delta_i\int_0^t \mathbf{1}_{(V_k^{(n)}(s)>b_i)}\mathrm{d}s\Big)\\ &= X(t) + R_k^{(\infty)} - \sum_{i=1}^{k}\delta_i\int_0^t \mathbf{1}_{(V_k^{(\infty)}(s)>b_i)}\mathrm{d}s,\end{aligned}$$

where $R_k^{(n)}(t) = (-\inf_{0\leq s\leq t}(V_k^{(n)}(s))) \vee 0$. This result gives us $V_k := V_k^{(\infty)}$ solves Eq. (3.6.1), which completes the proof. $\square$

After proving the strong approximation for the SDE given in Eq. (3.6.1), we present potential measure and one-sided exit problem of such processes. We define the first passage times for the reflected multi-refracted process for $a > 0$ and $k \geq 1$ as follows.

$$T_k^{a,+} = \inf\{t > 0 : V_k(t) > a\}, \quad \text{and} \quad T_k^{a,-} = \inf\{t > 0 : V_k(t) < a\}. \tag{3.6.10}$$

Theorem 61 (Resolvent for reflected multi-refracted SNLP)**.** *For $q \geq 0$, $y \leq x \leq a$, and $y < b_1 < b_2 < \cdots < b_k \leq a$, and a Borel set $\mathfrak{B}$ on $[0, a]$, we have*

$$\mathbb{E}_x\Big[\int_0^{T_k^{a,+}} e^{-qt}\mathbf{1}_{(V_k(t)\in\mathfrak{B})}\mathrm{d}t\Big] = \int_{\mathfrak{B}} \Xi_{\phi_k}(y)^{-1}\Big(\mathbb{W}_k^{(q)}(a;y)\frac{\mathbb{Z}_k^{(q)}(x)}{\mathbb{Z}_k^{(q)}(a)} - \mathbb{W}_k^{(q)}(x;y)\Big)\mathrm{d}y, \tag{3.6.11}$$

where $\mathbb{W}_k^{(q)}(x;y)$ *and* $\mathbb{Z}_k^{(q)}(x)$ *are given in Eq. (3.4.7) and Eq. (3.4.6), respectively*

Proof. We shall prove the result for $q > 0$ because the case $q = 0$ can be obtained by monotone convergence theorem and the continuity of the scale functions in q as given in [Lemma 8.3 in [24]]. We prove for the case when X has paths of bounded variation, then to deal with the case of X has unbounded variation, we consider, as shown in [13, 25], a strongly approximating sequence $X^{(n)}$. For the proof of unbounded variation case, we also need to consider for any $k \geq 1$ the convergence of multi-refracted scale functions associated with $V_k^{(n)}$ given below

$$\mathbb{W}_k^{(q),(n)}(x;d) = \mathbb{W}_{k-1}^{(q),(n)}(x;d) + \delta_k \int_{b_k}^{x} W_k^{(q),(n)}(x-y)\mathbb{W}_{k-1}^{(q),(n)\prime}(y;d)\mathrm{d}y, \quad (3.6.12)$$

$$\mathbb{Z}_k^{(q),(n)}(x) = \mathbb{Z}_{k-1}^{(q),(n)}(x) + \delta_k \int_{b_k}^{x} W_k^{(q)}(x-y)\mathbb{Z}_{k-1}^{(q),(n)\prime}(y)\mathrm{d}y. \quad (3.6.13)$$

(i) Bounded Variation: The proof is inductive. First, for $k = 1$, it is easy to check, the formula Eq. (3.6.11) agrees with [31] in Theorem 4.1 (after using Eq. (32) of Lemma 14 in [13]). Now, we assume that Eq. (3.6.11) holds for $k-1$. We follow the main idea of the proof of Theorem (4.1) given in [31] and then prove for each $k \geq 1$. Let us define

$$f^{(q)}(x,a,\mathfrak{B}) := \mathbb{E}_x\Big[\int_0^{T_k^{a,+}} e^{-qt}\mathbf{1}_{(V_k(t)\in\mathfrak{B})}\mathrm{d}t\Big].$$

Then, by using strong Markov property we have for $x < b_k$ that

$$\begin{aligned} f^{(q)}(x,a,\mathfrak{B}) &= \mathbb{E}_x\Big[\int_0^{T_{k-1}^{b_k,+}} e^{-qt}\mathbf{1}_{(V_{k-1}(t)\in\mathfrak{B})}\mathrm{d}t\Big] && (3.6.14)\\ &\quad + \mathbb{E}_x\Big[e^{-qT_{k-1}^{b_k,+}}\mathbf{1}_{(T_{k-1}^{b_k,+}<\infty)}\Big]f^{(q)}(b_k,a,\mathfrak{B}) \\ &= \int_0^{\infty}\sum_{i=0}^{k-1}\frac{\mathbb{W}_{k-1}^{(q)}(b_k;y)\frac{\mathbb{Z}_{k-1}^{(q)}(x)}{\mathbb{Z}_{k-1}^{(q)}(b_k)} - \mathbb{W}_{k-1}^{(q)}(x;y)}{\Xi_{\phi_i}(y)}\mathbf{1}_{(y\in(b_i,b_{i+1}])}\mathrm{d}y \\ &\quad + \frac{\mathbb{Z}_{k-1}^{(q)}(x)}{\mathbb{Z}_{k-1}^{(q)}(b_k)}f^{(q)}(b_k,a,\mathfrak{B}). && (3.6.15)\end{aligned}$$

Notice that in the above equation, the last identity is the one-sided exit problem for the reflected multi-refracted Lévy process which holds by inductive hypothesis. Since in [31] the one-sided exit problem for reflected-refracted spectrally negative Lévy processes given in Theorem 39 considers the case $k = 1$, equivalently may formulate it as

$$\mathbb{E}_x\big[e^{-qT_1^{a,+}}\mathbf{1}_{(T_1^{a,+}<\infty)}\big] = \frac{\mathbb{Z}_1^{(q)}(x)}{\mathbb{Z}_1^{(q)}(a)},$$

is proved by using the resolvent result given in Theorem 38. Since by inductive hypothesis the resolvent holds for $k-1$, it can be concluded that one-sided exit for reflected multi-refracted holds for $k-1$ as well.

On the other hand, for $b_k \le x \le a$, again by using strong Markov property, we have

$$\begin{aligned} f^{(q)}(x,a,\mathfrak{B}) = \mathbb{E}_x\Big[&\int_0^{\tau_k^{a,+}\wedge\tau_k^{b_k,-}} e^{-qt}\mathbf{1}_{(X_k(t)\in\mathfrak{B})}\mathrm{d}t\Big] \\ &+ \mathbb{E}_x\Big[e^{-q\tau_k^{b_k,-}} f^{(q)}(X_k(\tau_k^{b_k,-}),a,\mathfrak{B})\mathbf{1}_{(\tau_k^{b_k,-}<\tau_k^{a,-})}\Big]. \end{aligned}$$

By using Theorem 41, we get

$$\begin{aligned} f^{(q)}(x,a,\mathfrak{B}) = &\int_0^\infty \Big[\frac{W_k^{(q)}(x-b_k)W_k^{(q)}(a-y)}{W_k^{(q)}(a-b_k)} - W_k^{(q)}(x-y)\Big]\mathbf{1}_{(y\in(b_k,a])}\mathrm{d}y \\ &+ \int_0^\infty\int_z^\infty f^{(q)}(z-\theta+b_k,a,\mathfrak{B}) \\ &\quad\times\Big[\frac{W_k^{(q)}(x-b_k)W_k^{(q)}(a-b_k-z)}{W_k^{(q)}(a-b_k)} - W_k^{(q)}(x-b_k-z)\Big]\nu(\mathrm{d}\theta)\mathrm{d}z. \end{aligned}$$

Then, substituting Eq. (3.6.15) for the second expression in the above equation gives that

$$\begin{aligned} &f^{(q)}(x,a,\mathfrak{B}) \\ &= \int_0^\infty \Big[\frac{W_k^{(q)}(x-b_k)W_k^{(q)}(a-y)}{W_k^{(q)}(a-b_k)} - W_k^{(q)}(x-y)\Big]\mathbf{1}_{(y\in(b_k,a])}\mathrm{d}y \\ &+ \int_0^\infty\int_z^\infty\int_0^\infty\sum_{i=0}^{k-1}\frac{\mathbb{W}_{k-1}^{(q)}(b_k;y)\frac{\mathbb{Z}_{k-1}^{(q)}(z-\theta+b_k)}{\mathbb{Z}_{k-1}^{(q)}(b_k)} - \mathbb{W}_{k-1}^{(q)}(z-\theta+b_k;y)}{\Xi_{\phi_i}(y)}\mathbf{1}_{(y\in(b_i,b_{i+1}])}\mathrm{d}y \\ &\quad\times\Big[\frac{W_k^{(q)}(x-b_k)W_k^{(q)}(a-b_k-z)}{W_k^{(q)}(a-b_k)} - W_k^{(q)}(x-b_k-z)\Big]\nu(\mathrm{d}\theta)\mathrm{d}z \\ &+ \frac{f^{(q)}(b_k,a,\mathfrak{B})}{\mathbb{Z}_{k-1}^{(q)}(b_k)}\int_0^\infty\int_z^\infty \mathbb{Z}_{k-1}^{(q)}(z-\theta+b_k) \\ &\quad\times\Big[\frac{W_k^{(q)}(x-b_k)W_k^{(q)}(a-b_k-z)}{W_k^{(q)}(a-b_k)} - W_k^{(q)}(x-b_k-z)\Big]\nu(\mathrm{d}\theta)\mathrm{d}z \\ &= \int_0^\infty \Big[\frac{W_k^{(q)}(x-b_k)W_k^{(q)}(a-y)}{W_k^{(q)}(a-b_k)} - W_k^{(q)}(x-y)\Big]\mathbf{1}_{(y\in(b_k,a])}\mathrm{d}y \end{aligned}$$

$$
\begin{aligned}
&+\int_0^\infty\int_z^\infty\int_0^\infty \frac{\mathbb{W}_{k-1}^{(q)}(b_k;y)}{\mathbb{Z}_{k-1}^{(q)}(b_k)}\sum_{i=0}^{k-1}\frac{\mathbb{Z}_{k-1}^{(q)}(z-\theta+b_k)}{\Xi_{\phi_i}(y)}\mathbf{1}_{(y\in(b_i,b_{i+1}])}\mathrm{d}y\\
&\quad\times\Big[\frac{W_k^{(q)}(x-b_k)W_k^{(q)}(a-b_k-z)}{W_k^{(q)}(a-b_k)}-W_k^{(q)}(x-b_k-z)\Big]\nu(\mathrm{d}\theta)\mathrm{d}z\\
&-\int_0^\infty\int_z^\infty\int_0^\infty\sum_{i=0}^{k-1}\frac{1}{\Xi_{\phi_i}(y)}\mathbb{W}_{k-1}^{(q)}(z-\theta+b_k;y)\mathbf{1}_{(y\in(b_i,b_{i+1}])}\mathrm{d}y\\
&\quad\times\Big[\frac{W_k^{(q)}(x-b_k)W_k^{(q)}(a-b_k-z)}{W_k^{(q)}(a-b_k)}-W_k^{(q)}(x-b_k-z)\Big]\nu(\mathrm{d}\theta)\mathrm{d}z\\
&+\frac{f^{(q)}(b_k,a,\mathfrak{B})}{\mathbb{Z}_{k-1}^{(q)}(b_k)}\int_0^\infty\int_z^\infty\mathbb{Z}_{k-1}^{(q)}(z-\theta+b_k)\\
&\quad\times\Big[\frac{W_k^{(q)}(x-b_k)W_k^{(q)}(a-b_k-z)}{W_k^{(q)}(a-b_k)}-W_k^{(q)}(x-b_k-z)\Big]\nu(\mathrm{d}\theta)\mathrm{d}z.
\end{aligned}
$$

Next, we use Remark 46 in order to simplify the above equation, then we get that

$$
\begin{aligned}
&f^{(q)}(x,a,\mathfrak{B})\\
&=\int_0^\infty\Big[\frac{W_k^{(q)}(x-b_k)W_k^{(q)}(a-y)}{W_k^{(q)}(a-b_k)}-W_k^{(q)}(x-y)\Big]\mathbf{1}_{(y\in(b_k,a])}\mathrm{d}y\\
&+\int_0^\infty\sum_{i=0}^{k-1}\frac{1}{\Xi_{\phi_i}(y)}\Big(\frac{W_k^{(q)}(x-b_k)}{W_k^{(q)}(a-b_k)}\mathbb{W}_k^{(q)}(a;y)-\mathbb{W}_k^{(q)}(x;y)\Big)\mathbf{1}_{(y\in(b_i,b_{i+1}])}\mathrm{d}y\\
&+\int_0^\infty\sum_{i=0}^{k-1}\frac{1}{\Xi_{\phi_i}(y)}\Big[\frac{\mathbb{W}_{k-1}^{(q)}(b_k;y)}{\mathbb{Z}_{k-1}^{(q)}(b_k)}\Big(-\frac{W_k^{(q)}(x-b_k)}{W_k^{(q)}(a-b_k)}\mathbb{Z}_k^{(q)}(a)-\mathbb{Z}_k^{(q)}(x)\Big)\Big]\mathbf{1}_{(y\in(b_i,b_{i+1}])}\mathrm{d}y\\
&+\frac{f^{(q)}(b_k,a,\mathfrak{B})}{\mathbb{Z}_{k-1}^{(q)}(b_k)}\Big(-\frac{W_k^{(q)}(x-b_k)}{W_k^{(q)}(a-b_k)}\mathbb{Z}_k^{(q)}(a)-\mathbb{Z}_k^{(q)}(x)\Big).
\end{aligned}
$$

By setting $x=b_k$ and noticing that $\mathbb{W}_{k-1}^{(q)}(b_k)=\mathbb{W}_k^{(q)}(b_k)$ and $\mathbb{Z}_{k-1}^{(q)}(b_k)=\mathbb{Z}_k^{(q)}(b_k)$ by Eqs. (3.4.7) – (3.4.6), we obtain that

$$
\begin{aligned}
&f^{(q)}(b_k,a,\mathfrak{B})\\
&=\int_0^\infty\Big[W_k^{(q)}(a-y)\frac{\mathbb{Z}_k^{(q)}(b_k)}{\mathbb{Z}_k^{(q)}(a)}-\frac{W_k^{(q)}(b_k-y)W_k^{(q)}(a-b_k)}{W_k^{(q)}(0)}\frac{\mathbb{Z}_k^{(q)}(b_k)}{\mathbb{Z}_k^{(q)}(a)}\Big]\mathbf{1}_{(y\in(b_k,a])}\mathrm{d}y\\
&\quad+\int_0^\infty\sum_{i=0}^{k-1}\frac{1}{\Xi_{\phi_i}(y)}\Big[\frac{\mathbb{Z}_k^{(q)}(b_k)}{\mathbb{Z}_k^{(q)}(a)}\mathbb{W}_k^{(q)}(a;y)-\mathbb{W}_{k-1}^{(q)}(b_k;y)\Big]\mathbf{1}_{(y\in(b_i,b_{i+1}])}\mathrm{d}y.
\end{aligned}
$$

Finally, putting the expression for $f^{(q)}(b_k,a,\mathfrak{B})$ into Eq. (3.6.15), one gets

$$
f^{(q)}(x,a,\mathfrak{B})
$$

$$\begin{aligned}&= \int_0^\infty \Big[\frac{W_k^{(q)}(b_k - y)W_k^{(q)}(x-b_k)}{W_k^{(q)}(0)} - W_k^{(q)}(x-y)\\ &\quad + W_k^{(q)}(a-y)\frac{\mathbb{Z}_k^{(q)}(b_k)}{\mathbb{Z}_k^{(q)}(a)} - \frac{W_k^{(q)}(b_k-y)W_k^{(q)}(a-b_k)}{W_k^{(q)}(0)}\frac{\mathbb{Z}_k^{(q)}(b_k)}{\mathbb{Z}_k^{(q)}(a)}\Big]\mathbf{1}_{(y\in(b_k,a])}\mathrm{d}y\\ &\quad + \int_0^\infty \sum_{i=0}^{k-1}\frac{1}{\Xi_{\phi_i}(y)}\Big[\frac{\mathbb{Z}_k^{(q)}(x)}{\mathbb{Z}_k^{(q)}(a)}\mathbb{W}_k^{(q)}(a;y) - \mathbb{W}_k^{(q)}(x;y)\Big]\mathbf{1}_{(y\in(b_i,b_{i+1}])}\mathrm{d}y.\end{aligned}$$

Note that when $y \in (b_k, a]$, the scale function of drift changed process $W_k^{(q)}(b_k - y)$ vanishes since it is not defined on $\mathbb{R}^-$. Thus, the above equation becomes

$$\begin{aligned}f^{(q)}(x,a,\mathfrak{B}) &= \int_0^\infty \Big[W_k^{(q)}(a-y)\frac{\mathbb{Z}_k^{(q)}(b_k)}{\mathbb{Z}_k^{(q)}(a)} - W_k^{(q)}(x-y)\Big]\mathbf{1}_{(y\in(b_k,a])}\mathrm{d}y\\ &\quad + \int_0^\infty \sum_{i=0}^{k-1}\frac{1}{\Xi_{\phi_i}(y)}\Big[\frac{\mathbb{Z}_k^{(q)}(x)}{\mathbb{Z}_k^{(q)}(a)}\mathbb{W}_k^{(q)}(a;y) - \mathbb{W}_k^{(q)}(x;y)\Big]\mathbf{1}_{(y\in(b_i,b_{i+1}])}\mathrm{d}y.\end{aligned}$$

Since $\mathbb{W}_k^{(q)}(x;y) = \Xi_{\phi_k}(y)W_k^{(q)}(x-y)$ for $y \in (b_k, a]$ by Lemma 45, we get that

$$\begin{aligned}f^{(q)}(x,a,\mathfrak{B}) &= \int_0^\infty \sum_{i=0}^{k}\frac{1}{\Xi_{\phi_i}(y)}\Big[\frac{\mathbb{Z}_k^{(q)}(x)}{\mathbb{Z}_k^{(q)}(a)}\mathbb{W}_k^{(q)}(a;y) - \mathbb{W}_k^{(q)}(x;y)\Big]\\ &\qquad \times \mathbf{1}_{(y\in(b_i,b_{i+1}]\bigcap[0,a))}\mathrm{d}y\\ &= \int_0^\infty \frac{1}{\Xi_{\phi_k}(y)}\Big[\frac{\mathbb{Z}_k^{(q)}(x)}{\mathbb{Z}_k^{(q)}(a)}\mathbb{W}_k^{(q)}(a;y) - \mathbb{W}_k^{(q)}(x;y)\Big]\mathbf{1}_{(y\in[0,a))}\mathrm{d}y,\end{aligned}$$

which Eq. (3.6.11) follows by using Definition 44.

(ii) Unbounded Variation: We now extended the result to the case when X with paths of unbounded variation. Let $V_k^{(n)}(t)$ is be the reflected multi-refracted process associated with $X^{(n)}$ which is strongly approximating sequence (of bounded variation) for X and $T_k^{a,(n),+}$ is the first passage time above a of $V_k^{(n)}$. It is clear that $T_k^{a,(n),+}$ and $T_k^{a,+}$ are both finite a.s. because they are bounded from above by the upcrossing times at a for the reflected processes of drift changed process $X_k(t)$ for any $k \geq 0$ (This also seen by Theorem 62 below). In order to prove that

$$T_k^{a,(n),+} \xrightarrow{n\uparrow\infty} T_k^{a,+}, \quad \text{a.s.},$$

it is enough to show that $T_k^{a^-,+} = T_k^{a,+}$ a.s. as in p. 212 of [8] or in [31], which can be obtained immediately following the same arguments as in the proof of Theorem 4.1 in [31]. Therefore, by dominated convergence theorem and Proposition 60, we have

$$\lim_{n\to\infty}\mathbb{E}_x\Big[\int_0^{T_k^{a,(n),+}} e^{-qt}\mathbf{1}_{(V_k^{(n)}(t)\in\mathfrak{B})}\mathrm{d}t\Big] = \mathbb{E}_x\Big[\int_0^{T_k^{a,+}} e^{-qt}\mathbf{1}_{(V_k(t)\in\mathfrak{B})}\mathrm{d}t\Big], \quad a.s..$$

Moreover, as mentioned earlier, using exactly the same arguments of Lemma 4.1 in [31] it follows that $\mathbb{P}_x(V_k(t) \in \partial\mathfrak{B}) = \mathbb{P}_x(\sup_{0\leq s\leq t} V_k(s) = a) = 0$ for Lebesgue a.e. $t > 0$, which concludes the convergence of the LHS of Eq. (3.6.11).

It remains to show the convergence of the RHS of Eq. (3.6.11). This is proven in [13] [see Theorem 7(i)] for any $k \geq 1$, the sequence $(\mathbb{W}_k^{(q),(n)})_{n\geq 1}$ and $(\mathbb{Z}_k^{(q),(n)})_{n\geq 1}$ associated with $X^{(n)}$ (given in Eq. (3.6.12) and Eq. (3.6.13)) converges pointwise to scale functions $\mathbb{W}_k^{(q)}$ and $\mathbb{Z}_k^{(q)}$, respectively. Hence, Eq. (3.6.11) holds for unbounded variation case. ☐

Next, we derive an identity for the one-sided exit problem for reflected multi-refracted spectrally negative Lévy process, given in the next theorem.

Theorem 62. *For any $q \geq 0$ and $x \leq a$, we have*

$$\mathbb{E}_x\left[e^{-qT_k^{a,+}}\mathbf{1}_{(T_k^{a,+}<\infty)}\right] = \frac{\mathbb{Z}_k^{(q)}(x)}{\mathbb{Z}_k^{(q)}(a)},$$

where $\mathbb{Z}_k^{(q)}(x)$ is defined in Eq. (3.4.6).

In particular, $T_k^{a,+} < \infty$ $\mathbb{P}_x$-a.s..

Proof. By using the definition $\mathbb{Z}_k^{(q)}(x)$ is given in Eq. (3.4.12) in [13], we have

$$\mathbb{Z}_k^{(q)}(x) := 1 + q\sum_{i=0}^{k}\int_{b_i}^{b_{i+1}} \frac{\mathbb{W}_k^{(q)}(x;y)}{\Xi_{\phi_i}(y)}\mathrm{d}y, \qquad \text{where} \qquad b_0 = -\infty, \quad b_{k+1} = x, \tag{3.6.16}$$

or equivalently,

$$\begin{aligned}\frac{\mathbb{Z}_k^{(q)}(x) - 1}{q} &= \sum_{i=0}^{k}\int_{b_i}^{b_{i+1}} \frac{\mathbb{W}_k^{(q)}(x;y)}{\Xi_{\phi_i}(y)}\mathrm{d}y \\ &= \int_0^\infty \mathbb{W}_k^{(q)}(x;y)\sum_{i=0}^{k}\frac{1}{\Xi_{\phi_i}(y)}\mathbf{1}_{(y\in(b_i,b_{i+1}])}\mathrm{d}y \\ &= \int_0^x \Xi_{\phi_k}(y)^{-1}\mathbb{W}_k^{(q)}(x;y)\mathrm{d}y. \end{aligned}\tag{3.6.17}$$

Therefore,

$$\mathbb{E}_x\left[\int_0^{T_k^{a,+}} e^{-qt}\mathbf{1}_{(V_k(t)\in\mathfrak{B})}\mathrm{d}t\right]$$

$$
\begin{aligned}
&= \int_{\mathfrak{B}} \Xi_{\phi_k}(y)^{-1}\Big(\mathbb{W}_k^{(q)}(a;y)\frac{\mathbb{Z}_k^{(q)}(x)}{\mathbb{Z}_k^{(q)}(a)} - \mathbb{W}_k^{(q)}(x;y)\Big)\mathrm{d}y,\\
&= \frac{\mathbb{Z}_k^{(q)}(x)}{\mathbb{Z}_k^{(q)}(a)}\int_0^a \Xi_{\phi_k}(y)^{-1}\mathbb{W}_k^{(q)}(a;y)\mathrm{d}y - \int_0^x \Xi_{\phi_k}(y)^{-1}\mathbb{W}_k^{(q)}(x;y)\mathrm{d}y\\
&= \frac{\mathbb{Z}_k^{(q)}(x)}{\mathbb{Z}_k^{(q)}(a)}\frac{\mathbb{Z}_k^{(q)}(a)-1}{q} - \frac{\mathbb{Z}_k^{(q)}(x)-1}{q} = \frac{1}{q}\Big(1-\frac{\mathbb{Z}_k^{(q)}(x)}{\mathbb{Z}_k^{(q)}(a)}\Big).
\end{aligned}
$$

where the last equation holds by using Eq. (3.6.17).

Then, the results follows by noting that for an e_q, independent and exponentially distributed random variable with parameter $q > 0$, it holds that

$$
\begin{aligned}
\mathbb{E}_x\big[e^{-qT_k^{a,+}}\mathbf{1}_{(T_k^{a,+}<\infty)}\big] &= \mathbb{P}_x(\overline{V}_{e_q} > a) = 1 - \mathbb{P}_x(\overline{V}_{e_q} \leq a)\\
&= 1 - q\int_0^\infty e^{-qt}\mathbb{P}_x(V_t \in \mathfrak{B}, t < T_k^{a,+})\mathrm{d}t\\
&= 1 - q\mathbb{E}_x\Big[\int_0^{T_k^{a,+}} e^{-qt}\mathbf{1}_{(V_k(t)\in\mathfrak{B})}\mathrm{d}t\Big]\\
&= 1 - q\Big[\frac{1}{q}\Big(1-\frac{\mathbb{Z}_k^{(q)}(x)}{\mathbb{Z}_k^{(q)}(a)}\Big)\Big].
\end{aligned}
$$

Finally, for the finiteness of $T_k^{a,+}$ is shown by setting $q = 0$ and noticing that $\mathbb{Z}_k^{(0)}(x) = 1$. □

As we allueded earlier, our second aim in this section is to derive the potential measure for the reflected level-dependent Lévy process which does not exist in the current literature.

3.6.2 Reflected Level-dependent Spectrally Negative Lévy Processes

In this subsection, we extend the theory of reflected multi-refracted Lévy processes for reflected level-dependent Lévy processes with a general rate function ϕ given as follows

$$
V_t = X_t + R_t - \int_0^t \phi(V_s)\mathrm{d}s, \tag{3.6.18}
$$

where $R_t := \sup_{0\leq s\leq t}(-V_s) \vee 0 = (-\inf_{0\leq s\leq t}(V_s)) \vee 0$, which modifies the process by pushing upward whenever V_t attempts to go below 0. Also, the general rate function $\phi(\cdot)$ satisfies the assumption (same assumption with the level-dependent without reflection case, i.e. Assumption 47) given below.

Assumption 63. *For a fixed $d' \in \mathbb{R}$, non-decreasing function ϕ is either locally Lipschitz continuous or in the form of Eq. (3.4.1). Moreover, $\phi(x) = 0$ for $x \leq d'$, and for the bounded variation case, it is assumed that $\phi(x) < c$ for all $x \in \mathbb{R}$.*

For the existence of the solution of Eq. (3.6.18), we approximate a general function ϕ by a sequence of a rate function $(\phi_n)_{n\geq 1}$ of the form in Eq. (3.4.1). Thereby, we are able to define a sequence of reflected multi-refracted Lévy processes which provides existence as a limiting argument. In order to prove this approximating argument monotonicity property of the solution based on their driving rate function is essential.

We first consider a sequence of functions $(\phi_n)_{n\geq 1}$ that satisfies the following same conditions with level-dependent case given in Section 3.5.

(i) $\lim_{n\to\infty} \phi_n = \phi$ uniformly on the compact sets,

(ii) $\phi_1(x) \leq \phi_2(x) \leq \cdots \leq \phi(x)$ for all $x \in \mathbb{R}$, (non-decreasing sequence),

(iii) For each $n \geq 1$, the rate function $\phi_n(x) = \sum_{j=1}^{m_n} \delta_j^n \mathbf{1}_{(x>b_j^n)}$ for some $m_n \in \mathbb{N}$, $0 < b_1^n < \cdots < b_{m_n}^n$ and $\delta_j^n > 0$ for each $j = 1, \ldots, m_n$.

By recalling the Remark 50, for each $n \geq 1$ we denote the solution of Eq. (3.6.18) with the rate function ϕ_n by $V_n(t)$ as follows

$$V_n(t) = X(t) + \sup_{0\leq s\leq t} (-V_n(s)) \vee 0 - \int_0^t \phi_n(V_n(s))\mathrm{d}s.$$

Before showing existence of a solution to $V_n(t)$ given above for any $n \geq 1$, we prove that the sequence of $(V_n(t))_{n\geq 1}$ is non-increasing for any $t \geq 0$ with the following lemma.

Lemma 64. *Suppose that for each $n \geq 1$, $\phi_n(x) \leq \phi_{n+1}(x)$ for all $x \in \mathbb{R}$. Then, $V_{n+1}(t) \leq V_n(t)$ for all $t \geq 0$.*

Proof. We define the function $\phi_{n+1}^{\varepsilon}(x) := \phi_{n+1}(x) + \varepsilon$ for $\varepsilon > 0$. Then, we have $\phi_n(x) < \phi_{n+1}^{\varepsilon}(x)$ for all $x \in \mathbb{R}$. Consider the process $V_{n+1}^{\varepsilon}(t)$ which is a solution to the following SDE

$$\begin{aligned}V^{\varepsilon}_{n+1}(t) &= X(t) + R^{\varepsilon}_{n+1}(t) - \int_0^t \phi^{\varepsilon}_{n+1}(V^{\varepsilon}_{n+1}(s))\mathrm{d}s,\\ &= X(t) + \sup_{0\le s\le t}(-V^{\varepsilon}_{n+1}(s)) \vee 0 - \int_0^t \phi^{\varepsilon}_{n+1}(V^{\varepsilon}_{n+1}(s))\mathrm{d}s, \quad t \ge 0.\end{aligned}$$

Moreover, we define

$$\varsigma := \inf\{t > 0 : V_n(t) < V^{\varepsilon}_{n+1}(t)\},$$

and assume that $\varsigma < \infty$. Note that since V_n and V^{ε}_{n+1} have the same jumps, the crossing of the path cannot occur at a jump instant. Thus, we obtain that $V_n(t) - V^{\varepsilon}_{n+1}(t)$ is non-increasing in some $[\varsigma - \epsilon, \varsigma)$ for small enough ϵ. Further, in $[\varsigma - \epsilon, \varsigma)$ we have also that $\sup_{0\le s\le t}(-V_n(s)) \vee 0 - \sup_{0\le s\le t}(-V^{\varepsilon}_{n+1}(s)) \vee 0$ is increasing. Then, we have

$$\begin{aligned}0 \ge \frac{\mathrm{d}}{\mathrm{d}t}(V_n(t) - V^{\varepsilon}_{n+1}(t))\Big|_{t=\varsigma^-} &\\ = \frac{\mathrm{d}}{\mathrm{d}t}\big[X(t) + \sup_{0\le s\le t}(-V_n(s)) \vee 0 - \int_0^t \phi_n(V_n(s))\mathrm{d}s &\\ - \big(X(t) + \sup_{0\le s\le t}(-V^{\varepsilon}_{n+1}(s)) \vee 0 - \int_0^t \phi^{\varepsilon}_{n+1}(V^{\varepsilon}_{n+1}(s))\mathrm{d}s\big)\big] &\\ = \frac{\mathrm{d}}{\mathrm{d}t}\big(\sup_{0\le s\le t}(-V_n(s)) \vee 0 - \sup_{0\le s\le t}(-V^{\varepsilon}_{n+1}(s)) \vee 0\big) &\\ + \phi^{\varepsilon}_{n+1}(V^{\varepsilon}_{n+1}(\varsigma^-)) - \phi_n(V_n(\varsigma^-)) > 0,&\end{aligned}$$

that yields to $\varsigma = \infty$, which implies that $V^{\varepsilon}_{n+1}(t) \le V_n(t)$.

Further, since $\phi_{n+1}(x) < \phi^{\varepsilon}_{n+1}(x)$ for all $x \in \mathbb{R}$, following the same argument as above, we obtain that $V^{\varepsilon}_{n+1}(t) \le V_{n+1}(t)$ for all $t \ge 0$.

On the other hand, let

$$\begin{aligned}\Delta_{\varepsilon}(t) &:= V_{n+1}(t) - V^{\varepsilon}_{n+1}(t)\\ &= X(t) + \sup_{0\le s\le t}(-V_{n+1}(s)) \vee 0 - \int_0^t \phi_{n+1}(V_{n+1}(s))\mathrm{d}s\\ &\quad - \big[X(t) + \sup_{0\le s\le t}(-V^{\varepsilon}_{n+1}(s)) \vee 0 - \int_0^t \phi^{\varepsilon}_{n+1}(V^{\varepsilon}_{n+1}(s))\mathrm{d}s\big]\\ &= \sup_{0\le s\le t}(-V_{n+1}(s)) \vee 0 - \sup_{0\le s\le t}(-V^{\varepsilon}_{n+1}(s)) \vee 0\\ &\quad + \int_0^t \big[\phi^{\varepsilon}_{n+1}(V^{\varepsilon}_{n+1}(s)) - \phi_{n+1}(V_{n+1}(s))\big]\mathrm{d}s.\end{aligned}$$

By noting that $\sup_{0\le s\le t}(-V_{n+1}(s))\vee 0-\sup_{0\le s\le t}(-V^{\varepsilon}_{n+1}(s))\vee 0\le 0$ (as we have shown $V^{\varepsilon}_{n+1}(t)\le V_{n+1}(t)$ for all $t\ge 0$), then we get

$$\Delta_{\varepsilon}(t)\le\int_0^t\big(\phi^{\varepsilon}_{n+1}(V^{\varepsilon}_{n+1}(s))-\phi_{n+1}(V_{n+1}(s))\big)\mathrm{d}s. \tag{3.6.19}$$

Using the classical calculus (integration by parts) and also recalling that $\phi^{\varepsilon}_{n+1}(x):=\phi_{n+1}(x)+\varepsilon$ gives that

$$\begin{aligned}(\Delta_{\varepsilon}(t))^2&\le 2\int_0^t\Delta_{\varepsilon}(s)\big(\phi^{\varepsilon}_{n+1}(V^{\varepsilon}_{n+1}(s))-\phi_{n+1}\left(V_{n+1}(s)\right)\big)\mathrm{d}s\\&=2\int_0^t\Delta_{\varepsilon}(s)\big(\phi_{n+1}(V^{\varepsilon}_{n+1}(s))-\phi_{n+1}(V_{n+1}(s))\big)\mathrm{d}s+2\varepsilon\int_0^t\Delta_{\varepsilon}(s)\mathrm{d}s.\end{aligned}$$

Since $V^{\varepsilon}_{n+1}(t)\le V_{n+1}(t)$ for all $t\ge 0$ and ϕ_{n+1} is a non-decreasing function, and also by definition $\Delta_{\varepsilon}(s)>0$, we obtain that

$$\int_0^t\Delta_{\varepsilon}(s)\left(\phi_{n+1}\left(V^{\varepsilon}_{n+1}(s)\right)-\phi_{n+1}\left(V_{n+1}(s)\right)\right)ds\le 0.$$

After these observations, we get

$$(\Delta_{\varepsilon}(t))^2\le 2\varepsilon\int_0^t\Delta_{\varepsilon}(s)\mathrm{d}s\xrightarrow{\varepsilon\downarrow 0}0,$$

or equivalently,

$$0\le\lim_{\varepsilon\downarrow 0}(\Delta_{\varepsilon}(t))^2\le 2\lim_{\varepsilon\downarrow 0}\varepsilon\int_0^t\Delta_{\varepsilon}(s)\mathrm{d}s=0,$$

which implies that $\lim_{\varepsilon\downarrow 0}\Delta_{\varepsilon}(t)=\lim_{\varepsilon\downarrow 0}(V_{n+1}(t)-V^{\varepsilon}_{n+1}(t))\equiv 0$. Hence, we conclude that

$$\lim_{\varepsilon\downarrow 0}V_{n+1}(t)=\lim_{\varepsilon\downarrow 0}V^{\varepsilon}_{n+1}(t)\le\lim_{\varepsilon\downarrow 0}V_n(t),$$

which completes the proof. □

In the next proposition, we prove the existence of a solution to the SDE in Eq. (3.6.18) with a general rate function ϕ satisfying Assumption 63. We shall show existence by proving uniform convergence on compact sets of a sequence of reflected m_n-multi-refracted Lévy processes $(V_n)_{n\ge 1}$ defined by the approximating sequence of rate functions $(\phi_n)_{n\ge 1}$ to the solution V of Eq. (3.6.18) with corresponding rate function ϕ.

For the proof of existence of a solution, we need the following lemma.

Lemma 65 (Gronwall's Lemma [10])**.** *Let $f, g : [0,\alpha] \to [0,\infty)$ be continuous and let c be a non-negative constant. If*

$$f(t) \leq c + \int_0^t g(s)f(s)\mathrm{d}s, \quad 0 \leq t < \alpha,$$

then

$$f(t) \leq c \exp \int_0^t g(s)\mathrm{d}s, \quad 0 \leq t < \alpha.$$

Moreover, for proof of the following proposition, we introduce the notations below

$$\overline{X}(t) := \sup_{0\leq s\leq t} X(s) \quad \text{and } \underline{X}(t) := \inf_{0\leq s\leq t} X(s),$$

for any $t > 0$ and we assume that ϕ is continuously differentiable since its statement $\phi = \phi_k$ is proved.

Proposition 66. *Suppose that the locally Lipschitz continous rate function ϕ satisfies Assumption 63. Then, there exists a solution $V(t)$ to the SDE given in Eq. (3.6.18). Moreover, the sequence $V_n(t)$ for each $n \geq 1$ converges uniformly to $V(t)$ a.s. on compact time intervals.*

Proof. We prove existence of the solution to SDE by demonstrating uniform convergence of the sequence $(V_n)_{n\geq 1}$ on compact sets to a solution of V with the corresponding rate function ϕ.

Let $(\phi_n)_{n\geq 1}$ be the non-decreasing approximating sequence for ϕ. For each $n \geq 1$, we consider V_n is a solution to the following equation

$$V_n(t) = X(t) + \sup_{0\leq s\leq t} (-V_n(s)) \vee 0 - \int_0^t \phi_n(V_n(s))\mathrm{d}s, \quad t \geq 0.$$

Recall that by Lemma 64, we have $\phi_1(x) \leq \phi_2(x) \leq \ldots \leq \phi(x)$ for all $x \in \mathbb{R}$, and $V_1(t) \geq V_2(t) \geq \ldots$. Now, fix an arbitrary $T > 0$, it follows that

$$\begin{aligned}
V_n(t) &= X(t) + \sup_{0\leq s\leq t} (-V_n(s)) \vee 0 - \int_0^t \phi_n(V_n(s))\mathrm{d}s \\
&\leq X(t) + \sup_{0\leq s\leq t} (-X(s)) \vee 0 \\
&\leq \left|\overline{X}(T)\right| + \sup_{0\leq s\leq t} (-X(s)) \vee 0 \\
&\leq \left|\overline{X}(T)\right| + \sup_{0\leq s\leq T} (-X(s)) \vee 0, \qquad 0 \leq t \leq T,
\end{aligned}$$

as $\sup_{0\le s\le t}(-X(s))$ is right-continuous and non-decreasing. Then, we have

$$V_n(t) \le \big|\overline{X}(T)\big| + \sup_{0\le s\le T}(-X(s)) \vee 0 + \phi\big(\big|\overline{X}(T)\big| + \sup_{0\le s\le T}(-X(s)) \vee 0\big)T,$$

where the last term artificially added in the above equation makes the upper bound of $V_n(t)$ in such form that will be used in the next step of the proof.

On the other hand, since $\phi(x) \ge 0$ for all $x \in \mathbb{R}$,

$$\begin{aligned} V_n(t) &\ge X(t) + \sup_{0\le s\le t}(-V_n(s)) \vee 0 - \int_0^T \phi\big(\big|\overline{X}(T)\big| + \sup_{0\le s\le T}(-X(s)) \vee 0\big)\mathrm{d}s \\ &= X(t) + \sup_{0\le s\le t}(-V_n(s)) \vee 0 - \phi\big(\big|\overline{X}(T)\big| + \sup_{0\le s\le T}(-X(s)) \vee 0\big)T. \end{aligned}$$

By the fact that $\sup_{0\le s\le t}(-V_n(s)) \vee 0 \ge \sup_{0\le s\le t}(-X(s)) \vee 0$, we get

$$\begin{aligned} V_n(t) &\ge X(t) + \sup_{0\le s\le t}(-X(s)) \vee 0 - \phi\big(\big|\overline{X}(T)\big| + \sup_{0\le s\le T}(-X(s)) \vee 0\big)T \\ &\ge X(t) - \sup_{0\le s\le T}(-X(s)) \vee 0 - \phi\big(\big|\overline{X}(T)\big| + \sup_{0\le s\le T}(-X(s)) \vee 0\big)T \\ &\ge -\big|\overline{X}(T)\big| - \sup_{0\le s\le T}(-X(s)) \vee 0 - \phi\big(\big|\overline{X}(T)\big| + \sup_{0\le s\le T}(-X(s)) \vee 0\big)T, \end{aligned}$$

for $0 \le t \le T$.

Hence,

$$\begin{aligned} \big|V_n(t)\big| \le \big(\big|\overline{X}(T)\big| \vee \big|\underline{X}(T)\big|\big) &+ \sup_{0\le s\le T}(-X(s)) \vee 0 \\ &+ \phi\big(\big|\overline{X}(T)\big| + \sup_{0\le s\le T}(-X(s)) \vee 0\big)T := K_T. \end{aligned}$$

for $n \ge 1$ and $0 \le t \le T$.

Since the sequence $n \mapsto V_n$ is non-increasing by Lemma 64 and bounded below, we can define $V(t) := \lim_{n\to\infty} V_n(t)$ (pointwise using by monotone convergence theorem). Further, by the uniform convergence of $\phi_n \to \phi$ on compact sets and pointwise convergence of $V_n \to V$, we have $\phi_n(V_n) \to \phi(V)$ pointwise.

To show uniform convergence of V_n to V on compact time intervals, we consider

$$\begin{aligned} V_n(t) - V(t) = X(t) &+ \sup_{0\le s\le t}(-V_n(s)) \vee 0 - \int_0^t \phi_n(V_n(s))\mathrm{d}s \\ &- \big(X(t) + \sup_{0\le s\le t}(-V(s)) \vee 0 - \int_0^t \phi(V(s))\mathrm{d}s\big) \end{aligned}$$

$$
\begin{aligned}
&= \sup_{0\le s\le t}(-V_n(s))\vee 0-\sup_{0\le s\le t}(-V(s))\vee 0\\
&\quad+\int_0^t \big(\phi(V(s))-\phi_n(V_n(s))\big)\mathrm{d}s\\
&\le \int_0^t \big(\phi(V(s))-\phi_n(V_n(s))\big)\mathrm{d}s,
\end{aligned}
$$

since by monotonicity of $(V_n)_{n\ge 1}$ (see Lemma 64) and $\phi(\cdot)\ge 0$, we obtain for each $n\ge 1$,

$$
\sup_{0\le s\le t}(-V_n(s))\vee 0-\sup_{0\le s\le t}(-V(s))\vee 0\le 0.
$$

Then, we have

$$
\begin{aligned}
|V_n(t)-V(t)| &\le \int_0^t \big|\phi(V(s))-\phi_n(V_n(s))\big|\mathrm{d}s\\
&= \int_0^t \big|\phi(V(s))\pm\phi(V_n(s))-\phi_n(V_n(s))\big|\mathrm{d}s
\end{aligned}
$$

Using triangular inequality gives that

$$
|V_n(t)-V(t)|\le \int_0^t \big|\phi(V(s))-\phi(V_n(s))\big|\mathrm{d}s+\int_0^t \big|\phi(V_n(s))-\phi_n(V_n(s))\big|\mathrm{d}s.
$$

Then, by the assumption of ϕ is a locally Lipschitz function, we have

$$
|V_n(t)-V(t)|\le L_I\int_0^t \big|V(s)-V_n(s)\big|\mathrm{d}s+\int_0^t \big|\phi(V_n(s))-\phi_n(V_n(s))\big|\mathrm{d}s,
$$

where $I:=[-K_T,K_T]$ for the K_T defined above. Then, by using Mean Value Theorem, we obtain for $0\le t\le T$,

$$
\begin{aligned}
|V_n(t)-V(t)| &\le L_I\int_0^t \big|V(s)-V_n(s)\big|\mathrm{d}s+\int_0^T \sup_{s\in I}\big|\phi(s)-\phi_n(s)\big|\mathrm{d}s\\
&= L_I\int_0^t \big|V(s)-V_n(s)\big|\mathrm{d}s+T\sup_{s\in I}\big|\phi(s)-\phi_n(s)\big|.
\end{aligned}
$$

Thus, using Gronwall's Inequality (in Lemma 65) gives that

$$
|V_n(t)-V(t)|\le T\sup_{s\in I}\big|\phi(s)-\phi_n(s)\big|e^{L_I\int_0^t 1\mathrm{d}s}=T\sup_{s\in I}\big|\phi(s)-\phi_n(s))\big|e^{L_I t},
$$

which implies that

$$
\sup_{0\le t\le T}|V_n(t)-V(t)|\le T\sup_{s\in I}\big|\phi(s)-\phi_n(s)\big|e^{L_I T}.
$$

Then, taking limit of both side when $n\to\infty$, we have

$$
\lim_{n\to\infty}\sup_{0\le t\le T}|V_n(t)-V(t)|\le Te^{L_I T}\lim_{n\to\infty}\sup_{s\in I}\big|\phi(s)-\phi_n(s)\big|.
$$

Since $\lim_{n\to\infty}\phi_n=\phi$ uniformly a.s., we conclude that $V_n(t)$ converges uniformly to $V(t)$ a.s. on compact time intervals. □

Next, we derive an explicit expression for resolvent measure of reflected level-dependent Lévy processes. To do this, we first give an preliminary result, which follows verbatim from [31].

Lemma 67. *Let $\overline{V}(t) = \sup_{0\leq s\leq t} V(s)$. For each $x, a \in \mathbb{R}$, the reflected level-dependent Lévy process with associated rate function ϕ satisfies $\mathbb{P}_x(\overline{V}(t) = a) = 0$ for Lebesgue almost every $t > 0$.*

The following theorem explains that the resolvent of level-dependent Lévy processes reflected at its infimum which is killed on exiting $[0, a]$. We first define the first passage time for the reflected level-dependent Lévy process exits above a level $a \in \mathbb{R}$ as

$$K_a = \inf\{t > 0 : V_t > a\}, \text{ for } a > 0. \tag{3.6.20}$$

Theorem 68 (Resolvent of Reflected Level-Dependent at Infimum)**.** *For $q \geq 0$, $x \leq a$, and a Borel set $\mathfrak{B}$ on $[0, a)$ we have*

$$\mathbb{E}_x\Big[\int_0^{K_a} e^{-qt}\mathbf{1}_{(V_t\in\mathfrak{B})}\mathrm{d}t\Big] = \int_{\mathfrak{B}} \Xi_\phi(y)^{-1}\Big(\mathbb{W}^{(q)}(a;y)\frac{\mathbb{Z}^{(q)}(x)}{\mathbb{Z}^{(q)}(a)} - \mathbb{W}^{(q)}(x;y)\Big)\mathrm{d}y, \tag{3.6.21}$$

where $\mathbb{W}^{(q)}(x; y)$ and $\mathbb{Z}^{(q)}(x)$ are given in Eq. (3.5.8) and Eq. (3.5.3) respectively.

Proof. Consider an approximating sequence $(\phi_n)_{n\geq 1}$ for the rate function ϕ of the reflected level-dependent Lévy process. We also define corresponding sequence of non-increasing reflected multi-refracted Lévy process $(V_n)_{n\geq 0}$. Then, by Proposition 66, it is proven that the sequence $(V_n)_{n\geq 0}$ converges uniformly on compact sets to V.

Let $\overline{V}_n(t) := \sup_{0\leq s\leq t} V_n(s)$ and $\overline{V}(t) := \sup_{0\leq s\leq t} V(s)$ and note that by triangular inequality we have

$$\big|\overline{V}_n(t) - \overline{V}(t)\big| \leq \sup_{s\in[0,t]} \big|V_n(s) - V(s)\big|, \quad t \geq 0,$$

and therefore for $t \geq 0$,

$$\lim_{n\uparrow\infty} \big(V_n(t), \overline{V}_n(t)\big) = (V(t), \overline{V}(t)),$$

in almost sure sense.

Now, by using Lemma 64, for each $t > 0$, the sequence $(V_n)_{n\geq 1}$ is non-increasing, then we have for $a, y \geq 0$

$$\{\overline{V}(t) \geq a\} = \bigcap_{n\geq 1} \{\overline{V}_n(t) \geq a\} \quad \text{and} \quad \{V(t) \geq y\} = \bigcap_{n\geq 1} \{V_n(t) \geq y\}.$$

This tells us for any $x \in \mathbb{R}$, and $t > 0$,

$$\begin{aligned}\mathbb{P}_x(V(t) \geq y, \overline{V}(t) \geq a) &= \mathbb{P}_x\Big(\bigcap_{n\geq 1} \{V_n(t) \geq y, \overline{V}_n(t) \geq a\}\Big) \\ &= \lim_{n\to\infty} \mathbb{P}_x\big(V_n(t) \geq y, \overline{V}_n(t) \geq a\big).\end{aligned}$$

Then, by Lemma 67, we have that $\mathbb{P}_x(\overline{V}(t) = a) = 0$ for Lebesgue almost every $t > 0$, which implies that

$$\begin{aligned}\mathbb{P}_x(V(t) \geq y, \overline{V}(t) \leq a) &= \mathbb{P}_x(V(t) \geq y) - \mathbb{P}_x(V(t) \geq y, \overline{V}(t) \geq a) \\ &= \lim_{n\to\infty} \mathbb{P}_x\big(V_n(t) \geq y\big) - \lim_{n\to\infty} \mathbb{P}_x\big(V_n(t) \geq y, \overline{V}_n(t) \geq a\big) \\ &= \lim_{n\to\infty} \mathbb{P}_x\big(V_n(t) \geq y, \overline{V}_n(t) \leq a\big).\end{aligned}$$

Thus, by Dominated Convergence Theorem we obtain that

$$\begin{aligned}\lim_{n\to\infty} \mathbb{E}_x\Big[\int_0^{T_n^{a,+}} e^{-qt}\mathbf{1}_{\{V_n(t)\geq y\}}\mathrm{d}t\Big] &= \lim_{n\to\infty} \int_0^\infty e^{-qt}\mathbb{P}_x\big(V_n(t) \geq y, \overline{V}_n(t) \leq a\big)\mathrm{d}t \\ &= \int_0^\infty e^{-qt}\mathbb{P}_x\big(V(t) \geq y, \overline{V}(t) \leq a\big)\mathrm{d}t \\ &= \mathbb{E}_x\Big[\int_0^{K_a} e^{-qt}\mathbf{1}_{\{V(t)\geq y\}}\mathrm{d}t\Big].\end{aligned}$$

Recalling Eq. (3.6.11) gives that

$$\begin{aligned}&\mathbb{E}_x\Big[\int_0^{K_a} e^{-qt}\mathbf{1}_{\{V(t)\geq y\}}\mathrm{d}t\Big] \\ &\qquad= \lim_{n\to\infty} \mathbb{E}_x\Big[\int_0^{T_n^{a,+}} e^{-qt}\mathbf{1}_{\{V_n(t)\geq y\}}\mathrm{d}t\Big] \\ &\qquad= \lim_{n\to\infty} \int_{[y,\infty]} \Xi_{\phi_n}(z)^{-1}\Big(\mathbb{W}_n^{(q)}(a;z)\frac{\mathbb{Z}_n^{(q)}(x)}{\mathbb{Z}_n^{(q)}(a)} - \mathbb{W}_n^{(q)}(x;z)\Big)\mathrm{d}z.\end{aligned}$$

Then, by using the fact the scale functions $\mathbb{W}_n^{(q)}$ and $\mathbb{Z}_n^{(q)}$ converges to $\mathbb{W}^{(q)}$ and $\mathbb{Z}^{(q)}$ as given Remark 54 (also see Theorem 27 in [13]) and by the assumption of $\lim_{n\to\infty} \phi_n = \phi$ uniformly a.s., we conclude that

$$\mathbb{E}_x\Big[\int_0^{K_a} e^{-qt}\mathbf{1}_{\{V(t)\geq y\}}\mathrm{d}t\Big] = \int_{[y,\infty]} \Xi_\phi(z)^{-1}\Big(\mathbb{W}^{(q)}(a;z)\frac{\mathbb{Z}^{(q)}(x)}{\mathbb{Z}^{(q)}(a)} - \mathbb{W}^{(q)}(x;z)\Big)\mathrm{d}z,$$

which completes the proof. □

Theorem 69 (One-sided exit for reflected level-dependent SNLP)**.** *For $x \leq a$ and $q \geq 0$, we have*

$$\mathbb{E}_x\left[e^{-qK_a}\mathbf{1}_{(K_a<\infty)}\right] = \frac{\mathbb{Z}^{(q)}(x)}{\mathbb{Z}^{(q)}(a)}. \tag{3.6.22}$$

Proof. In this proof, one obtains that

$$\begin{aligned}\mathbb{E}_x\left[e^{-qK_a}\mathbf{1}_{(K_a<\infty)}\right] &= 1 - q\int_0^\infty e^{-qt}\mathbb{P}_x(V(t) \in [0,a), t < K_a)\mathrm{d}t \\ &= \lim_{n\to\infty}\left(1 - q\int_0^\infty e^{-qt}\mathbb{P}_x(V_n(t) \in [0,a), t < T_n^{a,+})\mathrm{d}t\right) \\ &= \lim_{n\to\infty}\mathbb{E}_x\left[e^{-qT_n^{a,+}}\mathbf{1}_{(T_n^{a,+}<\infty)}\right] = \lim_{n\to\infty}\frac{\mathbb{Z}_n^{(q)}(x)}{\mathbb{Z}_n^{(q)}(a)} = \frac{\mathbb{Z}^{(q)}(x)}{\mathbb{Z}^{(q)}(a)},\end{aligned}$$

by noting that in the last line, we use Remark 54 to conclude the proof. ☐

We provide the following remark which will be used in Section 3.7.4.

Remark 70. *Note that for the case of $q = 0$, density of resolvent for level-dependent Lévy process reflected at its infimum is equal to $\Xi_\phi(y)^{-1}(\mathbb{W}(a;y) - \mathbb{W}(x;y))$ since $\mathbb{Z}^{(0)}(x) := \mathbb{Z}(x) = 1$. It is not obvious to see why $\mathbb{Z}(x) = 1$ in Eq. (3.5.3), but since $\mathbb{Z}_n(x)$ given in Eq. (3.6.16) when $q = 0$ converges to $\mathbb{Z}(x)$, then we directly get $\mathbb{Z}(x) = 1$.*

3.7 Fluctuations of Omega-Killed Level-Dependent Spectrally Lévy Processes

Previously, we addressed the essential background related to fluctuation identities for the spectrally negative Lévy processes in terms of scale functions. Further extensions (see [1, 18, 26]) to the classical Lévy process can be done by considering ω-killing in which the process is killed with intensity rate function $\omega(x)$ depending on the level of process. The fluctuation identities of ω-killed spectrally Lévy process is well-studied in [26] which is an outstanding generalization of the classical case with an arbitrary non-negative locally bounded measurable function $\omega(x)$. They obtain a suite of ω-killed fluctuation identities such as one and two-sided exit problems and one-sided reflected exit identities and their corresponding resolvents.

Let $\omega : \mathbb{R} \to \mathbb{R}^+$ be a non-negative, locally bounded measurable function and X_t is a spectrally negative Lévy process which is killed exponentially with rate/intensity depending on the present level of the process. For such killing feature we use "ω-killing", which could be interpreted as a (level-dependent) Laplace argument, a discount factor, a bankruptcy rate or weight of occupation time in different context. The fluctuation theory for ω-killed spectrally negative Lévy processes are characterized in terms of a new generalization of scale functions namely ω-scale functions in [26].

In this subsection, we develop fluctuation theory for ω-killed level-dependent spectrally negative Lévy process by introducing a new family of scale functions, namely ω-killed level-dependent scale functions.

3.7.1 Omega-Killed Level-Dependent Scale Functions

In this study, we introduce a suite of fluctuation identities for ω-killed level-dependent Lévy processes such as one and two-sided exit problems and their resolvents in the pursuit of the studies related to ω-killed spectrally negative Lévy processes introduced in [26] and level-dependent Lévy processes introduced in [13] (reviewed also in Section 3.5). In order to express these identities, we develop a new family of scale functions.

Given the previous results, our main aim is to analyze the general version of level-dependent (analogously multi-refracted) Lévy processes so-called *ω-killed level dependent Lévy process*, which are exponentially killed with intensity depends upon the present state of the process. As mentioned before and seen in [12, 26], *omega-killing* covers to the idea of level (state) dependent Laplace argument of a level-dependent Lévy process with general rate function ϕ which can be seen as a function changing the drift of the ω-killed process according to its level.

Recall from Eq. (3.5.1) that level-dependent Lévy process $\{U_t : t \geq 0\}$ is solution to stochastic differential equation (SDE) given as

$$\mathrm{d}U_t = \mathrm{d}X_t - \phi(U_t)\mathrm{d}t,$$

where X_t is a spectrally negative Lévy process and general rate function $\phi(\cdot)$ satisfies the Assumption 63. Note that as discussed in the previous section by Theorem 48, the above SDE has a unique strong solution.

Remark 71. *Note that by Lemma 3 in [13], for multi-refracted case unique solution of SDE has strong Markov property. Since SDE for general rate function* ϕ *converges uniformly almost surely on compact time intervals, one can conclude that strong Markov property is satisfied for the unique solution to SDE for general case as well.*

All fluctuation identities derived below are based on a new family of scale functions, $\{\mathcal{W}^{(\omega)}(x),\ x\in\mathbb{R}\}$ and $\{\mathcal{Z}^{(\omega)}(x),\ x\in\mathbb{R}\}$, which we call as ω-killed level-dependent scale functions, are defined as the unique solutions to the following integral equations

$$\mathcal{W}^{(\omega)}(x,y)=\mathbb{W}^{(q)}(x;y)+\int_y^x \mathbb{W}^{(q)}(x;z)\Xi_\phi(z)^{-1}(\omega(z)-q)\mathcal{W}^{(\omega)}(z,y)\mathrm{d}z, \quad (3.7.1)$$

$$\mathcal{Z}^{(\omega)}(x,y)=\mathbb{Z}^{(q)}(x;y)+\int_y^x \mathbb{W}^{(q)}(x;z)\Xi_\phi(z)^{-1}(\omega(z)-q)\mathcal{Z}^{(\omega)}(z,y)\mathrm{d}z, \quad (3.7.2)$$

where $\mathbb{W}^{(q)}(x;y)$, $\mathbb{Z}^{(q)}(x;y)$ and $\Xi_\phi(x)$ are given in Eqs.(3.5.8) – (3.5.9) respectively, and Eq. (3.5.6) with $\mathcal{W}^{(\omega)}(y,y)=W^{(q)}(0)$ and $\mathcal{Z}^{(\omega)}(y,y)=1$. When $y=0$ and $q=0$, we denote $\mathcal{W}^{(\omega)}(x,0):=\mathcal{W}^{(\omega)}(x)$ and $\mathcal{Z}^{(\omega)}(x,0):=\mathcal{Z}^{(\omega)}(x)$ and the above equations are reduced to

$$\mathcal{W}^{(\omega)}(x)=\mathbb{W}(x)+\int_0^x \mathbb{W}(x;y)\Xi_\phi(y)^{-1}\omega(y)\mathcal{W}^{(\omega)}(y)\mathrm{d}y, \quad (3.7.3)$$

$$\mathcal{Z}^{(\omega)}(x)=1+\int_0^x \mathbb{W}(x;y)\Xi_\phi(y)^{-1}\omega(y)\mathcal{Z}^{(\omega)}(y)\mathrm{d}y, \quad (3.7.4)$$

since $\mathbb{Z}(x;0)=Z(x)=1$.

In the remaining of the study, we shall use extensively integral equations of the form in Eqs. (3.7.1) – (3.7.4). The existence and uniqueness of a solution of such integral equations is given by the proposition below. The existence and uniqueness of the solutions to Eqs. (3.7.1) – (3.7.2) and consequently to Eqs. (3.7.3) – (3.7.4) are assured by the following proposition. Note that Eqs. (3.7.3) – (3.7.4) are the special case of Eqs. (3.7.1) – (3.7.2) for $y=0$ and $q=0$.

Proposition 72. *Let* $h(x,y)$ *be a locally bounded function on* $\mathbb{R}^2$ *and* $\omega(\cdot)$ *is a locally bounded function on* $\mathbb{R}$. *The following equation*

$$H^{(\omega)}(x,y)=h(x,y)+\int_y^x \mathbb{W}^{(q)}(x;z)\Xi_\phi(z)^{-1}(\omega(z)-q)H^{(\omega)}(z,y)\mathrm{d}z, \quad (3.7.5)$$

admits a unique locally bounded solution on $\mathbb{R}^2$ *satisfying* $H^{(\omega)}(x,y)=h(x,y)$ *for* $x\le y$.

Proof. We start with existence of the solution. Notice that $\mathcal{W}^{(q)}(x,y)$ is well-defined (see Eq. (3.7.10)) and satisfies Eq. (3.7.1). For any $h(x,y)$, we define

$$H^{(\omega)}(x,y) := h(x,y) + \int_y^x \mathcal{W}^{(\omega)}(x,z)\Xi_\phi(z)^{-1}(\omega(z)-q)h(z,y)\mathrm{d}z. \tag{3.7.6}$$

Let $x > y$, based on the above equation, we have

$$\begin{aligned}
&\int_y^x \mathbb{W}^{(q)}(x;z)\Xi_\phi(z)^{-1}(\omega(z)-q)H^{(\omega)}(z,y)\mathrm{d}z \\
&= \int_y^x \mathbb{W}^{(q)}(x;z)\Xi_\phi(z)^{-1}(\omega(z)-q)h(z,y)\mathrm{d}z \\
&\quad + \int_y^x \mathbb{W}^{(q)}(x;z)\Xi_\phi(z)^{-1}(\omega(z)-q)\int_y^z \mathcal{W}^{(\omega)}(z,u)\Xi_\phi(u)^{-1}(\omega(u)-q)h(u,y)\mathrm{d}u\mathrm{d}z \\
&= \int_y^x \mathbb{W}^{(q)}(x;z)\Xi_\phi(z)^{-1}(\omega(z)-q)h(z,y)\mathrm{d}z \\
&\quad + \int_y^x\int_y^z \mathbb{W}^{(q)}(x;z)\Xi_\phi(z)^{-1}(\omega(z)-q)\mathcal{W}^{(\omega)}(z,u)\Xi_\phi(u)^{-1}(\omega(u)-q)h(u,y)\mathrm{d}u\mathrm{d}z.
\end{aligned}$$

Using Fubini's Theorem, the above equation becomes

$$\begin{aligned}
&\int_y^x \mathbb{W}^{(q)}(x;z)\Xi_\phi(z)^{-1}(\omega(z)-q)h(z,y)\mathrm{d}z \\
&\quad + \int_y^x\int_u^x \mathbb{W}^{(q)}(x;z)\Xi_\phi(z)^{-1}(\omega(z)-q)\mathcal{W}^{(\omega)}(z,u)\Xi_\phi(u)^{-1}(\omega(u)-q)h(u,y)\mathrm{d}z\mathrm{d}u \\
&= \int_y^x \mathbb{W}^{(q)}(x;z)\Xi_\phi(z)^{-1}(\omega(z)-q)h(z,y)\mathrm{d}z \\
&\quad + \int_y^x \Xi_\phi(u)^{-1}(\omega(u)-q)h(u,y)\int_u^x \mathbb{W}^{(q)}(x;z)\Xi_\phi(z)^{-1}(\omega(z)-q)\mathcal{W}^{(\omega)}(z,u)\mathrm{d}z\mathrm{d}u.
\end{aligned}$$

Using Eq. (3.7.1), the above equation yields to

$$\begin{aligned}
&\int_y^x \mathbb{W}^{(q)}(x;z)\Xi_\phi(z)^{-1}(\omega(z)-q)h(z,y)\mathrm{d}z \\
&\qquad + \int_y^x \Xi_\phi(u)^{-1}(\omega(u)-q)h(u,y)\big[\mathcal{W}^{(\omega)}(x,u)-\mathbb{W}^{(q)}(x;u)\big]\mathrm{d}u \\
&= \int_y^x \mathbb{W}^{(q)}(x;z)\Xi_\phi(z)^{-1}(\omega(z)-q)h(z,y)\mathrm{d}z \\
&\qquad + \int_y^x \mathcal{W}^{(\omega)}(x,u)\Xi_\phi(u)^{-1}(\omega(u)-q)h(u,y)\mathrm{d}u \\
&\qquad - \int_y^x \mathbb{W}^{(q)}(x;u)\Xi_\phi(u)^{-1}(\omega(u)-q)h(u,y)\mathrm{d}u \\
&= \int_y^x \mathcal{W}^{(\omega)}(x,u)\Xi_\phi(u)^{-1}(\omega(u)-q)h(u,y)\mathrm{d}u \\
&= H^{(\omega)}(x,y) - h(x,y),
\end{aligned}$$

after using Eq. (3.7.6), which completes the proof of existence of $\mathcal{W}^{(q)}(x,y)$.

Furthermore, in order to show the uniqueness of Eq. (3.7.5) suppose that there exist two solutions $H_1^{(\omega)}$ and $H_2^{(\omega)}$, then $\widetilde{H}^{(\omega)} = H_1^{(\omega)} - H_2^{(\omega)}$ is solution to

$$|\widetilde{H}^{(\omega)}(x,y)| = \int_y^x \mathbb{W}^{(q)}(x;z)\Xi_\phi(z)^{-1}|\omega(z)-q||\widetilde{H}^{(\omega)}(z,y)|\mathrm{d}z.$$

Then, by using the Gronwall's Lemma (see Lemma 65), for a non-negative function $g(z) = \mathbb{W}^{(q)}(x;z)\Xi_\phi(z)^{-1}|\omega(z)-q|$ we conclude that

$$|\widetilde{H}^{(\omega)}(x,y)| \le 0 \times e^{\int_y^x g(z)\mathrm{d}z},$$

and thus $\widetilde{H}^{(\omega)}(x,y) \equiv 0$. □

The next proposition can be used to show the Volterra equations Eqs. (3.7.1) – (3.7.2) are true. Consider the general version of Eqs. (3.7.3) – (3.7.4),

$$\mathcal{W}^{(\omega)}(x,y) = \mathbb{W}(x;y) + \int_y^x \mathbb{W}(x;z)\Xi_\phi(z)^{-1}\omega(z)\mathcal{W}^{(\omega)}(z,y)\mathrm{d}z, \tag{3.7.7}$$

$$\mathcal{Z}^{(\omega)}(x,y) = 1 + \int_y^x \mathbb{W}(x;y)\Xi_\phi(z)^{-1}\omega(z)\mathcal{Z}^{(\omega)}(z,y)\mathrm{d}z.$$

Proposition 73. *Let* $(\mathcal{W}^{(\omega_1)}, \mathcal{Z}^{(\omega_1)})$ *and* $(\mathcal{W}^{(\omega_2)}, \mathcal{Z}^{(\omega_2)})$ *be some generalised auxiliary functions with respect to* $\omega_1(\cdot) \ge 0$ *and* $\omega_2(\cdot) \ge 0$*, respectively. Then, for* x*,* $y \in \mathbb{R}$*,*

$$\mathcal{W}^{(\omega_1)}(x,y) - \mathcal{W}^{(\omega_2)}(x,y) = \int_y^x \mathcal{W}^{(\omega_1)}(x,z)\Xi_\phi(z)^{-1}\big(\omega_1(z)-\omega_2(z)\big)\mathcal{W}^{(\omega_2)}(z,y)\mathrm{d}z,$$

$$\mathcal{Z}^{(\omega_1)}(x,y) - \mathcal{Z}^{(\omega_2)}(x,y) = \int_y^x \mathcal{Z}^{(\omega_1)}(x,z)\Xi_\phi(z)^{-1}\big(\omega_1(z)-\omega_2(z)\big)\mathcal{Z}^{(\omega_2)}(z,y)\mathrm{d}z.$$

Proof. Using Eq. (3.7.6), the Volterra equation for $\mathcal{W}^{(\omega)}(x,y)$ has solution of the form

$$\mathcal{W}^{(\omega)}(x,y) = \mathbb{W}(x;y) + \int_y^x \mathcal{W}^{(\omega)}(x,z)\Xi_\phi(z)^{-1}\omega(z)\mathbb{W}(z;y)\mathrm{d}z.$$

Applying the above equation in the first and third line below and Eq. (3.7.7) in second line below, we have

$$\begin{aligned}
&\int_{\mathbb{R}} \mathcal{W}^{(\omega_1)}(x,z)\Xi_\phi(z)^{-1}\omega_2(z)\mathcal{W}^{(\omega_2)}(z,y)\mathrm{d}z \\
&= \int_{\mathbb{R}} \Big[\mathbb{W}(x;z) + \int_{\mathbb{R}} \mathcal{W}^{(\omega_1)}(x,u)\Xi_\phi(u)^{-1}\omega_1(u)\mathbb{W}(u;z)\mathrm{d}u\Big] \\
&\qquad\qquad \times \Xi_\phi(z)^{-1}\omega_2(z)\mathcal{W}^{(\omega_2)}(z,y)\mathrm{d}z
\end{aligned}$$

$$\begin{aligned}&= \mathcal{W}^{(\omega_2)}(x,y) - \mathbb{W}(x;y) \\ &\qquad + \int_{\mathbb{R}} \mathcal{W}^{(\omega_1)}(x,u)\Xi_\phi(u)^{-1}\omega_1(u)\big[\mathcal{W}^{(\omega_2)}(u,y) - \mathbb{W}(u;y)\big]\mathrm{d}u \\ &= \mathcal{W}^{(\omega_2)}(x,y) - \mathcal{W}^{(\omega_1)}(x,y) + \int_{\mathbb{R}} \mathcal{W}^{(\omega_1)}(x,z)\Xi_\phi(z)^{-1}\omega_1(z)\mathcal{W}^{(\omega_2)}(z,y)\mathrm{d}z,\end{aligned}$$

which proves the first equation of the proposition. The second equation can be proved similarly. ☐

Using Proposition 73 and repeating the reasoning of Proposition 2 in [27], Eq. (3.7.1) follows immediately. Same line of logic can be used to show Eq. (3.7.2).

3.7.2 Two-Sided Exit Problems and One-Sided Downwards Problem

In this subsection, we derive fluctuation identities in terms of ω-killed level-dependent scale functions defined in the previous subsection.

Let $\omega : \mathbb{R} \to \mathbb{R}^+$ be a (locally) bounded, non-negative measurable function and first passage stopping times given in Eq. (3.5.10), i.e.

$$\kappa_x^+ := \inf\{t > 0 : U_t \geq x\} \quad \text{and} \quad \kappa_x^- := \inf\{t > 0 : U_t < x\}.$$

In the following theorems we analyze the two-sided exit above and below by extending the arguments in [26] for a level-dependent Lévy process.

Theorem 74 (Two-Sided Exit Problem Above). *For $0 \leq x \leq a$, we have*

$$\mathcal{A}(x,a) := \mathbb{E}_x\Big[e^{-\int_0^{\kappa_a^+}\omega(U_t)\mathrm{d}t}\mathbf{1}_{(\kappa_a^+<\kappa_0^-)}\Big] = \frac{\mathcal{W}^{(\omega)}(x)}{\mathcal{W}^{(\omega)}(a)}, \tag{3.7.8}$$

where $\mathcal{W}^{(q)}(\cdot)$ is solution to Eq. (3.7.3).

Proof. For $x < 0$, it is straightforward since $\mathcal{W}^{(\omega)}(x) = \mathbb{W}(x) = 0$. For $x \geq 0$, applying strong Markov property [see Lemma 3 in [13]] and using the fact that U has no positive jumps gives that

$$\begin{aligned}\mathbb{E}_x\Big[e^{-\int_0^{\kappa_a^+}\omega(U_t)\mathrm{d}t}\mathbf{1}_{(\kappa_a^+<\kappa_0^-)}\Big] &= \mathbb{E}_x\Big[e^{-\int_0^{\kappa_y^+}\omega(U_t)\mathrm{d}t}\mathbf{1}_{(\kappa_y^+<\kappa_0^-)}\Big]\mathbb{E}_{U_{\kappa_y^+}}\Big[e^{-\int_0^{\kappa_a^+}\omega(U_t)\mathrm{d}t}\mathbf{1}_{(\kappa_a^+<\kappa_0^-)}\Big] \\ &= \mathbb{E}_x\Big[e^{-\int_0^{\kappa_y^+}\omega(U_t)\mathrm{d}t}\mathbf{1}_{(\kappa_y^+<\kappa_0^-)}\Big]\mathbb{E}_y\Big[e^{-\int_0^{\kappa_a^+}\omega(U_t)\mathrm{d}t}\mathbf{1}_{(\kappa_a^+<\kappa_0^-)}\Big],\end{aligned}$$

or equivalently by using the definition $\mathcal{A}(x,\cdot)$, we get

$$\mathcal{A}(x,a) = \mathcal{A}(x,y)\mathcal{A}(y,a), \quad \text{for all} \quad 0 \le x \le y \le a. \qquad (3.7.9)$$

Now, we define $E = \{E_t,\ t \ge 0\}$ to be a Poisson point process with measure $\eta(\mathrm{d}t \times \mathrm{d}x) = \lambda \mathrm{d}t \times F(\mathrm{d}x)$. That is $E = \{(T_i, M_i),\ i = 1, 2, \dots\}$ is a marked Poisson process with intensity λ at jump epochs T_i and (independent) marks M_i. Further, let

$$N_A := \#\{i : (T_i, M_i) \in A\},$$

is the number of pairs (times and marks) in the set of A.

Under the above set up

$$\mathbb{P}(N_A = 0) = e^{-\eta(A)} = e^{-\int_A \lambda \mathrm{d}t \times F(\mathrm{d}x)}.$$

Next, for a stopping time κ_a^+, we define the event of A to be of the form

$$A := \{T_i < \kappa_a^+,\ M_i < \omega(U_{T_i}) \text{ for } i = 1, 2 \dots\},$$

and thus using the above arguments, we have that

$$\mathbb{P}_x(N_A = 0, \kappa_a^+ < \kappa_0^-) = \mathbb{E}_x\Big[e^{-\int_0^{\kappa_a^+}\int_0^{\omega(U_{T_i})} \lambda \mathrm{d}t \times F(\mathrm{d}x)} \mathbf{1}_{(\kappa_a^+ < \kappa_0^-)} \,\Big|\, T_1\Big].$$

Now, considering that M_i ~Unif$(0,\lambda)$, i.e. $F(\mathrm{d}x) = \frac{1}{\lambda}$ with $\omega(\cdot) \le \lambda$, the above equation becomes

$$\mathbb{P}_x(N_A = 0, \kappa_a^+ < \kappa_0^-) = \mathbb{E}_x\Big[e^{-\int_0^{\kappa_a^+} \omega(U_t)\mathrm{d}t} \mathbf{1}_{(\kappa_a^+ < \kappa_0^-)}\Big] = \mathcal{A}(x,a).$$

Based on the above, the crucial observation is that

$$\mathcal{A}(x,a) = \mathbb{P}_x\big(\#_i\{M_i < \omega(U_{T_i}) \text{ for all } T_i \le \kappa_a^+ \text{ and } \{\kappa_a^+ < \kappa_0^-\}\} = 0\big).$$

In this case, either there is no T_i which occurs before reaching level a or a jump at time T_1 occurs and the process renews. Hence,

$$\begin{aligned}
\mathcal{A}(x,a) &= \mathbb{E}_x\big[\mathbb{P}_x\big(\#_i\{M_i < \omega(U_{T_i}) \text{ for all } T_i < \kappa_a^+ \text{ and } \kappa_a^+ < \kappa_0^-\} = 0 \,\big|\, T_1\big)\big] \\
&= \mathbb{E}_x\big[\mathbf{1}_{(T_1 > \kappa_a^+)}\mathbf{1}_{(\kappa_a^+ < \kappa_0^-)}\big] + \mathbb{E}_x\big[\mathcal{A}(U_{T_1}, a)\mathbf{1}_{(T_1 < \kappa_a^+ \wedge \kappa_0^-,\, M_1 > \omega(U_{T_1}))}\big] \\
&= \mathbb{E}_x\big[\mathbf{1}_{(\kappa_a^+ < \kappa_0^-)}\mathbb{P}_x(T_1 > \kappa_a^+)\big] + \mathbb{E}_x\big[\mathcal{A}(U_{T_1}, a)\mathbf{1}_{(T_1 < \kappa_a^+ \wedge \kappa_0^-,\, M_1 > \omega(U_{T_1}))}\big]
\end{aligned}$$

$$
\begin{aligned}
&= \mathbb{E}_x\big[\mathbf{1}_{(\kappa_a^+<\kappa_0^-)}(1-\mathbb{P}_x(T_1\le\kappa_a^+))\big]\\
&\quad+\int_0^\infty \mathbb{E}_x\big[\mathcal{A}(U_t,a)\mathbf{1}_{(t<\kappa_a^+\wedge\kappa_0^-,\,M_1>\omega(U_t))}\big]\mathbb{P}_x(T_1\in \mathrm{d}t)\\
&= \mathbb{E}_x\big[e^{-\lambda\kappa_a^+}\mathbf{1}_{(\kappa_a^+<\kappa_0^-)}\big]+\int_0^\infty \mathbb{E}_x\big[\mathcal{A}(U_t,a)\mathbf{1}_{(t<\kappa_a^+\wedge\kappa_0^-,\,M_1>\omega(U_t))}\big]\lambda e^{-\lambda t}\mathrm{d}t.
\end{aligned}
$$

Using Eq. (3.5.11), we immediately have

$$
\begin{aligned}
\mathcal{A}(x,a) &= \frac{\mathbb{W}^{(\lambda)}(x)}{\mathbb{W}^{(\lambda)}(a)}+\int_0^\infty \lambda e^{-\lambda t}\mathbb{E}_x\big[\mathcal{A}(U_t,a)\mathbf{1}_{(t<\kappa_a^+\wedge\kappa_0^-,\,M_1>\omega(U_t))}\big]\mathrm{d}t\\
&= \frac{\mathbb{W}^{(\lambda)}(x)}{\mathbb{W}^{(\lambda)}(a)}\\
&\quad+\int_0^\infty\int_0^a \lambda e^{-\lambda t}\mathcal{A}(y,a)\mathbb{P}_x(U_t\in\mathrm{d}y,\ t<\kappa_a^+\wedge\kappa_0^-)\mathbb{P}_x(M_1>\omega(y))\mathrm{d}t\\
&= \frac{\mathbb{W}^{(\lambda)}(x)}{\mathbb{W}^{(\lambda)}(a)}+\int_0^\infty\int_0^a e^{-\lambda t}(\lambda-\omega(y))\mathcal{A}(y,a)\mathbb{P}_x(U_t\in\mathrm{d}y,\ t<\kappa_a^+\wedge\kappa_0^-)\mathrm{d}t\\
&= \frac{\mathbb{W}^{(\lambda)}(x)}{\mathbb{W}^{(\lambda)}(a)}+\int_0^a(\lambda-\omega(y))\mathcal{A}(y,a)\int_0^\infty e^{-\lambda t}\mathbb{P}_x(U_t\in\mathrm{d}y,\ t<\kappa_a^+\wedge\kappa_0^-)\mathrm{d}t,
\end{aligned}
$$

where the last integral is the resolvent of the process U killed on exiting the interval $[0,a]$ given in Eq. (3.5.14). Thus, substituting Eq. (3.5.14) for $d=0$ in the above equation, it yields to

$$
\begin{aligned}
\mathcal{A}(x,a) &= \frac{\mathbb{W}^{(\lambda)}(x)}{\mathbb{W}^{(\lambda)}(a)}\\
&\quad+\int_0^a(\lambda-\omega(y))\mathcal{A}(y,a)\Xi_\phi(y)^{-1}\Big(\frac{\mathbb{W}^{(\lambda)}(x)}{\mathbb{W}^{(\lambda)}(a)}\mathbb{W}^{(\lambda)}(a;y)-\mathbb{W}^{(\lambda)}(x;y)\Big)\mathrm{d}y\\
&= \frac{\mathbb{W}^{(\lambda)}(x)}{\mathbb{W}^{(\lambda)}(a)}\Big(1+\int_0^a(\lambda-\omega(y))\mathcal{A}(y,a)\Xi_\phi(y)^{-1}\mathbb{W}^{(\lambda)}(a;y)\mathrm{d}y\Big)\\
&\quad-\int_0^x(\lambda-\omega(y))\mathcal{A}(y,a)\Xi_\phi(y)^{-1}\mathbb{W}^{(\lambda)}(x;y)\mathrm{d}y,
\end{aligned}
$$

By using Eq. (3.7.9), the above equation becomes

$$
\begin{aligned}
\mathcal{A}(x,a) &= \frac{\mathbb{W}^{(\lambda)}(x)}{\mathbb{W}^{(\lambda)}(a)}\Big(1+\int_0^a(\lambda-\omega(y))\mathcal{A}(y,a)\Xi_\phi(y)^{-1}\mathbb{W}^{(\lambda)}(a;y)\mathrm{d}y\Big)\\
&\quad-\int_0^x(\lambda-\omega(y))\mathcal{A}(y,x)\mathcal{A}(x,a)\Xi_\phi(y)^{-1}\mathbb{W}^{(\lambda)}(x;y)\mathrm{d}y.
\end{aligned}
$$

Rearranging the above equation yields to

$$
\begin{aligned}
&\mathcal{A}(x,a)\Big(1+\int_0^x\mathcal{A}(y,x)(\lambda-\omega(y))\Xi_\phi(y)^{-1}\mathbb{W}^{(\lambda)}(x;y)\mathrm{d}y\Big)\\
&\qquad\qquad= \frac{\mathbb{W}^{(\lambda)}(x)}{\mathbb{W}^{(\lambda)}(a)}\Big(1+\int_0^a\mathcal{A}(y,a)(\lambda-\omega(y))\Xi_\phi(y)^{-1}\mathbb{W}^{(\lambda)}(a;y)\mathrm{d}y\Big).
\end{aligned}
$$

Then, we define

$$\mathcal{W}^{(\omega)}(x) := \mathbb{W}^{(\lambda)}(x)\Big(1+\int_0^x \mathcal{A}(y,x)(\lambda-\omega(y))\Xi_\phi(y)^{-1}\mathbb{W}^{(\lambda)}(x;y)\mathrm{d}y\Big)^{-1}, \quad (3.7.10)$$

however, since $\mathcal{A}(x,a)$ does not depend on λ, we may write the above equation by taking $\lambda = 0$, thus we have that

$$\begin{aligned}\mathcal{A}(x,a)\Big(1-\int_0^x \mathcal{A}(y,x)\omega(y)\Xi_\phi(y)^{-1}\mathbb{W}(x;y)\mathrm{d}y\Big)& \\ = \frac{\mathbb{W}(x)}{\mathbb{W}(a)}\Big(1-\int_0^a \mathcal{A}(y,a)\omega(y)\Xi_\phi(y)^{-1}\mathbb{W}(a;y)\mathrm{d}y\Big).&\end{aligned}$$

Defining

$$\mathcal{W}^{(\omega)}(x) := \mathbb{W}(x)\Big(1-\int_0^x \mathcal{A}(y,x)\omega(y)\Xi_\phi(y)^{-1}\mathbb{W}(x;y)\mathrm{d}y\Big)^{-1},$$

and substituting $\mathcal{A}(y,x) = \mathcal{W}^{(\omega)}(y)/\mathcal{W}^{(\omega)}(x)$, then we obtain the required ω-killed level-dependent scale function given in Eq. (3.7.3). □

Taking the limits of the two-sided exit above given in Theorem 74 as $a \to \infty$ we derive the one-sided exit problem for ω-killed level-dependent Lévy process. For this, we need first the following lemma.

Lemma 75. *(Monotonocity) The scale function $\mathcal{W}^{(q)}$ defined in Eq. (3.7.3) is an increasing function for $x \geq 0$.*

Proof. The monotonocity follows directly from the definition. For any $x > y$, by using Eq. (3.7.3) we obtain

$$\begin{aligned}\mathcal{W}^{(\omega)}(x) - \mathcal{W}^{(\omega)}(y) &= \mathbb{W}(x) - \mathbb{W}(y) + \int_0^x \mathbb{W}(x;z)\Xi_\phi(z)^{-1}\omega(z)\mathcal{W}^{(\omega)}(z)\mathrm{d}z \\ &\quad - \int_0^y \mathbb{W}(y;z)\Xi_\phi(z)^{-1}\omega(z)\mathcal{W}^{(\omega)}(z)\mathrm{d}z \\ &= \mathbb{W}(x) - \mathbb{W}(y) \\ &\quad + \int_0^y [\mathbb{W}(x;z) - \mathbb{W}(y;z)]\Xi_\phi(z)^{-1}\omega(z)\mathcal{W}^{(\omega)}(z)\mathrm{d}z \\ &\quad + \int_y^x \mathbb{W}(x;z)\Xi_\phi(z)^{-1}\omega(z)\mathcal{W}^{(\omega)}(z)\mathrm{d}z > 0,\end{aligned}$$

since $\mathbb{W}(\cdot)$ is monotonic function by Lemma 10 in [13]. □

Corollary 76. *Let* $a_{\mathcal{W}(\infty)^{-1}} := \lim_{a\to\infty} \mathcal{W}^{(\omega)}(a)^{-1}$ *exists and finite. Then, for all* $x \geq 0$*, we have*

$$\mathbb{E}_x\Big[e^{-\int_0^\infty \omega(U_t)\mathrm{d}t}\mathbf{1}_{(\kappa_0^-=\infty)}\Big] = a_{\mathcal{W}(\infty)^{-1}}\mathcal{W}^{(\omega)}(x).$$

Proof. The proof of this corollary is trivial since it is the limiting case of Theorem 74 as $a \to \infty$. However, we discuss the finiteness of $\lim_{a\to\infty} \mathcal{W}^{(\omega)}(a)^{-1}$. The limit is well-defined and finite, since $\mathcal{W}^{(\omega)}(a)$ is the monotonic function of a by Lemma 75.

□

In the next theorem, we derive identities for the resolvent of ω-killed level-dependent SNLP before exiting from some interval. To do this we recall that for a Markov process $Z = \{Z_t, t \geq 0\}$ with killing time ζ, and f a non-negative bounded continuous function on $\mathbb{R}$ such that $\int_0^\infty \mathbb{E}_x[f(Z_t)1_{(t<\zeta)}]\mathrm{d}t < \infty$, ω-type resolvent, $K^{(\omega)}$, is given by

$$K^{(\omega)}f(x) := \int_0^\infty Q_t^{(\omega)}f(x)\mathrm{d}t, \quad \text{with} \quad Q_t^{(\omega)}f(x) := \mathbb{E}_x\Big[e^{-\int_0^t \omega(Z_s)\mathrm{d}s}f(Z_t)\mathbf{1}_{(t<\zeta)}\Big].$$

Then, (see Lemma 4.1 in [26]) $K^{(\omega)}f(x)$ is finite and satisfies the following equation

$$K^{(\omega)}f(x) = K^{(0)}\big(f - \omega K^{(\omega)}f\big)(x). \tag{3.7.11}$$

We point out that we choose similar approach as in [26], i.e. using the ω-type resolvent identity in Eq. (3.7.11), which is significantly simpler than the corresponding arguments in [13].

Theorem 77 (Resolvent of ω-killed level-dependent)**.** *For* $x, y \in [0, a]$*, we have*

$$\begin{aligned}\mathcal{U}^{(\omega)}(x, \mathrm{d}y) &:= \int_0^\infty \mathbb{E}_x\big[e^{-\int_0^t \omega(U_s)\mathrm{d}s}\mathbf{1}_{(U_t\in\mathrm{d}y,\, t<\kappa_0^-\wedge\kappa_a^+)}\big]\mathrm{d}t \\ &= \Xi_\phi(y)^{-1}\Big(\frac{\mathcal{W}^{(\omega)}(x)}{\mathcal{W}^{(\omega)}(a)}\mathcal{W}^{(\omega)}(a, y) - \mathcal{W}^{(\omega)}(x, y)\Big)\mathrm{d}y, \end{aligned} \tag{3.7.12}$$

where $\mathcal{W}^{(\omega)}(\cdot, \cdot)$ *is solution to Eq. (3.7.1) for* $q = 0$.

Proof. Let f is non-negative bounded continuous function, we define

$$\mathcal{U}^{(\omega)}f(x) := \int_0^\infty \mathbb{E}_x\Big[f(U_t)e^{-\int_0^t \omega(U_s)\mathrm{d}s}\mathbf{1}_{(t<\kappa_0^-\wedge\kappa_a^+)}\Big]\mathrm{d}t.$$

Then, by using Eq. (3.7.11) for $\zeta = \kappa_0^- \wedge \kappa_a^+$, we have

$$\begin{aligned}
\mathcal{U}^{(\omega)} f(x) &= \int_0^\infty Q_t^{(\omega)} f(x) \mathrm{d}t \,\big|_{\zeta=\kappa_0^- \wedge \kappa_a^+} = K^{(\omega)} f(x) \\
&= K^{(0)}\big(f - \omega(\cdot) K^{(\omega)} f\big)(x) \\
&= \int_0^\infty \mathbb{E}_x\Big[\big(f(U_t) - \omega(U_t) K^{(\omega)} f(U_t)\big) \mathbf{1}_{(t<\kappa_0^- \wedge \kappa_a^+)}\Big] \mathrm{d}t \\
&= \int_0^\infty \mathbb{E}_x\Big[\big(f(U_t) - \omega(U_t) \mathcal{U}^{(\omega)} f(U_t)\big) \mathbf{1}_{(t<\kappa_0^- \wedge \kappa_a^+)}\Big] \mathrm{d}t \\
&= \int_0^\infty \int_0^a \big(f(y) - \omega(y) \mathcal{U}^{(\omega)} f(y)\big) \mathbb{P}_x(U_t \in \mathrm{d}y,\ t < \kappa_0^- \wedge \kappa_a^+) \mathrm{d}t \\
&= \int_0^a \big(f(y) - \omega(y) \mathcal{U}^{(\omega)} f(y)\big) \int_0^\infty \mathbb{P}_x(U_t \in \mathrm{d}y,\ t < \kappa_0^- \wedge \kappa_a^+) \mathrm{d}t,
\end{aligned}$$

where the last integral in the above equation is the resolvent measure of level-dependent Lévy process killed on exiting the interval $[0, a]$ which is known by Eq. (3.5.14) with $d = 0$ and $q = 0$. Hence,

$$\begin{aligned}
\mathcal{U}^{(\omega)} f(x) &= \int_0^a \big(f(y) - \omega(y) \mathcal{U}^{(\omega)} f(y)\big) \Xi_\phi(y)^{-1} \Big(\frac{\mathbb{W}(x)}{\mathbb{W}(a)} \mathbb{W}(a; y) - \mathbb{W}(x; y)\Big) \mathrm{d}y \\
&= \mathbb{W}(x) \int_0^a \big(f(y) - \omega(y) \mathcal{U}^{(\omega)} f(y)\big) \Xi_\phi(y)^{-1} \frac{\mathbb{W}(a; y)}{\mathbb{W}(a)} \mathrm{d}y \\
&\quad - \int_0^x \mathbb{W}(x; y) \Xi_\phi(y)^{-1} f(y) \mathrm{d}y + \int_0^x \mathbb{W}(x; y) \Xi_\phi(y)^{-1} \omega(y) \mathcal{U}^{(\omega)} f(y) \mathrm{d}y,
\end{aligned}$$

where the limits of the last integral follow from the fact that $\mathbb{W}(x; y) = 0$ for $y \geq x$.

Defining

$$a_U := \int_0^a \big(f(y) - \omega(y) \mathcal{U}^{(\omega)} f(y)\big) \Xi_\phi(y)^{-1} \frac{\mathbb{W}(a; y)}{\mathbb{W}(a)} \mathrm{d}y,$$

the above equation becomes

$$\begin{aligned}
\mathcal{U}^{(\omega)} f(x) = a_U \mathbb{W}(x) &- \int_0^x \mathbb{W}(x; y) \Xi_\phi(y)^{-1} f(y) \mathrm{d}y \\
&+ \int_0^x \mathbb{W}(x; y) \Xi_\phi(y)^{-1} \omega(y) \mathcal{U}^{(\omega)} f(y) \mathrm{d}y. \quad (3.7.13)
\end{aligned}$$

Now, we define the operator

$$\mathcal{R}^{(\omega)} f(x) := \int_0^x f(y) \Xi_\phi(y)^{-1} \mathcal{W}^{(\omega)}(x, y) \mathrm{d}y, \quad x > 0, \tag{3.7.14}$$

with $\mathcal{R}^{(\omega)} f(x) = 0$ for $x \leq 0$. Then, by recalling the (general version) ω-killed level-dependent scale function $\mathcal{W}^{(\omega)}(x, y)$ given in Eq. (3.7.1) for $q = 0$, we obtain

$$\mathcal{R}^{(\omega)} f(x) = \int_0^x f(y) \Xi_\phi(y)^{-1} \mathbb{W}(x; y) \mathrm{d}y$$

$$
\begin{aligned}
&+\int_0^x f(y)\Xi_\phi(y)^{-1}\int_y^x \mathbb{W}(x;z)\Xi_\phi(z)^{-1}\omega(z)\mathcal{W}^{(\omega)}(z,y)\mathrm{d}z\mathrm{d}y\\
&=\int_0^x f(y)\Xi_\phi(y)^{-1}\mathbb{W}(x;y)\mathrm{d}y\\
&\quad+\int_0^x\int_0^z f(y)\Xi_\phi(y)^{-1}\mathbb{W}(x;z)\Xi_\phi(z)^{-1}\omega(z)\mathcal{W}^{(\omega)}(z,y)\mathrm{d}y\mathrm{d}z\\
&=\int_0^x f(y)\Xi_\phi(y)^{-1}\mathbb{W}(x;y)\mathrm{d}y\\
&\quad+\int_0^x \mathbb{W}(x;z)\omega(z)\Xi_\phi(z)^{-1}\int_0^z f(y)\Xi_\phi(y)^{-1}\mathcal{W}^{(\omega)}(z,y)\mathrm{d}y\mathrm{d}z\\
&=\int_0^x f(y)\Xi_\phi(y)^{-1}\mathbb{W}(x;y)\mathrm{d}y+\int_0^x \mathbb{W}(x;z)\Xi_\phi(z)^{-1}\omega(z)\mathcal{R}^{(\omega)}f(z)\mathrm{d}z,
\end{aligned}
$$

or equivalently we write

$$
\mathcal{R}^{(\omega)}f(x)=\int_0^x \mathbb{W}(x;y)\Xi_\phi(y)^{-1}f(y)\mathrm{d}y+\int_0^x \mathbb{W}(x;y)\Xi_\phi(y)^{-1}\omega(y)\mathcal{R}^{(\omega)}f(y)\mathrm{d}y, \tag{3.7.15}
$$

which proves that the operator $\mathcal{R}^{(\omega)}f(x)$ uniquely defines a renewal-type equation.

Further, using Eq. (3.7.3) and Eq. (3.7.15), we have that

$$
\begin{aligned}
a_u\mathcal{W}^{(q)}(x)-\mathcal{R}^{(\omega)}f(x)&=a_U\Big(\mathbb{W}(x)+\int_0^x \mathbb{W}(x;y)\omega(y)\Xi_\phi(y)^{-1}\mathcal{W}^{(\omega)}(y)\mathrm{d}y\Big)\\
&\quad-\int_0^x \mathbb{W}(x;y)\Xi_\phi(y)^{-1}f(y)\mathrm{d}y-\int_0^x \mathbb{W}(x;y)\omega(y)\Xi_\phi(y)^{-1}\mathcal{R}^{(\omega)}f(y)\mathrm{d}y\\
&=a_U\mathbb{W}(x)-\int_0^x \mathbb{W}(x;y)\Xi_\phi(y)^{-1}f(y)\mathrm{d}y\\
&\quad+\int_0^x \mathbb{W}(x;y)\omega(y)\Xi_\phi(y)^{-1}[a_U\mathcal{W}^{(\omega)}(y)-\mathcal{R}^{(\omega)}f(y)]\mathrm{d}y.
\end{aligned}
$$

Since the form of the above equation is same with Eq. (3.7.13), and also using Proposition 72 gives that both renewal equations are uniquely defined, we then conclude that

$$
\mathcal{U}^{(\omega)}f(x)=a_U\mathcal{W}^{(\omega)}(x)-\mathcal{R}^{(\omega)}f(x), \tag{3.7.16}
$$

with boundary condition $\mathcal{U}^{(\omega)}f(a)=0$ since the resolvent measure of level-dependent Lévy process killed on exiting $[0,a]$ given in Eq. (3.5.14) is zero. Thus, employing the boundary condition, we find the constant a_U, given by

$$
a_U=\frac{\mathcal{R}^{(\omega)}f(a)}{\mathcal{W}^{(\omega)}(a)}=\frac{\int_0^a f(y)\Xi_\phi(y)^{-1}\mathcal{W}^{(\omega)}(a,y)\mathrm{d}y}{\mathcal{W}^{(\omega)}(a)}.
$$

Now substituting the above form of a_U into Eq. (3.7.16), we obtain that

$$\mathscr{U}^{(\omega)}f(x) = \int_0^\infty f(y)\Xi_\phi(y)^{-1}\Big(\frac{\mathscr{W}^{(\omega)}(a,y)}{\mathscr{W}^{(\omega)}(a)}\mathscr{W}^{(\omega)}(x) - \mathscr{W}^{(\omega)}(x,y)\Big)\mathrm{d}y. \tag{3.7.17}$$

Finally, by definition of $\mathscr{U}^{(\omega)}f(x)$

$$\begin{aligned}\mathscr{U}^{(\omega)}f(x) &= \int_0^\infty \mathbb{E}_x\Big[f(U_t)e^{-\int_0^t \omega(U_s)\mathrm{d}s}\mathbf{1}_{(t<\kappa_0^-\wedge\kappa_a^+)}\Big]\mathrm{d}t\\ &= \int_0^a f(y)\int_0^\infty \mathbb{E}_x\Big[e^{-\int_0^t \omega(U_s)\mathrm{d}s}\mathbf{1}_{(U_t\in\mathrm{d}y,\ t<\kappa_0^-\wedge\kappa_a^+)}\Big]\mathrm{d}t\\ &= \int_0^a f(y)\,\mathscr{U}^{(\omega)}(x,\mathrm{d}y),\end{aligned}$$

and comparing the above equation with Eq. (3.7.17) gives the required identity by denoting density of $\mathscr{U}^{(\omega)}f(x)$ as $\mathscr{U}^{(\omega)}(x,\mathrm{d}y)$. ☐

Now, using Theorem 77, we obtain the following identity for the two-sided exit from below.

Theorem 78 (Two-Sided Exit Problem Below)**.** *For $x \le a$, we have*

$$\mathscr{B}(x,a) := \mathbb{E}_x\Big[e^{-\int_0^{\kappa_0^-}\omega(U_t)\mathrm{d}t}\mathbf{1}_{(\kappa_0^-<\kappa_a^+)}\Big] = \mathscr{Z}^{(\omega)}(x) - \frac{\mathscr{W}^{(\omega)}(x)}{\mathscr{W}^{(\omega)}(a)}\mathscr{Z}^{(\omega)}(a), \tag{3.7.18}$$

where $\mathscr{W}^{(\omega)}(\cdot)$ and $\mathscr{Z}^{(\omega)}(\cdot)$ are given in Eq. (3.7.3) and Eq. (3.7.4), respectively.

Proof. First note that recalling the definition of $\mathscr{Z}^{(\omega)}(x)$ and using Proposition 72, we have that

$$\mathscr{Z}^{(\omega)}(x) = 1 + \int_0^x \mathscr{W}^{(\omega)}(x,z)\Xi_\phi(z)^{-1}\omega(z)\mathrm{d}z. \tag{3.7.19}$$

Further, using Eq. (3.7.17), it yields that

$$\begin{aligned}\mathscr{U}^{(\omega)}\omega(x) &= \int_0^a \omega(y)\Xi_\phi(y)^{-1}\Big(\frac{\mathscr{W}^{(\omega)}(x)}{\mathscr{W}^{(\omega)}(a)}\mathscr{W}^{(\omega)}(a,y) - \mathscr{W}^{(\omega)}(x,y)\Big)\mathrm{d}y\\ &= \frac{\mathscr{W}^{(\omega)}(x)}{\mathscr{W}^{(\omega)}(a)}\big(\mathscr{Z}^{(\omega)}(a)-1\big) - \big(\mathscr{Z}^{(\omega)}(x)-1\big),\end{aligned}$$

where the last equation follows from Eq. (3.7.19). On the other hand, using Theorem 74, noticing that

$$\begin{aligned}\mathscr{U}^{(\omega)}\omega(x) &= \int_0^\infty \mathbb{E}_x\Big[\omega(U_t)e^{-\int_0^t \omega(U_s)\mathrm{d}s}\mathbf{1}_{(t<\kappa_0^-\wedge\kappa_a^+)}\Big]\mathrm{d}t\\ &= \mathbb{E}_x\Big[\int_0^{\kappa_0^-\wedge\kappa_a^+}\omega(U_t)e^{-\int_0^t \omega(U_s)\mathrm{d}s}\mathrm{d}t\Big]\end{aligned}$$

$$
\begin{aligned}
&= \mathbb{E}_x\Big[-\int_0^{\kappa_0^- \wedge \kappa_a^+} \mathrm{d}\big(e^{-\int_0^t \omega(U_s)\mathrm{d}s}\big)\Big] \\
&= \mathbb{E}_x\Big[-e^{-\int_0^{\kappa_0^- \wedge \kappa_a^+} \omega(U_s)\mathrm{d}s} + e^0\Big] = 1 - \mathbb{E}_x\Big[e^{-\int_0^{\kappa_0^- \wedge \kappa_a^+} \omega(U_s)\mathrm{d}s}\Big] \\
&= 1 - \mathbb{E}_x\Big[e^{-\int_0^{\kappa_a^+} \omega(U_s)\mathrm{d}s}\mathbf{1}_{(\kappa_a^+ < \kappa_0^-)}\Big] - \mathbb{E}_x\Big[e^{-\int_0^{\kappa_0^-} \omega(U_s)\mathrm{d}s}\mathbf{1}_{(\kappa_0^- < \kappa_a^+)}\Big] \\
&= 1 - \frac{\mathcal{W}^{(\omega)}(x)}{\mathcal{W}^{(\omega)}(a)} - \mathcal{B}(x,a),
\end{aligned}
$$

and combining the above two equations, the result follows. □

Remark 79. *We point out that by considering the Eqs. (3.7.1) – (3.7.2), when we choose* $\omega(x) = q$ *the two-sided exit from above and below given in Theorem 74 and Theorem 78 turn out to be the two-sided exit identities of level-dependent Lévy processes with general rate function* $\phi(\cdot)$ *(see Theorem 31 in [13]).*

Now, by applying Theorem 74 and Theorem 78 and shifting arguments, we obtain the exit identities for and interval $[y, z]$. The remark above allows us to present the following corollary.

Corollary 80. *For* $y \leq x \leq z$

$$
\mathbb{E}_x\Big[e^{-\int_0^{\kappa_z^+} \omega(U_t)\mathrm{d}t}\mathbf{1}_{(\kappa_z^+ < \kappa_y^-)}\Big] = \frac{\mathcal{W}^{(\omega)}(x,y)}{\mathcal{W}^{(\omega)}(z,y)}, \tag{3.7.20}
$$

$$
\mathbb{E}_x\Big[e^{-\int_0^{\kappa_y^-} \omega(U_t)\mathrm{d}t}\mathbf{1}_{(\kappa_y^- < \kappa_z^+)}\Big] = \mathcal{Z}^{(\omega)}(x,y) - \frac{\mathcal{W}^{(\omega)}(x,y)}{\mathcal{W}^{(\omega)}(z,y)}\mathcal{Z}^{(\omega)}(z,y), \tag{3.7.21}
$$

where $\mathcal{W}^{(\omega)}(x,y)$ *and* $\mathcal{Z}^{(\omega)}(x,y)$ *are defined in Eq. (3.7.1) and Eq. (3.7.2) respectively for* $q = 0$.

3.7.3 One-Sided Upwards Exit Problem and Its Resolvent

To solve one-sided upwards exit problem, we additionally assume that

$$
\omega(x) = p, \quad \text{for all} \quad x \leq 0, \tag{3.7.22}
$$

and we define the function $\mathcal{H}^{(\omega)}(x)$ on $\mathbb{R}$ satisfying the following integral equation

$$
\mathcal{H}^{(\omega)}(x) = u^{(p)}(x) + \int_0^x \mathbb{W}^{(p)}(x;y)\Xi_\phi(y)^{-1}(\omega(y) - p)\mathcal{H}^{(\omega)}(y)\mathrm{d}y. \tag{3.7.23}
$$

where $u^{(p)}(x) := \lim_{\gamma\to\infty} \frac{\mathbb{W}^{(p)}(x;-\gamma)}{W^{(p)}(\gamma)}$ is given by Theorem 56.

Theorem 81. *Assume that Eq. (3.7.22) holds. Then, for $x \le a$,*

$$\mathbb{E}_x\Big[e^{-\int_0^{\kappa_a^+}\omega(U_s)\mathrm{d}s}\mathbf{1}_{(\kappa_a^+<\infty)}\Big] = \frac{\mathcal{H}^{(\omega)}(x)}{\mathcal{H}^{(\omega)}(a)}, \tag{3.7.24}$$

and the corresponding resolvent for $x, y \le a$,

$$\begin{aligned}\chi^{(\omega)}(x,\mathrm{d}y) &:= \int_0^\infty \mathbb{E}_x\Big[e^{-\int_0^t \omega(U_s)\mathrm{d}s}\mathbf{1}_{(U_t\in\mathrm{d}y,\, t<\kappa_a^+)}\Big]\mathrm{d}t \\ &= \Xi_\phi(y)^{-1}\Big(\frac{\mathcal{H}^{(\omega)}(x)}{\mathcal{H}^{(\omega)}(a)}\mathcal{W}^{(\omega)}(a,y) - \mathcal{W}^{(\omega)}(x,y)\Big)\mathrm{d}y, \end{aligned}\tag{3.7.25}$$

where $\mathcal{H}^{(\omega)}(\cdot)$ is given in Eq. (3.7.23).

Proof. For the one-sided exit upwards problem, we first shift the level below from 0 to $-\gamma$ and then we take limit as $\gamma \to \infty$. Therefore, we employ the following (more general) ω-killed level-dependent scale function by recalling Eq. (3.7.1), for $y = -\gamma$

$$\mathcal{W}^{(\omega)}(x,-\gamma) = \mathbb{W}^{(p)}(x;-\gamma) + \int_{-\gamma}^{x}\mathbb{W}^{(p)}(x;y)\Xi_\phi(y)^{-1}(\omega(y)-p)\mathcal{W}^{(\omega)}(y,-\gamma)\mathrm{d}y,$$

which can be rewritten as

$$\mathcal{W}^{(\omega)}(x,-\gamma) = \mathbb{W}^{(p)}(x;-\gamma) + \int_{0}^{x}\mathbb{W}^{(p)}(x;y)\Xi_\phi(y)^{-1}(\omega(y)-p)\mathcal{W}^{(\omega)}(y,-\gamma)\mathrm{d}y, \tag{3.7.26}$$

after adjusting the limits in the integral as $y \in [-\gamma, 0]$, $\omega(x) = p$ and thus the term $\omega(x) - p$ is equal to zero.

Now, recalling from Corollary 80 that

$$\mathbb{E}_x\Big[e^{-\int_0^{\kappa_a^+}\omega(U_t)\mathrm{d}t}\mathbf{1}_{(\kappa_a^+<\kappa_{-\gamma}^-)}\Big] = \frac{\mathcal{W}^{(\omega)}(x,-\gamma)}{\mathcal{W}^{(\omega)}(a,-\gamma)},$$

and taking the limit as $\gamma \to \infty$, we get that

$$\lim_{\gamma\to\infty}\mathbb{E}_x\Big[e^{-\int_0^{\kappa_a^+}\omega(U_t)\mathrm{d}t}\mathbf{1}_{(\kappa_a^+<\kappa_{-\gamma}^-)}\Big] = \mathbb{E}_x\Big[e^{-\int_0^{\kappa_a^+}\omega(U_t)\mathrm{d}t}\mathbf{1}_{(\kappa_a^+<\infty)}\Big] = \lim_{\gamma\to\infty}\frac{\mathcal{W}^{(\omega)}(x,-\gamma)}{\mathcal{W}^{(\omega)}(a,-\gamma)},$$

Hence, it suffices to prove that

$$\lim_{\gamma\to\infty}\frac{\mathcal{W}^{(\omega)}(x,-\gamma)}{\mathcal{W}^{(\omega)}(a,-\gamma)} = \frac{\mathcal{H}^{(\omega)}(x)}{\mathcal{H}^{(\omega)}(a)}. \tag{3.7.27}$$

In order to show this, from Corollary 80, we have

$$\mathbb{E}\Big[e^{-\int_0^{\kappa_x^+}\omega(U_t)\mathrm{d}t}\mathbf{1}_{(\kappa_x^+<\kappa_{-\gamma}^-)}\Big] = \frac{\mathcal{W}^{(\omega)}(0,-\gamma)}{\mathcal{W}^{(\omega)}(x,-\gamma)},$$

or equivalently,

$$\mathcal{W}^{(\omega)}(x,-\gamma) = \mathcal{W}^{(\omega)}(0,-\gamma)\Big(\mathbb{E}\Big[e^{-\int_0^{\kappa_x^+}\omega(U_t)\mathrm{d}t}\mathbf{1}_{(\kappa_x^+<\kappa_{-\gamma}^-)}\Big]\Big)^{-1}. \tag{3.7.28}$$

Hence, the limit of the above equation should be identified. To calculate this limit, by employing general scale function given in Eq. (3.7.26), we observe that for $x \in [-\gamma, 0]$, the integral vanishes thus we have

$$\mathcal{W}^{(\omega)}(x,-\gamma) = \mathbb{W}^{(p)}(x;-\gamma).$$

Then, multiplying both sides of the above equation with $1/W^{(p)}(\gamma)$ and taking limit as $\gamma \to \infty$ by using Eq. (3.5.13) gives that

$$\lim_{\gamma\to\infty}\frac{\mathcal{W}^{(\omega)}(x,-\gamma)}{W^{(p)}(\gamma)} = \lim_{\gamma\to\infty}\frac{\mathbb{W}^{(p)}(x;-\gamma)}{W^{(p)}(\gamma)} = u^{(p)}(x),$$

and for $x = 0$, it yields that

$$\lim_{\gamma\to\infty}\frac{\mathcal{W}^{(\omega)}(0,-\gamma)}{W^{(p)}(\gamma)} = u^{(p)}(0) = 1.$$

Moreover, since the expectation in Eq. (3.7.28) is (monotone) increasing with respect to γ, and the limit is well defined and finite for $x \geq -\gamma$, which gives

$$\lim_{\gamma\to\infty}\frac{\mathcal{W}^{(\omega)}(x,-\gamma)}{W^{(p)}(\gamma)} = \Big(\mathbb{E}\Big[e^{-\int_0^{\kappa_x^+}\omega(U_t)\mathrm{d}t}\mathbf{1}_{(\kappa_x^+<\infty)}\Big]\Big)^{-1}.$$

Finally, defining

$$\mathcal{H}^{(\omega)}(x) := \lim_{\gamma\to\infty}\frac{\mathcal{W}^{(\omega)}(x,-\gamma)}{W^{(p)}(\gamma)},$$

completes the first part of the theorem, i.e. proof of Eq. (3.7.27).

Furthermore, we show that the above-defined $\mathcal{H}^{(\omega)}(x)$ satisfies Eq. (3.7.23) by multiplying Eq. (3.7.26) with $1/W^{(p)}(\gamma)$ and taking limit as $\gamma \to \infty$

$$\begin{aligned}\lim_{\gamma\to\infty}\frac{\mathcal{W}^{(\omega)}(x,-\gamma)}{W^{(p)}(\gamma)} &= \lim_{\gamma\to\infty}\frac{\mathbb{W}^{(p)}(x;-\gamma)}{W^{(p)}(\gamma)}\\ &\quad + \int_0^x \mathbb{W}^{(p)}(x;y)\Xi_\phi(y)^{-1}(\omega(y)-p)\lim_{\gamma\to\infty}\frac{\mathcal{W}^{(\omega)}(y,-\gamma)}{W^{(p)}(\gamma)}\mathrm{d}y,\end{aligned}$$

by using in the last line dominated convergence theorem, we obtain the required equation defined in Eq. (3.7.23).

Finally, to prove the resolvent identity in Eq. (3.7.25), recalling Eq. (3.7.12) and shifting to $-\gamma$, we have that

$$\int_0^\infty \mathbb{E}_x\big[e^{-\int_0^t \omega(U_s)\mathrm{d}s}\mathbf{1}_{(U_t\in \mathrm{d}y,\, t<\kappa^-_{-\gamma}\wedge\kappa^+_a)}\big]\mathrm{d}t$$
$$= \Xi_\phi(y)^{-1}\Big(\frac{\mathcal{W}^{(\omega)}(x,-\gamma)}{\mathcal{W}^{(\omega)}(a,-\gamma)}\mathcal{W}^{(\omega)}(a,y)-\mathcal{W}^{(\omega)}(x,y)\Big)\mathrm{d}y,$$

and taking limit of the both sides as $\gamma\to\infty$ of the above equation and using Eq. (3.7.27) yields to required result. □

Remark 82. *Note that $\mathcal{H}^{(\omega)}(x)=u^{(p)}(x)$ for $x\leq 0$ due to (the assumption given in) Eq. (3.7.22). In this case, Eq. (3.7.24) simply implies to one-sided upwards problem for the level-dependent Lévy processes (see Theorem 3.2(ii) in [13]).*

3.7.4 Reflected Omega-Killed Level-Dependent Spectrally Negative Lévy Processes

In this subsection, we derive exit times and resolvents of ω-killed level-dependent Lévy process reflected at its infimum. In Section 3.6.2, we define reflected level-dependent Lévy processes with a general ϕ as

$$V_t = X_t + R_t - \int_0^t \phi(V_s)\mathrm{d}s,$$

where $R_t := \sup_{0\leq s\leq t}(-V_s)\vee 0 = (-\inf_{0\leq s\leq t}(V_s))\vee 0$. Then, we obtained the exit identities of such processes.

In the literature, exit identities for the reflected spectrally negative Lévy processes is studied in [4, 32] by using combination of excursion theory, Itô calculus and martingale property of the scale functions and in [14] by using directly excursion theory. For ω-killed spectrally negative Lévy processes, all the identities for reflected processes are generalized to ω-killed version in [26] by using Eq. (3.7.11).

Now, we present one-sided exit problem for the ω-killed level-dependent Lévy process reflected at its infimum and the corresponding resolvent.

Theorem 83 (One-sided exit for reflected at infimum)**.** *For $0\leq x\leq a$, we have*

$$\mathcal{C}(x,a) := \mathbb{E}_x\big[e^{-\int_0^{K_a}\omega(V_t)\mathrm{d}t}\mathbf{1}_{(K_a<\infty)}\big] = \frac{\mathcal{Z}^{(\omega)}(x)}{\mathcal{Z}^{(\omega)}(a)}, \qquad (3.7.29)$$

where $\mathcal{Z}^{(\omega)}(\cdot)$ is defined in Eq. (3.7.4).

Proof. For all $0 \le x \le y \le z$, by applying strong Markov property of V_t and using the fact that V_t has no positive jumps, we have

$$\begin{aligned}\mathbb{E}_x\big[e^{-\int_0^{K_z}\omega(V_t)\mathrm{d}t}\mathbf{1}_{(K_z<\infty)}\big] &= \mathbb{E}_x\big[e^{-\int_0^{K_y}\omega(V_t)\mathrm{d}t}\mathbf{1}_{(K_y<\infty)}\big]\mathbb{E}_{V_{K_y}}\big[e^{-\int_0^{K_z}\omega(V_t)\mathrm{d}t}\mathbf{1}_{(K_z<\infty)}\big] \\ &= \mathbb{E}_x\big[e^{-\int_0^{K_y}\omega(V_t)\mathrm{d}t}\mathbf{1}_{(K_y<\infty)}\big]\mathbb{E}_y\big[e^{-\int_0^{K_z}\omega(V_t)\mathrm{d}t}\mathbf{1}_{(K_z<\infty)}\big],\end{aligned}$$

or equivalently by using the definition of $\mathcal{C}(x,\cdot)$

$$\mathcal{C}(x,z) = \mathcal{C}(x,y)\mathcal{C}(y,z), \quad \text{for all} \quad 0 \le x \le y \le z. \tag{3.7.30}$$

Moreover, with the similar fashion with proof of Theorem 78, we obtain

$$\begin{aligned}1 - \mathcal{C}(x,a) &= \mathbb{E}_x\Big[\int_0^{K_a} e^{-\int_0^t \omega(V_s)\mathrm{d}s}\omega(V_t)\mathrm{d}t\Big] \\ &= \int_0^\infty \mathbb{E}_x\Big[\omega(V_t)e^{-\int_0^t \omega(V_s)\mathrm{d}s}\mathbf{1}_{(t<K_a)}\Big]\mathrm{d}t.\end{aligned}$$

Then, by using Eq. (3.7.11) with $\zeta = K_a$ and $f(\cdot) = \omega(\cdot)$, we have

$$\begin{aligned}1 - \mathcal{C}(x,a) &= K^{(0)}\big(\omega(\cdot) - \omega(\cdot)K^{(\omega)}\omega(\cdot)\big)(x) = K^{(\omega)}\omega(x) \\ &= \int_0^\infty \int_0^a \big(\omega(y) - \omega(y)(1-\mathcal{C}(y,a))\big)\mathbb{P}_x(V_t \in \mathrm{d}y,\ t < K_a)\mathrm{d}t \\ &= \int_0^a \big(\omega(y) - \omega(y)(1-\mathcal{C}(y,a))\big)\int_0^\infty \mathbb{P}_x(V_t \in \mathrm{d}y,\ t < K_a)\mathrm{d}t.\end{aligned}$$

Thus,

$$\mathcal{C}(x,a) = 1 - \int_0^a \omega(y)\mathcal{C}(y,a)\int_0^\infty \mathbb{P}_x(V_t \in \mathrm{d}y,\ t < K_a)\mathrm{d}t,$$

where $\mathbb{P}_x(V_t \in \mathrm{d}y,\ t < K_a)$ is the resolvent measure of level-dependent Lévy process reflected at its infimum which is given by Eq. (3.6.21) when $q = 0$. Thus, we have

$$\begin{aligned}\mathcal{C}(x,a) &= 1 - \int_0^a \omega(y)\mathcal{C}(y,a)\Xi_\phi(y)^{-1}\big(\mathbb{W}(a;y) - \mathbb{W}(x;y)\big)\mathrm{d}y \\ &= 1 - \int_0^a \mathbb{W}(a;y)\Xi_\phi(y)^{-1}\omega(y)\mathcal{C}(y,a)\mathrm{d}y \\ &\qquad - \int_0^x \mathbb{W}(x;y)\Xi_\phi(y)^{-1}\omega(y)\mathcal{C}(y,a)\mathrm{d}y \\ &= 1 - \int_0^a \mathbb{W}(a;y)\Xi_\phi(y)^{-1}\omega(y)\mathcal{C}(y,a)\mathrm{d}y \\ &\qquad - \mathcal{C}(x,a)\int_0^x \mathbb{W}(x;y)\Xi_\phi(y)^{-1}\omega(y)\mathcal{C}(y,x)\mathrm{d}y.\end{aligned}$$

Rearranging the above equation yields to

$$\mathcal{C}(x,a)\big[1-\int_0^x \mathbb{W}(x;y)\Xi_\phi(y)^{-1}\omega(y)\mathcal{C}(y,x)\mathrm{d}y\big]$$
$$=1-\int_0^a \mathbb{W}(a;y)\Xi_\phi(y)^{-1}\omega(y)\mathcal{C}(y,a)\mathrm{d}y.$$

Defining

$$\mathcal{Z}^{(\omega)}(x):=\Big(1-\int_0^x \mathbb{W}(x;y)\Xi_\phi(y)^{-1}\omega(y)\mathcal{C}(y,x)\mathrm{d}y\Big)^{-1}, \tag{3.7.31}$$

gives that

$$\mathcal{C}(x,a)=\frac{\mathcal{Z}^{(\omega)}(x)}{\mathcal{Z}^{(\omega)}(a)}, \quad \text{for} \quad 0\le x\le a.$$

Replacing $\mathcal{C}(y,x)$ by $\mathcal{Z}^{(\omega)}(y)/\mathcal{Z}^{(\omega)}(x)$ in Eq. (3.7.31) produces Eq. (3.7.4), that is

$$\mathcal{Z}^{(\omega)}(x)=1+\int_0^x \mathbb{W}(x;y)\Xi_\phi(y)^{-1}\omega(y)\mathcal{Z}(y)\mathrm{d}y,$$

which completes the proof. □

Theorem 84 (Resolvent of ω-killed level-dependent reflected at infimum)**.** *For $x,y\in[0,a]$, we have*

$$\mathcal{L}^{(\omega)}(x,\mathrm{d}y):=\int_0^\infty \mathbb{E}_x\Big[e^{-\int_0^t\omega(V_s)\mathrm{d}s}\mathbf{1}_{(V_t\in\mathrm{d}y,\ t<K_a)}\Big]\mathrm{d}t, \tag{3.7.32}$$

which is a measure absolutely continuous with respect to the Lebesgue measure and has density

$$\ell^{(\omega)}(x,y)=\Xi_\phi(y)^{-1}\Big(\frac{\mathcal{Z}^{(\omega)}(x)}{\mathcal{Z}^{(\omega)}(a)}\mathcal{W}^{(\omega)}(a,y)-\mathcal{W}^{(\omega)}(x,y)\Big),$$

where $\mathcal{Z}^{(\omega)}(\cdot)$ and $\mathcal{W}^{(\omega)}(\cdot,\cdot)$ are defined in Eq. (3.7.4) and Eq. (3.7.1) for $q=0$, respectively.

Proof. In this proof, we follow the same steps with the proof of Theorem 77. Let f is a non-negative bounded continuous function, we define

$$\mathcal{L}^{(\omega)}f(x):=\int_0^\infty \mathbb{E}_x\Big[f(V_t)e^{-\int_0^t\omega(V_s)\mathrm{d}s}\mathbf{1}_{(t<K_a)}\Big]\mathrm{d}t,$$

Then, using Eq. (3.7.11), we have

$$\begin{aligned}\mathcal{L}^{(\omega)}f(x)&=\int_0^\infty Q_t^{(\omega)}f(x)\mathrm{d}t\ \Big|_{\zeta=K_a}=K^{(\omega)}f(x)\\&=K^{(0)}\big(f-\omega(\cdot)K^{(\omega)}f\big)(x)\end{aligned}$$

$$
\begin{aligned}
&= \int_0^\infty \mathbb{E}_x\Big[\big(f(V_t) - \omega(V_t)K^{(\omega)}f(V_t)\big)\mathbf{1}_{(t<K_a)}\Big]\mathrm{d}t \\
&= \int_0^\infty \mathbb{E}_x\Big[\big(f(V_t) - \omega(V_t)\mathcal{L}^{(\omega)}f(V_t)\big)\mathbf{1}_{(t<K_a)}\Big]\mathrm{d}t \\
&= \int_0^\infty \int_0^a \big(f(y) - \omega(y)\mathcal{L}^{(\omega)}f(y)\big)\mathbb{P}_x(V_t \in \mathrm{d}y,\ t < K_a)\mathrm{d}t \\
&= \int_0^a \big(f(y) - \omega(y)\mathcal{L}^{(\omega)}f(y)\big) \int_0^\infty \mathbb{P}_x(V_t \in \mathrm{d}y,\ t < K_a)\mathrm{d}t \\
&= \int_0^a \big(f(y) - \omega(y)\mathcal{L}^{(\omega)}f(y)\big)\Xi_\phi(y)^{-1}\big(\mathbb{W}(a;y) - \mathbb{W}(x;y)\big)\mathrm{d}y,
\end{aligned}
$$

where the last equality follows from Theorem 68 for $q = 0$ and also by taking into account to Remark 70.

Further algebraic manipulations bring the above equation to

$$
\begin{aligned}
\mathcal{L}^{(\omega)}f(x) = &\int_0^a \mathbb{W}(a;y)\Xi_\phi(y)^{-1}\big(f(y) - \omega(y)\mathcal{L}^{(\omega)}f(y)\big)\mathrm{d}y \\
&- \int_0^a \mathbb{W}(x;y)\Xi_\phi(y)^{-1}f(y)\mathrm{d}y + \int_0^a \mathbb{W}(x;y)\Xi_\phi(y)^{-1}\omega(y)\mathcal{L}^{(\omega)}f(y)\mathrm{d}y.
\end{aligned}
\tag{3.7.33}
$$

Defining

$$
a_L := \int_0^a \mathbb{W}(a;y)\Xi_\phi(y)^{-1}\big(f(y) - \omega(y)\mathcal{L}^{(\omega)}f(y)\big)\mathrm{d}y,
$$

and recalling $\mathcal{R}^{(\omega)}(x)$ given in Eq. (3.7.14) which is uniquely defined renewal type of equation (see Eq. (3.7.15)), we obtain

$$
\begin{aligned}
a_L\mathcal{Z}^{(\omega)}(x) - \mathcal{R}^{(\omega)}(x) = a_L\Big(1 &+ \int_0^x \mathbb{W}(x;y)\Xi_\phi(y)^{-1}\omega(y)\mathcal{Z}^{(\omega)}(y)\mathrm{d}y\Big) \\
&- \int_0^x \mathbb{W}(x;y)\Xi_\phi(y)^{-1}f(y)\mathrm{d}y \\
&- \int_0^x \mathbb{W}(x;y)\Xi_\phi(y)^{-1}\omega(y)\mathcal{R}^{(\omega)}(x)\mathrm{d}y.
\end{aligned}
$$

Since the form of the above equation is the same with Eq. (3.7.33) and by using Proposition 72, we conclude that

$$
\mathcal{L}^{(\omega)}f(x) = a_L\mathcal{Z}^{(\omega)}(x) - \mathcal{R}^{(\omega)}(x), \tag{3.7.34}
$$

with boundary condition $\mathcal{L}^{(\omega)}f(a) = 0$, since the resolvent defined in Eq. (3.6.21) is zero. Then, employing the boundary condition, we find the constant a_L, given by

$$
a_L = \frac{\int_0^a \mathcal{W}^{(\omega)}(a,y)\Xi_\phi(y)^{-1}f(y)\mathrm{d}y}{\mathcal{Z}^{(\omega)}(a)}.
$$

Now, substituting the above equation into Eq. (3.7.34), we obtain that

$$\mathcal{L}^{(\omega)}f(x) = \int_0^a f(y)\Xi_\phi(y)^{-1}\Big(\frac{\mathcal{Z}^{(\omega)}(x)}{\mathcal{Z}^{(\omega)}(a)}\mathcal{W}^{(\omega)}(a,y) - \mathcal{W}^{(\omega)}(x,y)\Big)\mathrm{d}y. \tag{3.7.35}$$

Finally, by definition of $\mathcal{L}^{(\omega)}f(x)$, we have

$$\begin{aligned}\mathcal{L}^{(\omega)}f(x) &= \int_0^\infty \mathbb{E}_x\Big[f(V_t)e^{-\int_0^t \omega(V_s)\mathrm{d}s}\mathbf{1}_{(t<K_a)}\Big]\mathrm{d}t\\ &= \int_0^a f(y)\int_0^\infty \mathbb{E}_x\Big[e^{-\int_0^t \omega(V_s)\mathrm{d}s}\mathbf{1}_{(V_t\in\mathrm{d}y,\, t<K_a)}\Big]\mathrm{d}t\\ &= \int_0^a f(y)\ell^{(\omega)}(x,y)\mathrm{d}y,\end{aligned}$$

and comparing the above equation with Eq. (3.7.35) gives the required identity by denoting density of $\mathcal{L}^{(\omega)}f(x)$ as $\ell^{(\omega)}(x,y)\mathrm{d}y$. □

3.7.5 Bankruptcy Probability for Omega-Killed Level-Dependent Surplus

In this subsection, we demonstrate one application of previously obtained results to the omega model by computing bankruptcy probability for an omega-killed level-dependent Lévy process U with rate function ϕ, under the assumption that $\mathbb{E}(X_1) = \psi'(0+) > 0$; this implies that the bankruptcy does not happen almost surely. In the omega model even with a negative surplus of the insurance company is allowed to continue its business till bankruptcy event occurs, see [18]. Bankruptcy happens either in the red zone, i.e. when the surplus process in $[-d, 0]$ and the bankruptcy rate is a function of the current level, or occur if the surplus process falls below some fixed level $-d < 0$. For further details related to the omega model and its applications to the different processes, we refer to [12, 18, 26].

We consider the omega-killed level-dependent Lévy process as the surplus process of an insurer and $\omega_b(x)$ as the bankruptcy rate function, which is defined as follows

$$\omega_b(x) = \begin{cases}\omega(x), & x\in[-d,0],\\ 0, & x\in(0,\infty).\end{cases} \tag{3.7.36}$$

As seen in the above function, bankruptcy function disappears on the positive half-line so that bankruptcy does not occur when the surplus is positive. For $x < 0$, the quantity of $\omega(x)dt$ describes the probability of bankruptcy within an infinitesimal time units, dt. Then, the bankruptcy probability is defined by

$$\mathcal{P}(x) = 1 - \mathbb{E}_x\left[e^{-\int_0^\infty \omega(U_s)\mathrm{d}s}\mathbf{1}_{(\tau_{-d}^- = \infty)}\right].$$

In the following proposition, we present the probability of bankruptcy.

Proposition 85. *Assume that $x \in [-d, \infty)$,*

(i) If $\mathbb{E}(X_1) \leq 0$, then $\mathcal{P}(x) = 1$ for all x.

(ii) If $\mathbb{E}(X_1) > 0$, and $\int_{-d}^\infty \phi(x)\mathbb{W}'(x;-d)\mathrm{d}x$ exists, then the bankruptcy probability is

$$\mathcal{P}(x) = 1 - a_{\mathcal{W}(\infty,-d)^{-1}}\mathcal{W}^{(\omega)}(x,-d),$$

where

$$a_{\mathcal{W}(\infty,-d)^{-1}} = \frac{1}{A(-d) + \int_{-d}^\infty A(z)\Xi_\phi(z)^{-1}\omega(z)\mathcal{W}^{(\omega)}(z)\mathrm{d}z},$$

with $A(x) := \frac{1}{\mathbb{E}(X_1)}\big(1 + \int_x^\infty \phi(y)\mathbb{W}'(y;-d)\mathrm{d}y\big)$.

Furthermore, $\mathcal{P}(x)$ satisfies the following integral equation

$$1 - \mathcal{P}(x) = a_{\mathcal{W}(\infty,-d)^{-1}}\mathbb{W}(x;-d) + \int_{-d}^x \mathbb{W}(x;z)\Xi_\phi(z)^{-1}\omega_b(z)(1 - \mathcal{P}(z))\mathrm{d}z,$$

and, when $\int_x^\infty \phi(y)\mathbb{W}'(y;-d)\mathrm{d}y = \infty$, it follows that $A(x) = \infty$, hence $\mathcal{P}(x) = 1$.

Proof. For (i) part: it is straightforward since if $\mathbb{E}(X_1) \leq 0$ then $\lim_{a\to\infty} W(a+d) = \infty$, which using the fact that $\mathcal{W}^{(\omega)}(a;-d) \geq \mathbb{W}(a;-d) \geq W(a+d)$, for every $a \geq 0$ implies that

$$\lim_{a\to\infty} \mathcal{W}^{(\omega)}(a;-d) = \infty,$$

Then, it gives $a_{\mathcal{W}(\infty,-d)^{-1}} = 0$ which results in $\mathcal{P}(x) = 1$.

For part (ii), recall that we have

$$\mathcal{W}^{(\omega)}(x,-d) = \mathbb{W}(x;-d) + \int_{-d}^x \mathbb{W}(x;z)\Xi_\phi(z)^{-1}\omega_b(z)\mathcal{W}^{(\omega)}(z,-d)\mathrm{d}z. \quad (3.7.37)$$

Since the bankruptcy rate function is defined as a piecewise function, the equation above will be evaluated as according to interval of x. In that case, we calculate $\mathcal{W}^{(\omega)}(x,-d)$ as

$$\mathcal{W}^{(\omega)}(x,-d) = \begin{cases} \mathcal{W}_\ell^{(\omega)}(x,-d), & x \in [-d,0], \\ \mathcal{W}_u^{(\omega)}(x,-d), & x \in (0,\infty). \end{cases} \tag{3.7.38}$$

Therefore, we have

$$a_{\mathcal{W}(\infty,-d)^{-1}} = \begin{cases} a_{\mathcal{W}_\ell(\infty,-d)^{-1}}, & x \in [-d,0], \\ a_{\mathcal{W}_u(\infty,-d)^{-1}}, & x \in (0,\infty). \end{cases} \tag{3.7.39}$$

For $x \in [-d,0]$, by considering bankruptcy rate function given in Eq. (3.7.36), we can write Eq. (3.7.37) as

$$\mathcal{W}_\ell^{(\omega)}(x,-d) = \mathbb{W}(x;-d) + \int_{-d}^{x} \mathbb{W}(x;z)\Xi_\phi(z)^{-1}\omega(z)\mathcal{W}_\ell^{(\omega)}(z,-d)\mathrm{d}z,$$

and taking limit $a \to \infty$ gives that

$$\lim_{a\to\infty} \mathcal{W}_\ell^{(\omega)}(a;-d) = \lim_{a\to\infty} \mathbb{W}(a;-d) + \lim_{a\to\infty} \int_{-d}^{a} \mathbb{W}(a;z)\Xi_\phi(z)^{-1}\omega(z)\mathcal{W}_\ell^{(\omega)}(z,-d)\mathrm{d}z. \tag{3.7.40}$$

Since the limit of level-dependent scale function when $a \to \infty$ is calculated in [13] (see Proposition 33, Eq.(66)) when $d = 0$, we obtain the shifting case similarly as

$$\lim_{a\to\infty} \mathbb{W}(a;-d) = \frac{1}{\mathbb{E}(X_1)}\big(1 + \int_{-d}^{\infty} \phi(y)\mathbb{W}'(y;-d)\mathrm{d}y\big). \tag{3.7.41}$$

Thus, Eq. (3.7.40) becomes

$$\begin{aligned} &\lim_{a\to\infty} \mathcal{W}_\ell^{(\omega)}(a;-d) \\ &\quad = \frac{1}{\mathbb{E}(X_1)}\big(1 + \int_{-d}^{\infty} \phi(y)\mathbb{W}'(y;-d)\mathrm{d}y\big) \\ &\quad\quad + \int_{-d}^{\infty} \frac{1}{\mathbb{E}(X_1)}\big(1 + \int_{z}^{\infty} \phi(y)\mathbb{W}'(y;-d)\mathrm{d}y\big)\Xi_\phi(z)^{-1}\omega(z)\mathcal{W}^{(\omega)}(z,-d)\mathrm{d}z. \end{aligned}$$

Defining $A(x) := \frac{1}{\mathbb{E}(X_1)}\big(1 + \int_x^\infty \phi(y)\mathbb{W}'(y;-d)\mathrm{d}y\big)$ gives that

$$\lim_{a\to\infty} \mathcal{W}_\ell^{(\omega)}(a;-d) = A(-d) + \int_{-d}^{\infty} A(z)\Xi_\phi(z)^{-1}\omega(z)\mathcal{W}_\ell^{(\omega)}(z,-d)\mathrm{d}z.$$

Thus, we have that

$$a_{\mathcal{W}_\ell(\infty,-d)^{-1}} = \frac{1}{A(-d) + \int_{-d}^{\infty} A(z)\Xi_\phi(z)^{-1}\omega(z)\mathcal{W}_\ell^{(\omega)}(z,-d)\mathrm{d}z}.$$

On the other hand, for $x > 0$, we rewritten Eq. (3.7.37) as

$$\mathcal{W}_u^{(\omega)}(x,-d) = \mathbb{W}(x;-d) + \int_{-d}^{0} \mathbb{W}(x;z)\Xi_\phi(z)^{-1}\omega(z)\mathcal{W}_u^{(\omega)}(z,-d)\mathrm{d}z,$$

since in the positive half-line $\omega_b(x) = 0$, the integral from $[0,x]$ vanishes. Then, similarly to the limiting argument on the interval $x \in [-d,0]$, we get

$$\begin{aligned}
\lim_{a\to\infty} \mathcal{W}_u^{(\omega)}(a;-d) &= \lim_{a\to\infty} \mathbb{W}(a;-d) \\
&\quad + \int_{-d}^{0} \lim_{a\to\infty} \mathbb{W}(a;z)\Xi_\phi(z)^{-1}\omega(z)\mathcal{W}_u^{(\omega)}(z,-d)\mathrm{d}z \\
&= A(-d) + \int_{-d}^{0} A(z)\Xi_\phi(z)^{-1}\omega(z)\mathcal{W}_u^{(\omega)}(z,-d)\mathrm{d}z.
\end{aligned}$$

Then, we also have

$$a_{\mathcal{W}_u(\infty,-d)^{-1}} = \frac{1}{A(-d) + \int_{-d}^{0} A(z)\Xi_\phi(z)^{-1}\omega(z)\mathcal{W}_u^{(\omega)}(z,-d)\mathrm{d}z}.$$

Finally by considering Eq. (3.7.38) and Eq. (3.7.39), we note that

$$\begin{aligned}
\mathcal{P}(x) &= 1 - a_{\mathcal{W}(\infty,-d)^{-1}}\mathcal{W}^{(\omega)}(x,-d) \\
&= 1 - a_{\mathcal{W}(\infty,-d)^{-1}}\big(\mathbb{W}(x;-d) + \int_{-d}^{x} \mathbb{W}(x;z)\Xi_\phi(z)^{-1}\omega_b(z)\mathcal{W}^{(\omega)}(z,-d)\mathrm{d}z\big) \\
&= 1 - a_{\mathcal{W}(\infty,-d)^{-1}}\mathbb{W}(x;-d) \\
&\qquad - \int_{-d}^{x} \mathbb{W}(x;z)\Xi_\phi(z)^{-1}\omega_b(z)a_{\mathcal{W}(\infty,-d)^{-1}}\mathcal{W}^{(\omega)}(z,-d)\mathrm{d}z \\
&= 1 - a_{\mathcal{W}(\infty,-d)^{-1}}\mathbb{W}(x;-d) - \int_{-d}^{x} \mathbb{W}(x;z)\Xi_\phi(z)^{-1}\omega_b(z)(1-\mathcal{P}(z))\mathrm{d}z,
\end{aligned}$$

which gives the following equation (as a survival probability of the bankruptcy)

$$(1-\mathcal{P}(x)) = a_{\mathcal{W}(\infty,-d)^{-1}}\mathbb{W}(x;-d) + \int_{-d}^{x} \mathbb{W}(x;z)\Xi_\phi(z)^{-1}\omega_b(z)(1-\mathcal{P}(z))\mathrm{d}z.$$

□

Remark 86. *The ruin probability for the level-dependent Lévy processes with general rate function ϕ is calculated in [13]. We note that when we consider $d = 0$ and instead of taking $x \geq -d$, if we use $x \geq 0$, Proposition 85 reduces immediately to Proposition 33 of [13].*

CHAPTER 4

CONCLUSION

In this thesis, we derive the fluctuation identities of omega-killed (reflected) Lévy processes, which are exponentially (or geometrically) killed with an intensity (or rate) depending on the current level of the processes.

In the first part of this thesis, we consider right-continuous random walk, which is discrete analogous to the spectrally negative Lévy processes, so-called upwards skip-free random walk. We solve exit identities for both non-reflected and reflected case by using first-step analysis in terms of q-killed scale functions which are recursive equations. The results of this study generalize exit problems of [5] via scale functions paradigm.

In the second part of the thesis, we analyze the exit identities of omega-killed level-dependent SNLP as well as the reflected case. We first study fluctuations of reflected level-dependent SNLP. In order to show the existence of SDE for level-dependent SNLP reflected at its infima, we need to prove the existence of the reflected multi-refracted case. Then, by using an approximating sequence, we prove the level-dependent case. Moreover, the potential measure for reflected level-dependent process is derived by considering the reflected multi-refracted SNLP of the bounded variation first and then showing a strong approximation of the sequence of bounded variation to unbounded variation. Finding potential measure makes it easier to derive the result for one-sided exit problem of both reflected multi-refracted and level-dependent SNLP. The results of this study generalize the study of [31].

Finally, we explore the fluctuation theory of omega-killed level-dependent SNLP in which we model both exponential killing and level-dependent premium via $\omega(\cdot)$ and

$\phi(\cdot)$ functions satisfying some specific conditions. Because of these, our derivations are more general versions of many studies such as [13, 25, 26, 31]. We present all exit identities with a new generalization of scale functions, which are integral equations. We follow classical fluctuation identities and the Poisson observation approach in all derivations. Additionally, we add exit identities, such as potential measure and one-sided exit of reflected omega-killed level-dependent SNLP.

REFERENCES

[1] H. Albrecher, H. U. Gerber, and E. S. Shiu. The optimal dividend barrier in the Gamma–Omega model. *European Actuarial Journal*, 1(1):43–55, 2011.

[2] S. Asmussen and H. Albrecher. *Ruin probabilities*, volume 14. World Scientific Publishing, 2010.

[3] F. Avram, D. Grahovac, and C. Vardar-Acar. The W, Z scale functions kit for first passage problems of spectrally negative Lévy processes, and applications to control problems. *ESAIM. Probability and Statistics*, 24:454–525, 2020.

[4] F. Avram, A. E. Kyprianou, and M. R. Pistorius. Exit problems for spectrally negative Lévy processes and applications to (canadized) russian options. *The Annals of Applied Probability*, 14(1):215–238, 2004.

[5] F. Avram and M. Vidmar. First passage problems for upwards skip-free random walks via the scale functions paradigm. *Advances in Applied Probability*, 51(2):408–424, 2019.

[6] C. Banderier and P. Flajolet. Basic analytic combinatorics of directed lattice paths. *Theoretical Computer Science*, 281(1):37–80, 2002.

[7] A. Bazyari. On the ruin probabilities in a discrete time insurance risk process with capital injections and reinsurance. *Sankhya A.*, 85(2):1623–1650, 2023.

[8] J. Bertoin. *Lévy processes*, volume 121. Cambridge university press Cambridge, 1996.

[9] M. Brown, E. A. Peköz, and S. M. Ross. Some results for skip-free random walk. *Probability in the Engineering and Informational Sciences*, 24(4):491–507, 2010.

[10] T. A. Burton. *Volterra integral and differential equations*, volume 202. Elsevier, 2005.

[11] S. Cheng, H. U. Gerber, and E. S. Shiu. Discounted probabilities and ruin theory in the compound binomial model. *Insurance: Mathematics and Economics*, 26(2):239–250, 2000.

[12] I. Czarna, A. Kaszubowski, S. Li, and Z. Palmowski. Fluctuation identities for Omega-killed spectrally negative Markov additive processes and dividend problem. *Advances in Applied Probability*, 52(2):404–432, 2020.

[13] I. Czarna, J.-L. Pérez, T. Rolski, and K. Yamazaki. Fluctuation theory for level-dependent Lévy risk processes. *Stochastic Processes and their Applications*, 129(12):5406–5449, 2019.

[14] R. A. Doney. Some excursion calculations for spectrally one-sided Lévy processes. In *Séminaire de Probabilités XXXVIII*, pages 5–15. Springer, 2005.

[15] R. A. Doney. *Fluctuation Theory for Levy Processes: Ecole D'Eté de Probabilités de Saint-Flour XXXV-2005*. Springer, 2007.

[16] H. U. Gerber. Mathematical fun with ruin theory. *Insurance: Mathematics and Economics*, 7(1):15–23, 1988.

[17] H. U. Gerber, E. S. Shiu, and H. Yang. An elementary approach to discrete models of dividend strategies. *Insurance: Mathematics and Economics*, 46(1):109–116, 2010.

[18] H. U. Gerber, E. S. Shiu, and H. Yang. The Omega model: from bankruptcy to occupation times in the red. *European Actuarial Journal*, 2(2):259–272, 2012.

[19] F. Hubalek and E. Kyprianou. Old and new examples of scale functions for spectrally negative Lévy processes. In *Seminar on stochastic analysis, random fields and applications VI*, pages 119–145. Springer, 2011.

[20] J. Ivanovs. A note on killing with applications in risk theory. *Insurance: Mathematics and Economics*, 52(1):29–34, 2013.

[21] M. Jacobsen. Exit times for a class of random walks exact distribution results. *Journal of Applied Probability*, 48(A):51–63, 2011.

[22] A. Kuznetsov, A. E. Kyprianou, and V. Rivero. The theory of scale functions for spectrally negative Lévy processes. In *Lévy matters II*, pages 97–186. Springer, 2012.

[23] A. E. Kyprianou. *Gerber–Shiu risk theory*. Springer Science & Business Media, 2013.

[24] A. E. Kyprianou. *Fluctuations of Lévy processes with applications: Introductory Lectures*. Springer Science & Business Media, Berlin, 2014.

[25] A. E. Kyprianou and R. L. Loeffen. Refracted Lévy processes. *Annales de l'Institut Henri Poincaré Probabilités et Statistiques*, 46(1):24–44, 2010.

[26] B. Li and Z. Palmowski. Fluctuations of Omega-killed spectrally negative Lévy processes. *Stochastic Processes and their Applications*, 128(10):3273–3299, 2018.

[27] B. Li and X. Zhou. On weighted occupation times for refracted spectrally negative Lévy processes. *Journal of Mathematical Analysis and Applications*, 466(1):215–237, 2018.

[28] Y. Li and X. Zhou. On pre-exit joint occupation times for spectrally negative Lévy processes. *Statistics & Probability Letters*, 94:48–55, 2014.

[29] Y. Li, X. Zhou, and N. Zhu. Two-sided discounted potential measures for spectrally negative Lévy processes. *Statistics & Probability Letters*, 100:67–76, 2015.

[30] P. Marchal. A combinatorial approach to the two-sided exit problem for left-continuous random walks. *Combinatorics, Probability and Computing*, 10(3):251–266, 2001.

[31] J.-L. Pérez and K. Yamazaki. On the refracted–reflected spectrally negative Lévy processes. *Stochastic Processes and their Applications*, 128(1):306–331, 2018.

[32] M. R. Pistorius. On exit and ergodicity of the spectrally one-sided Lévy process reflected at its infimum. *Journal of Theoretical Probability*, 17(1):183–220, 2004.

[33] M. R. Pistorius. A potential-theoretical review of some exit problems of spectrally negative Lévy processes. In *Séminaire de Probabilités XXXVIII*, pages 30–41. Springer, 2005.

[34] J.-L. Pérez and K. Yamazaki. Refraction–reflection strategies in the dual model. *ASTIN Bulletin*, 47(1):199–238, 2017.

[35] K.-i. Sato. *Lévy processes and infinitely divisible distributions*. Cambridge University Press, Cambridge, 1999.

[36] L. Takács. *Combinatorial methods in the theory of stochastic processes*, volume 126. Wiley, New York, 1966.

[37] R. van der Hofstad, S. Kapodistria, Z. Palmowski, and S. Shneer. Unified approach for solving exit problems for additive-increase and multiplicative-decrease processes. *Journal of Applied Probability*, 60(1):85–105, 2023.

[38] G. E. Willmot. Ruin probabilities in the compound binomial model. *Insurance Mathematics Econom.*, 12(2):133–142, 1993.

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- Şimşek, M., Ramsden, L., & Papaioannou, A. D. (2022). Fluctuations of an Omega-Type Killed Process in Discrete Time, (under review).

- Wang, J., Şimşek, M., Iradukunda, A. E., & Constantinescu, C. (2023) Pricing agriculture parametric insurance against floods in Uganda and Democratic Republic of the Congo: A comparison of Black-Scholes and cluster analysis, (submitted).

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