ISTANBUL TECHNICAL UNIVERSITY ⋆ INSTITUTE OF SCIENCE AND TECHNOLOGY

BOUND ENTANGLEMENT IN QUANTUM INFORMATION THEORY

MSc Thesis by

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Programme : Physics Engineering

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CONTENTS

ABBREVIATIONS

P	:	Positive
CP	:	Completely Positive
P_nCP	:	Positive but not completely positive
NPT	:	Positive partial tranpose
PPT	:	Negative partial tranpose
EW	:	Entanglement witness

LIST OF SYMBOLS

$\|\, . \,\rangle$	:	State vector
ρ, σ	:	Density matrices
$\otimes$	:	Tensor product
U	:	Unitary operator
I	:	Identity operator

SUMMARY

Quantum Information theory uses quantum entanglement as a resource to perform certain tasks. In this theory, density operators representing the states of quantum systems can be characterized as entangled or separable. To do this characterization, positive maps are used as mathematical tools. Besides, not all entangled states are useful in quantum information processing tasks. A taxonomy of entangled states as free (useful) and bound is the problem of distillability. Bound entangled states which cannot be distilled can be activated to produce a non-classical effect.

KUANTUM BİLGİ TEORİSİNDE BAĞLI DOLANIKLIK

ÖZET

Kuantum bilgi teorisi kuantum dolanıklığını, teori içinde yer alan belirli protokolleri uygulayabilmek için kaynak olarak kullanır. Kuantum sistemlerinin durumlarını temsil eden yoğunluk operatörleri, dolanık ya da ayrıştırılabilir olarak sınıflandırılabilir ve bu sınıflandırmayı yapabilmek için pozitif tasvirlerden yaralanılır. Bununla birlikte, tüm dolanık durumlar kuantum bilgi protokollerinde kullanılamaz. Dolanık durumların serbest (kullanışlı) ve bağlı olarak sınıflandırılması damıtılabilirlik problemidir. Damıtılamayan bağlı dolanıklık durumlar, klasik olmayan etkiler üretebilmeleri için aktive edilebilir.

1 INTRODUCTION

Quantum entanglement is the main concept in quantum information and communication, which is at the heart of the many quantum information processing tasks. In principle, we may define entanglement as: If two systems interacted in the past, it is not possible, in general, to know the individual state vectors of the subsystems exactly. The most famous example of entangled states is the singlet state

$$|\psi\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}. \quad (1.1)$$

This is a state of a composite system containing two 2-level subsystems. Since the reduced density matrices of the subsystems are proportional to the identity we can not say anything certain about any physical quantity belonging to the subsystems alone.

Entanglement was first realized by Einstein, Podolsky, and Rosen [1], they had concluded in their argument that an entangled wave function does not describe the physical reality in a complete way. Rather than considering entanglement as a mystery, like in the early years of quantum mechanics, it is nowadays viewed as a resource to communicate in a more secure way, to perform quantum teleportation, to produce faster algorithms, etc. Because of the interaction with environment, the pure state entanglement weakens, we have to deal with the mixed state entanglement [15] whose manifestation is much more subtle. Mainly, there are two fundamental problems while investigating the structure of mixed state entanglement: i) the separability problem is about whether a given density matrix is entangled or disentangled (separable). ii) the distillability problem : whether a given state is free(distillable)-useful for quantum communication, information and computation, or bound(nondistillable)-a very weak and mysterious type of entanglement.

In this thesis some, basic notions and and some important and illustrative protocols showing the crucial role of entanglement in quantum information theory has been given first. In the rest of the thesis we have dealt with the separability and the distillability problems. In the context of separability we have summarized some important structural and operational criteria to describe the entanglement mathematically, and in the context of distillability we have introduced the BBPSSW distillation protocol, some theorems about the definition of distillability, free and bound entanglement concepts, and a method using the bound entangled states as a tool showing a non-classical effect. Finally, some generalized results that we have found by using the quasidistillation protocol has been introduced.

2 PRELIMINARY NOTIONS

2.1 Postulates of Quantum Mechanics

Postulate 1 :State of any physical system is completely described by a unit vector in a Hilbert space (a complex vector space with inner product) acknowledged as the *state space* of the system. The unit vector is named as the *state vector* of the system.

Postulate 2: Measurable quantities of the system , which are called *observable*, are represented by Hermitian operators acting over the state vector in Hilbert space of the system. When a measurement of an observable is performed the state of the system collapses to one of the *eigenstates* of the Hermitian operator representing the observable, and the possible measurement outcomes are given by the corresponding *eigenvalues* of that Hermitian operator with corresponding probabilities equal to the absolute square of the overlap between the eigenstates and genuine state (the Born rule).

Postulate 3: Schrödinger's equation characterizes the time-evolution of a closed quantum system:

$$i\hbar \frac{d\left|\Psi\right\rangle}{dt} = H(t)\left|\Psi\right\rangle \tag{2.1}$$

where H is an Hermitian operator known as the Hamiltonian, defining the energy of the system. We know from operator theory that there is a one-to-one correspondence between unitary and Hermitian operators; for some Hermitian operator K there exists an unitary operator U such that $U = \exp(iK)$. If we take the Hamiltonian of the system time-independent, as the simplest case, we obtain that the state

$$\left|\Psi(t)\right\rangle = \exp[\frac{-iH(t-t_0)}{\hbar}]\left|\Psi(t_0)\right\rangle \tag{2.2}$$

is a solution of the Schrödinger equation. Then we can write as it is affirmed

$$U\ (t, t_0) = \exp[\frac{-iH(t-t_0)}{\hbar}] \tag{2.3}$$

As a conclusion, it is possible to reexpress the Postulate 3: The time-evolution of a closed quantum system is defined by a unitary operator connected to the Hamiltonian of the system as in (2.3)

$$|\Psi(t)\rangle = U\,|\Psi(t_0)\rangle\,. \tag{2.4}$$

Postulate 4: The Hilbert space of a composite physical system consisting of k subsystems $s_1, ..., s_k$ each with Hilbert space $H_1, ..., H_k$ respectively, is the tensor product (direct product) space $H_1 \otimes ... \otimes H_k$ of Hilbert spaces of the subsystems. If the subsystems are in states $|\psi_1\rangle\,, ..., |\psi_k\rangle$ respectively,then the joint state vector of the composite system is $|\psi_1\rangle \otimes ... \otimes |\psi_k\rangle$ [8].

2.2 The Qubits

A qubit (quantum bit) is the simplest nontrivial quantum-mechanical system whose dynamics is limited to be placed in a 2-dimensional Hilbert space. It is essentially a microscopic physical system, such as an electron , atom or photon or ion.The conceptual situation of qubit in quantum information and computation is equivalent to that of the bit in classical information and computation. The qubit is simply a two-level system. We can define an orthonormal basis for it as

$$\{|0\rangle\,,\ |1\rangle\}, \langle i|\, j\rangle = \delta_{ij}, \quad i, j = 0, 1. \tag{2.5}$$

These two basis vectors $|0\rangle$ and $|1\rangle$ of a 2-dimensional complex vector space correspond to states of a classical bit 0 and 1 and are known as *computational basis states*. Then the pure qubit state can be written as linear combinations of the basis states in the most general way

$$|\psi\rangle = \alpha_1\,|0\rangle + \alpha_2\,|1\rangle\,, \quad |\alpha_1|^2 + |\alpha_2|^2 = 1, \tag{2.6}$$

such that α_1 and α_2 are complex coefficients. These coefficients are known as *probability amplitudes*.

We can perform a measurement on the qubit given in the generic state (2.6). For instance, we can project the state onto the orthonormal basis $\{|0\rangle\,, |1\rangle\}$, then we get 0 or 1 as a result of the measurement with probabilities $|\alpha_1|^2$ and $|\alpha_2|^2$ respectively. After the measurement the state irreversibly *collapses* into one of those basis states $|0\rangle$ and $|1\rangle$. We can at most attain, with this single measurement, that the coefficient reciprocal to the acquired result is not zero.

We could rebuild the unknown quantum state of the qubit (2.6) probabilistically, if we had infinitely many identically prepared copies at our disposal. In another way, it would be possible to reconstruct the state if we could copy an unknown quantum state. Unfavorably, this is impossible according to the *no-cloning* theorem that will be mentioned later.

The constraint $|\alpha_1|^2 + |\alpha_2|^2 = 1$ implies that probabilities sum to one. Geometrically, it can be interpreted that the quantum state of the qubit is a unit (normalized) vector in a two dimensional complex vector space.

Because the overall (global) phase of a state vector does not have any physically noticeable effects, only the relative phase between $|0\rangle$ and $|1\rangle$ does have physical significance, and in order to involve the constraint $|\alpha_1|^2 + |\alpha_2|^2 = 1$ over the coefficients $|0\rangle$ and $|1\rangle$, the state vector of the qubit (2.6) can be written as

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle, \tag{2.7}$$

with real parameter pair (θ, ϕ) restricted to

$$0 \leq \theta \leq \pi, \quad 0 \leq \phi \leq 2\pi. \tag{2.8}$$

The real parameter pair (θ, ϕ) uniquely define a point on the unit sphere of the 3-dimensional real space. This geometrical representation of the state space of the qubit (or any two-level quantum mechanical system) is called the *Bloch sphere* , and is a very helpful tool to picture the states of a qubit.

Only at the poles where $\theta = 0, \pi$ there is no unique representation, in this case $|\psi\rangle$ is one of the basis vectors $|0\rangle$ or $|1\rangle$. Except the poles, (θ, ϕ) pair uniquely corresponds to the point whose Cartesian coordinates are

$$\begin{aligned} x &= \sin\theta\cos\phi \\ y &= \sin\theta\sin\phi, \\ z &= \cos\theta. \end{aligned} \tag{2.9}$$

As already mentioned, any two-level system can be used as a qubit. For example, we can use the polarization of a photon as well as the alignment of a spin-1/2 particle (an electron) on a chosen axis (say z-axis). Qubit plays the same role in

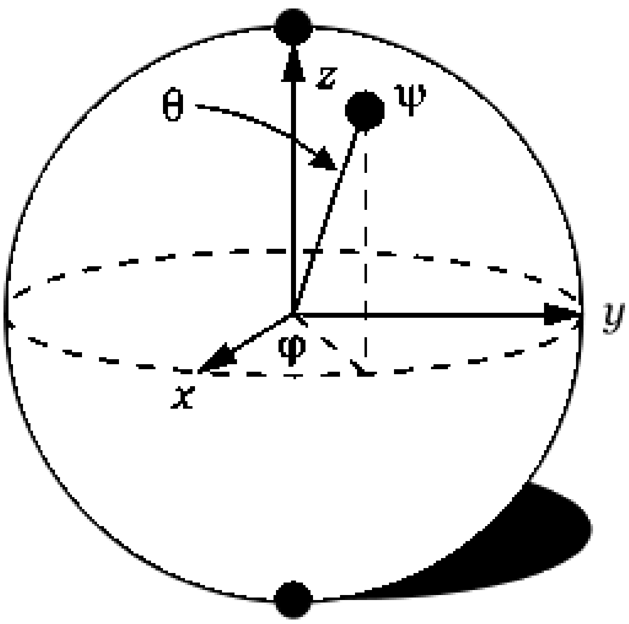

Figure 1: Bloch Sphere

quantum information and computation as the bit does in classical information and computation. The bit is a system which is able to store enough information just to answer one yes-or-no question, and it must always be either 0 or 1. Whereas a qubit can exist in a *continuum* of intermediate states (superpositions) between $|0\rangle$ and $|1\rangle$ in addition to having capability of being in the basis states in the way that a bit does. As an illustration, if we take photon's two linear polarization states as $|H\rangle = |0\rangle$ (horizontal polarization) and $|V\rangle = |1\rangle$ (vertical polarization) then the photon can be in

$$|R\rangle = \frac{1}{2}(|H\rangle + i\,|V\rangle) \tag{2.10}$$

right circular polarization state or

$$|L\rangle = \frac{1}{2}(i\,|H\rangle + |V\rangle) \tag{2.11}$$

left circular polarization state as a sample of superpositions corresponding to other polarizations of the photon . That is one of the crucial distinguishing features between a classical bit and a qubit. The other important difference is that two or more qubits can exhibit a nonlocal property allowing them to denote higher correlation than is capable of occurring between classical bits.

Now it is the proper time to turn our attention from the simplest quantum systems to the simplest composite quantum systems, namely joint systems consisting of two qubits. Let us label the subsystems as A and B. If the quantum states of two subsystems are

$$|\psi\rangle_A = a_1 |0\rangle + a_2 |1\rangle \ , \quad |\psi\rangle_B = b_1 |0\rangle + b_2 |1\rangle \, , \tag{2.12}$$

where $|a_1|^2 + |a_2|^2 = 1$ and $|b_1|^2 + |b_2|^2 = 1$. Then the joint quantum state of our two-qubit system is the tensor product of individual states of two systems,

$$|\Psi\rangle = |\psi\rangle_A \otimes |\psi\rangle_B \, . \tag{2.13}$$

We call such a state as "product state". We name a composite state having two parts "bipartite". In general any bipartite state can be written as any superposition of product states

$$|\Psi\rangle = \sum_{ij} a_{ij} |i\rangle_A \otimes |j\rangle_B \, , \tag{2.14}$$

where $\sum_{ij} |a_{ij}|^2 = 1$ and $\{|i\rangle_A \otimes |j\rangle_B\}$ is the set of orthonormal product basis states for the bipartite system. $\{|i\rangle_A\}$ and $\{|j\rangle_B\}$ are orthonormal basis for the two parts of the composite system.

Suppose that the Hamiltonian of our two-qubit system is of the form

$$H = H_1 \otimes I + I \otimes H_2. \tag{2.15}$$

Here the two terms $H_1 \otimes I$ and $I \otimes H_2$ act only on the individual parts of the system and commute. Then we get a product of unitary transformations with this Hamiltonian

$$\begin{aligned} U &= e^{-iHt/\hbar} = e^{-i(H_1 \otimes I + I \otimes H_2)t/\hbar} \\ &= e^{-i(H_1 \otimes I)t/\hbar} . e^{-i(I \otimes H_2)t/\hbar} = e^{-iH_1 t/\hbar} \otimes e^{-iH_2 t/\hbar} \\ &= U_1 \otimes U_2. \end{aligned} \tag{2.16}$$

Applying this tensor product unitary transformation to a product state, $|\psi_1\rangle \otimes |\psi_2\rangle$, will give us another product state. If the Hamiltonian contains a term acting on both subsystems

$$H = H_1 \otimes I + I \otimes H_2 + H_i, \tag{2.17}$$

then the unitary transformation U, in general, will not be a product unitary. In this context, we say that there is an "interaction" between our two subsystems. If we perform the non-product unitary transformation to an initial product state, a superposition of product states will be generated, namely the resultant state will be "entangled".

To illustrate this situation, we will investigate the composite system consisting of two spin-$\frac{1}{2}$ particles. The interaction term will be of the form

$$H_{int} = E_{int}\sigma_z \otimes \sigma_z \tag{2.18}$$

where σ_z is the Pauli spin-z matrix. Unitary transformation generated by H_{int} is

$$U(\tau) = \cos\left(\frac{\tau}{2}\right) I \otimes I + i \sin\left(\frac{\tau}{2}\right) \sigma_z \otimes \sigma_z \tag{2.19}$$

where $\tau = -E_{int}t/\hbar$.

Let our initial product state be

$$\begin{aligned} |\psi_1\rangle \otimes |\psi_2\rangle &= a_1b_1 |00\rangle + a_1b_2 |01\rangle + a_2b_1 |10\rangle + a_2b_2 |11\rangle \\ &= (a_1 |0\rangle + a_2 |1\rangle) \otimes (b_1 |0\rangle + b_2 |1\rangle). \end{aligned} \tag{2.20}$$

If this product state is transformed by

$$\begin{aligned} U(\tau)\left(|\psi_1\rangle \otimes |\psi_2\rangle\right) = e^{i\tau/2}a_1b_1 |00\rangle &+ e^{-i\tau/2}a_1b_2 |01\rangle \\ &+ e^{-i\tau/2}a_2b_1 |10\rangle + e^{i\tau/2}a_2b_2 |11\rangle \end{aligned} \tag{2.21}$$

then we get a state which is not a product state for $\tau \neq n\pi$, where n is integer. Entanglement has been generated by the interaction H_{int} [3, 8, 7].

2.3 Quantum Measurement

Quantum measurements are described by means of the sets of "measurement operators" $\{M_i\}$ as a general picture of measurement in Q.M. These operators act over the Hilbert space (state space) of the system being measured. m_i's are the possible outcomes of an experiment, which associate to each m_i respectively. It should be realized that m_i's need not to be the eigenvalue of an observable. If our quantum system is in the state $|\psi\rangle$ then the outcome m_i occurs with probability

$$P_i = \langle\psi| M_i^\dagger M_i |\psi\rangle \tag{2.22}$$

and the post-measurement state of the system is

$$\frac{M_i \left|\psi\right\rangle}{\sqrt{\left\langle\psi\right| M_i^\dagger M_i \left|\psi\right\rangle}}. \tag{2.23}$$

By the help of the fact that probabilities sum to one

$$1 = \sum_i P_i = \sum_i \left\langle\psi\right| M_i^\dagger M_i \left|\psi\right\rangle = \left\langle\psi\right| \sum_i M_i^\dagger M_i \left|\psi\right\rangle \tag{2.24}$$

we can say that the measurement operators satisfy the relation

$$\sum_i M_i^\dagger M_i = 1. \tag{2.25}$$

This relation is called the "completeness relation".

2.3.1 Projective Measurement

There is a special class of general quantum measurements. It is the most simple case of this general measurement scheme and known as "projective measurements". A projective measurement is relevant to an "observable" O of the measured system. The observable, O, is an Hermitian operator whose spectral decomposition can be written as

$$O = \sum_i \lambda_i \left|o_i\right\rangle \left\langle o_i\right| \tag{2.26}$$

where $\{\lambda_i\}$ are the eigenvalues and $\{\left|o_i\right\rangle\}$ are the corresponding eigenstates of the observable. If we define projectors onto the eigenspaces of the observable O with eigenvalues λ_i then the decomposition will take the form

$$O = \sum_i \lambda_i P_i \quad , \quad P_i = \left|o_i\right\rangle \left\langle o_i\right|. \tag{2.27}$$

The projectors P_i satisfy the orthogonality condition $P_i P_j = \delta_{ij} P_i$ since O is Hermitian and the completeness relation is

$$\sum_i P_i = I. \tag{2.28}$$

Hence, by equating the projectors to the measurement operators $M_i = P_i = \left|o_i\right\rangle \left\langle o_i\right|$, we get the probability of obtaining the result m_i as

$$\left\langle\psi\right| M_i^\dagger M_i \left|\psi\right\rangle = \left\langle\psi\right| P_i^\dagger P_i \left|\psi\right\rangle = \left\langle\psi\right| P_i \left|\psi\right\rangle = \left|\left\langle o_i|\psi\right\rangle\right|^2 \tag{2.29}$$

by using the properties $P_i^\dagger = P_i$ and $P_i^2 = P_i$. If the state of the system before the measurement is $|\psi\rangle$ the state of the system after the measurement is

$$\frac{P_i |\psi\rangle}{\sqrt{\langle\psi| P_i |\psi\rangle}} = \frac{\langle o_i|\psi\rangle |o_i\rangle}{|\langle o_i|\psi\rangle|} \tag{2.30}$$

where the factor $\frac{\langle o_i|\psi\rangle}{|\langle o_i|\psi\rangle|}$ has the absolute value equals to 1. Then we can conclude that the post-measurement state is an eigenstate of O up to a overall phase.

2.3.2 POVM Measurements

In the general measurement formalism, the probability of obtaining the outcome m_i is given by $\langle\psi| M_i^\dagger M_i |\psi\rangle = P_i$. If we make the definition

$$F_i = M_i^\dagger M_i \tag{2.31}$$

we acquire a set of Hermitian positive operators $\{F_i\}$ acting on the Hilbert space of the system, that sum to identity operator:

$$\sum_i F_i = I. \tag{2.32}$$

This equation is similar to the decomposition of the state space of the system into the set of orthogonal projectors corresponding to the eigenstates of an observable $\sum_i P_i = I$, $P_i P_j = \delta_{ij} P_i$. The complete set $\{F_i\}$ is named as "POVM" (Positive Operator-Valued Measure), and the operators F_i's are the "POVM elements". There is an important difference between the POVM and the projective measurement, it is that the POVM elements are not necessarily orthogonal. We attain the consequence that the number of POVM elements can be larger than the dimension of the Hilbert space of the quantum system.

Example: This example illustrates the use of the POVM formalism. Suppose that Alice gives Bob a qubit prepared in one of two states, $|\psi_1\rangle = |0\rangle$ or $|\psi_2\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. We know that Bob cannot reliably distinguish them. But it is sometimes possible for him to distinguish them with the help of a POVM whose elements are

$$E_1 = \frac{\sqrt{2}}{1+\sqrt{2}} |1\rangle \langle 1|, \tag{2.33}$$

$$E_2 = \frac{\sqrt{2}}{1+\sqrt{2}} \frac{(|0\rangle - |1\rangle)(\langle 0| - \langle 1|)}{2}, \tag{2.34}$$

$$E_2 = I - E_1 - E_2. \tag{2.35}$$

It is straightforward to verify that these are positive operators which satisfy the completeness relation $\sum_m E_m = I$, and therefore form a legitimate POVM.

Suppose Bob is given the state $|\psi_1\rangle = |0\rangle$. He performs the measurement described by the POVM (E_1, E_2, E_3). There is zero probability that he will observe the result E_1, since E_1 has been cleverly chosen to ensure that $\langle\psi_1| E_1 |\psi_1\rangle = 0$. Therefore, if the result of his measurement is E_1 then Bob can safely conclude that the state he has received must have been $|\psi_2\rangle$. A similar line of reasoning shows that if the measurement E_2 occurs then it must have been the state $|\psi_1\rangle$ that Bob received. Some of the time, however, Bob will obtain the measurement outcome E_3, and he can infer nothing about the identity of the state he has been given. The key point, however, is that Bob never makes a mistake identifying the state he has been given [3].

2.4 Density Operator: Pure and Mixed States

The state vector $|\psi\rangle$ of an isolated quantum system represents the maximal knowledge (information) about the physical state of the system. With this state vector $|\psi\rangle$ of the system we can completely determine the statistical behavior of an observable A, i.e. $\langle\psi| A |\psi\rangle$. We need "density operator" or "density matrix" formalism when we describe the statistical state of a quantum system. It is needed when we deal with either an "ensemble of systems" or a system whose preparation is ambiguous.

A quantum system may be prepared in one of the several states $\{|\psi_i\rangle\}$ with respective probabilities $\{P_i\}$, $\{P_i\,, |\psi_i\rangle\}$ establishes an "ensemble of pure states". In this situation, the density operator of the system is defined as

$$\rho = \sum_i P_i |\psi_i\rangle \langle\psi_i| \,. \tag{2.36}$$

This is the most general form and represents a statistical mixture of "pure states". A state is called "pure state" if it is represented by a unit vector. This means that we exactly know the state in which the quantum system is. The density operator is then (a one dimensional projection operator) $\rho = |\psi\rangle \langle\psi|$ for a pure state. Other states which are "convex combinations" of pure states are called "mixed states".

To detect whether a given density operator ρ is pure or mixed, it is sufficient to check the so called "purity factor." $Tr(\rho^2)$: $Tr(\rho^2) = 1$ holds for all pure states and for mixed states it is always smaller than one, $Tr(\rho^2) < 1$.

If we perform a general measurement scheme represented by the set of measurement operators $\{M_i\}$ to a quantum system, which was initially prepared in the state $|\psi_j\rangle$, the probability of obtaining the result m_i is

$$P_{ij} = p(i|j) = \langle\psi_j| M_i^\dagger M_i |\psi_j\rangle . \quad (2.37)$$

And, by means of the trace map, we can rewrite the probability as

$$P_{ij} = Tr(M_i^\dagger M_i |\psi_j\rangle \langle\psi_j|). \quad (2.38)$$

The total probability of getting the result m_i over all pure states belonging to the ensemble $\{P_j\ , |\psi_j\rangle\}$, realizing the density operator $\rho = \sum_j P_j |\psi_j\rangle \langle\psi_j|$, is then

$$\begin{aligned} P_i &= \sum_j P_j P_{ij} \\ &= \sum_j P_j Tr(M_i^\dagger M_i |\psi_j\rangle \langle\psi_j|) \\ &= Tr(M_i^\dagger M_i \sum_j |\psi_j\rangle \langle\psi_j|) \\ &= Tr(M_i^\dagger M_i \rho). \end{aligned} \quad (2.39)$$

After getting the measurement outcome m_i, the state of the system is

$$\begin{aligned} \rho_i &= \sum_i P_i \frac{M_i |\psi_j\rangle \langle\psi_j| M_i^\dagger}{Tr(M_i^\dagger M_i \rho)} \\ &= \frac{M_i \rho M_i^\dagger}{Tr(M_i^\dagger M_i \rho)}. \end{aligned} \quad (2.40)$$

If we do not register the measurement outcomes, the result is not known. Then we have to average over all possible outcomes,

$$\rho' = \sum_i P_i \rho_i = \sum_i P_i \frac{M_i \rho M_i^\dagger}{P_i} = \sum_i M_i^\dagger \rho M_i. \quad (2.41)$$

This is the post-measurement state of the system in the case where the measurement result is not recorded.

Now it is time to mention about the properties of the density operator, which can easily be deduced from its definition. These properties are

(1) ρ is a positive operator, namely for any vector $|\psi\rangle$ of the Hilbert space $\langle\psi|\,\rho\,|\psi\rangle \geqslant 0$. We can express this property by an equivalent statement that ρ is Hermitian with non-negative eigenvalues. Suppose $\rho = \sum_i P_i\,|\psi_i\rangle\,\langle\psi_i|$, then

$$\langle\psi|\,\rho\,|\psi\rangle = \sum_i P_i\langle\psi|\psi_i\rangle\langle\psi_i|\psi\rangle = \sum_i P_i\,|\langle\psi_i|\psi\rangle|^2 \geqslant 0. \qquad (2.42)$$

(2) The trace of a density operator ρ is equal to one.

$$Tr(\rho) = \sum_i P_i(|\psi_i\rangle\langle\psi_i|) = \sum_i P_i = 1. \qquad (2.43)$$

On the contrary, if ρ is any operator satisfying the previous two conditions, according to the spectral theorem it must be decomposed as,

$$\rho = \sum_i \mu_i|\mu_i\rangle\langle\mu_i|, \qquad (2.44)$$

because of its positivity. $\{\mu_i\}$ is the set of real, non-negative eigenvalues of ρ and $\{|\mu_i\rangle\}$ is the corresponding set of eigenvectors. $|\mu_i\rangle$'s are orthonormal, $\langle\mu_i|\mu_j\rangle = \delta_{ij}$ and $\mu_i \geqslant 0$ sum to one, $\sum_i \mu_i = 1$ by means of trace condition. Hence we can interpret the eigenvalue μ_i as the probability for the system to be in the state $|\mu_i\rangle$.

We can remark from the above discussion that the same density operator can be the realization of different ensembles like $\{P_j\,,|\psi_j\rangle\}$ and $\{\mu_i\,,|\mu_i\rangle\}$, so there is an unavoidable indistinguishability about the preparation of the mixed state ρ. Once ρ is described from an ensemble, we discard the information about which ensemble the density matrix was arranged from. For instance, we can prepare a density matrix ρ from computational basis $\{|0\rangle\,,|1\rangle\}$ by mixing them with equal probability, then $\rho = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1| = \frac{1}{2}I$. Also, the same mixed state can be realized by making a mixture of two states $|x\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ and $|y\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$ with equal probability , or the three states $\frac{1}{2}|0\rangle - \frac{\sqrt{3}}{2}|1\rangle$, $|x\rangle$ and $|0\rangle$ with respective probabilities $P_1 = 1 - \frac{\sqrt{3}}{3}$, $P_2 = \frac{\sqrt{3}}{3+\sqrt{3}}$ and $P_3 = \frac{1}{2} - \frac{\sqrt{3}}{6}$. (Unitary freedom in the ensemble for density matrices may be added.)

The utility of density operator formalism shows itself in locally describing a subsystem, which is a part of a composite quantum system. Let us consider now a composite system consisting of two subsystems A and B, whose state is ρ_{AB}, and we want to describe the states of individual parts of this composite physical system. For example, we can find the state of subsystem by ignoring the degrees of freedom related to subsystem A. Methodically, this can be done by means of the linear "partial trace" map Tr_A defined by

$$\begin{aligned} Tr_A(\rho_{AB}) = \sum_{ijkl} \rho_{ijkl} Tr(|\mu_i\rangle\langle\mu_j|) \otimes |\nu_k\rangle\langle\nu_l| &= \sum_{ijkl} \rho_{ijkl} \langle\mu_j|\mu_i\rangle |\nu_k\rangle\langle\nu_l| \\ &= \sum_{kl} \rho'_{kl} |\nu_k\rangle\langle\nu_l|, \end{aligned} \tag{2.45}$$

for any state $\sum_{ijkl} \rho_{ijkl} |\mu_i\rangle\langle\mu_j| \otimes |\nu_k\rangle\langle\nu_l|$ of our bipartite system.

$Tr_A(\rho_{AB}) = \rho_B$ is called the "reduced density operator" for the subsystem B, and similarly the reduced density matrix for subsystem A can be obtained by taking the partial trace over B, $Tr_B(\rho_{AB}) = \rho_A$. Both ρ_A and ρ_B give us measurement statistics for measurements performed on subsystems A and B respectively, i.e. $[Tr(A \otimes I\rho_{AB}) = Tr(A\rho_A)$, $Tr(I \otimes B\rho_{AB}) = Tr(B\rho_B)]$.

To illustrate the reduced density operator concept better, we are going to calculate the state of one of the two qubits prepared in the Bell state $|\psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$. The density matrix of this bipartite system

$$\begin{aligned} \rho_{AB} &= |\psi^-\rangle\langle\psi^-| \\ &= \frac{1}{2}(|01\rangle - |10\rangle)(\langle 01| - \langle 10|) \\ &= \frac{1}{2}(|01\rangle\langle 01| - |01\rangle\langle 10| - |10\rangle\langle 01| + |10\rangle\langle 10|). \end{aligned} \tag{2.46}$$

If we take the partial trace over qubit A, then we get the reduced density matrix, $\rho_B = Tr_A(\rho_{AB})$, of the qubit B as

$$\begin{aligned} \rho_B &= Tr_A[\frac{1}{2}(|0\rangle\langle 0| \otimes |1\rangle\langle 1| - |0\rangle\langle 1| \otimes |1\rangle\langle 0| - |1\rangle\langle 0| \otimes |0\rangle\langle 1| + |1\rangle\langle 1| \otimes |0\rangle\langle 0|)] \\ &= \frac{1}{2}[Tr(|0\rangle\langle 0|)|1\rangle\langle 1| - Tr(|0\rangle\langle 1|)|1\rangle\langle 0| - Tr(|1\rangle\langle 0|)|0\rangle\langle 1| + Tr(|1\rangle\langle 1|)|0\rangle\langle 0|] \\ &= \frac{1}{2}(\langle 0|0\rangle|1\rangle\langle 1| - \langle 1|0\rangle|1\rangle\langle 0| - \langle 0|1\rangle|0\rangle\langle 1| + \langle 1|1\rangle|0\rangle\langle 0|) = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|) \end{aligned} \tag{2.47}$$

As a result of the above calculations, we get the following result

$$\rho_B = \frac{1}{2}I. \tag{2.48}$$

We acquire the same result for qubit A, too. Bipartite states having this property are called "maximally entangled" states. The state of the joint system is a one-dimensional projector, namely a pure state that exactly define the composite system, but any subsystem is in a mixed state proportional to the identity. This means, for instance, if we take our qubits as spin-$\frac{1}{2}$ particles and perform a measurement on any subsystem to learn the orientation of its spin along any axis, we get completely casual result, i.e., measuring the qubit A or B locally does not give us any information about the preparation of the state. This situation is one of the remarkable features of the entangled state [3, 8, 7].

3 QUANTUM PROTOCOLS

In this part, we will deal with how to communicate information by using entanglement as a resource. We will describe some protocols [8, 7] that can be carried out if an EPR pair is shared and if we have capability of performing Bell measurement. These information processing protocols, like quantum teleportation and superdense coding, are not possible without entanglement and have no classical counterparts. Before explaining these protocols we will mention about a simple but important result of quantum mechanics known as no-cloning theorem .

3.1 No-cloning Theorem

It is possible to copy a file, on our computer screen, whose content is unknown just by clicking the right-hand side of our mouse and selecting the copy command. Is it possible to copy quantum information? This theorem is about the possibility of constructing a quantum photocopy machine. Let's assume that there is such a machine, then there must be a unitary operator U corresponding to the transformation of the initial input state $|\psi\rangle \otimes |i\rangle$, where $|i\rangle$ is some standard reference state and $|\psi\rangle$ is any arbitrary state to be copied, to

$$U(|\psi\rangle \otimes |i\rangle) = |\psi\rangle \otimes |\psi\rangle, \tag{3.1}$$

and for any other state $|\phi\rangle \neq |\psi\rangle$

$$U(|\phi\rangle \otimes |i\rangle) = |\phi\rangle \otimes |\phi\rangle. \tag{3.2}$$

By using the linearity property of the operation of our machine,

$$\begin{aligned} U(a|\psi\rangle + b|\phi\rangle) \otimes |i\rangle &= U(a|\psi\rangle \otimes |i\rangle + b|\phi\rangle \otimes |i\rangle) \qquad (3.3) \\ &= a|\psi\rangle \otimes |\psi\rangle + b|\phi\rangle \otimes |\phi\rangle \neq (a|\psi\rangle + b|\phi\rangle) \otimes (a|\psi\rangle + b|\phi\rangle). \end{aligned}$$

By contradiction we have concluded that there is no such unitary evolution U, i.e, we can not generate the identical copies of an arbitrary unknown state $|\psi\rangle$. Unlike classical information mentioned at the beginning, quantum information can not be copied.

3.2 Superdense Coding

A long time ago, a composite quantum system consisting of two qubits, say spin-1/2 particles, which is in one of the Bell states say $|\phi^+\rangle = \frac{1}{2}(|00\rangle + |11\rangle)$, is shared between two parties, say the first qubit is in possession of Alice and the second one belongs to Bob. These people are the leading actors of the Q.I.T. At a later time, Alice needs to transmit two classical bits to Bob, and she wants to manage this transmission by sending the qubit locally via performing one of the four unitary operators $\sigma_0, \sigma_1, \sigma_2$ or σ_3, where $\sigma_0 = I$ and $\sigma_1, \sigma_2, \sigma_3$ are Pauli matrices. By using the fact that Bell states can be transformed to each other via manipulating locally

$$(I \otimes I)|\phi^+\rangle = |\phi^+\rangle \tag{3.4}$$

$$(\sigma_1 \otimes I)|\phi^+\rangle = |\psi^+\rangle \tag{3.5}$$

$$(\sigma_2 \otimes I)|\phi^+\rangle = -i|\psi^-\rangle \tag{3.6}$$

$$(\sigma_3 \otimes I)|\phi^+\rangle = |\phi^-\rangle \tag{3.7}$$

Alice can encode her message to be sent. Alice mails her qubit to bob after performing one of the four local operations given above. Bob now can make a projective measurement on the Bell basis and decode the message including two classical bits. He gets one of the four possible outcomes $|\phi^+\rangle, |\phi^-\rangle, |\psi^+\rangle$ or $|\psi^-\rangle$ after the measurement. These outcomes contains two classical bits: one bit is the parity corresponds to whether it is ϕ or ψ and the other is the phase bit, if it is + or -. We can notice that no useful information can be obtained by the third person who acquires Alice's qubit illegally. Since it is a part of a composite system in a maximally entangled state, i.e., its reduces density operator equals to $\frac{1}{2}I$, it behaves like a random bit.

3.3 Quantum Teleportation

This quantum information processing protocol provides us an alternative way to transfer a quantum state from Alice to Bob other than mailing it as in previous protocol. Quantum teleportation scheme can be achieved by using local quantum operations and transmission of two bits of classical information. Before all, the spatially separated parties Alice and Bob must share maximally entangled state

$|\psi\rangle = \alpha_1|0\rangle + \alpha_2|1\rangle$ to be transmitted. So the joint state of those three qubits

$$
\begin{aligned}
|\psi\rangle_1 \otimes |\phi^+\rangle_{23} \quad &= (\alpha_1|0\rangle + \alpha_2|1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle) \\
&= \frac{1}{\sqrt{2}}(\alpha_1|000\rangle_{123} + \alpha_1|011\rangle_{123} + \alpha_2|100\rangle_{123} + \alpha_2|111\rangle_{123}) \\
&= \frac{1}{\sqrt{2}}(\alpha_1|00\rangle_{12} \otimes |0\rangle_3 + \alpha_1|01\rangle_{12} \otimes |1\rangle_3 \\
&\qquad + \alpha_2|10\rangle_{12} \otimes |0\rangle_3 + \alpha_2|11\rangle_{12} \otimes |1\rangle_3).
\end{aligned} \tag{3.8}
$$

Here, the qubits 1 and 2 are in possession of Alice and qubit 3 is in Bob's hands. If we rearrange Alice's qubits in the Bell basis by using the identities

$$|00\rangle = \frac{1}{\sqrt{2}}(|\phi^+\rangle + |\phi^-\rangle) \tag{3.9}$$

$$|01\rangle = \frac{1}{\sqrt{2}}(|\psi^+\rangle + |\psi^-\rangle) \tag{3.10}$$

$$|10\rangle = \frac{1}{\sqrt{2}}(|\psi^+\rangle - |\psi^-\rangle) \tag{3.11}$$

$$|11\rangle = \frac{1}{\sqrt{2}}(|\phi^+\rangle - |\phi^-\rangle) \tag{3.12}$$

we can rewrite the joint state by changing the Alice's part as

$$
\begin{aligned}
|\psi\rangle_1 \otimes |\phi^+\rangle_{23} \quad &= \frac{1}{2}|\phi^+\rangle_{12} \otimes |\psi\rangle_3 + \frac{1}{2}|\psi^+\rangle_{12} \otimes \sigma_x|\psi\rangle_3 \\
&\quad + \frac{1}{2}|\psi^-\rangle_{12} \otimes (-i\sigma_y)|\psi\rangle_3 + \frac{1}{2}|\phi^-\rangle_{12} \otimes \sigma_z|\psi\rangle_3.
\end{aligned} \tag{3.13}
$$

After all these rearrangements, Alice makes a projective measurement on the Bell basis $\{|\phi^+\rangle, |\phi^-\rangle, |\psi^+\rangle, |\psi^-\rangle\}$ and sends the result to Bob by communicating two classical bits. Depending on two-bits classical message he operates one of the unitaries $\sigma_0 = 1$, $\sigma_x, \sigma_y, \sigma_z$. Then by the property

$$\sigma^2_{x,y,z} = I \tag{3.14}$$

he makes the correction and recovers $|\psi\rangle$.

We can remark that quantum teleportation does not violate no-cloning theorem. Yes, Bob has obtained the copy of the state $|\psi\rangle$ but Alice had broken down the original one by performing a Bell measurement before Bob had.

3.4 Schmidt Decomposition

The general state vector of a composite system consisting of two subsystems A and B, is a linear superposition of products of individual states:

$$\psi\rangle = \sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}|a_i\rangle \otimes |b_j\rangle \tag{3.15}$$

where $\{|a_i\rangle\}_{i=1}^{m}$ and $\{|b_j\rangle\}_{j=1}^{n}$ are the orthonormal bases of Hilbert spaces, H_A and H_B, of the subsystems A and B respectively and $m = dim(H_A), n = dim(H_B)$. We can re-express this state via Schmidt decomposition procedure that gives the smallest possible number of teras for a biorthogonal basis. Assume $m \geqslant n$. The coefficients c_{ij} can be viewed as the inputs of a mxn matrix C. Recalling the singular value decomposition we can write matrix as

$$C = UC'V \tag{3.16}$$

where U is a mxm unitary matrix, C' is a mxn matrix whose submatrix is diagonal and occurs with real and positive elements, the others are all zero, V is a nxn unitary matrix. This implies that

$$c_{ij} = \sum_{k=1}^{l} u_{ik}c'_{kk}v_{kj} \tag{3.17}$$

where $l = dim(H_B)$. Now, we can rewrite the state vector

$$\begin{aligned} |\psi\rangle \quad &= \sum_{i=1}^{m}\sum_{j=1}^{n}(\sum_{k=1}^{l} u_{ik}c'_{kk}v_{kj})|a_i\rangle \otimes |b_j\rangle \qquad (3.18) \\ &= \sum_{k=1}^{l} c'_{kk}(\sum_{i=1}^{m} u_{ik}|a_i\rangle) \otimes (\sum_{j=1}^{n} v_{kj}|b_j\rangle) = \sum_{k=1}^{l} p_k|a'_k\rangle \otimes |b'_k\rangle \end{aligned}$$

where $\{p_k = c'_{kk}\}$'s are Schmidt coefficients, $\{|a'_k\rangle\}$ and $\{|b'_k\rangle\}$ are our new basis vectors called Schmidt basis. It can be noted that l is equal to n which is the minimum of $\{dimH_A, dimH_B\}$. The number of terms in Schmidt decomposition of $|\psi\rangle \in H_A \otimes H_B$ is called its Schmidt number or Schmidt rank.

If we take the partial trace of $\rho = |\psi\rangle\langle\psi|$ with respect to either subsystems A or B, we are obtain diagonal reduced density matrices whose elements are p_k^2, i.e, ρ_A ρ_A and ρ_B have the same positive spectrum:

$$\rho = \sum_{i,j} p_i p_j |a'_i\rangle\langle a'_j| \otimes |b'_i\rangle\langle b'_j| \tag{3.19}$$

$$\rho_A = \sum_k \langle b'_k|\rho|b'_k\rangle = \sum_i p_i^2|a'_i\rangle\langle a'_i|, \quad \rho_B = \sum_j p_j^2|b'_j\rangle\langle b'_j| \tag{3.20}$$

We will explain the importance of the Schmidt decomposition while qualifying pure state entanglement later.

4 CHARACTERIZATION OF BIPARTITE ENTANGLEMENT

In this section we will introduce some crucial theoretical results permitting us to categorize a given quantum state as entangled or separable (disentangled). Initially we will make definition of separability for both pure and mixed states and then some structural and practical separability criteria will be summarized. We will deal with finite dimensional bipartite case.

4.1 Qualifying Pure State Entanglement

A pure state $|\psi\rangle \in H_A \otimes H_B$ is called separable if and only if it can be written as a tensor product of two individual pure states of the subsystems $|\psi\rangle_A \in H_A$, $\dim(H_A) = m$ and $|\psi\rangle_B \in H_B$, $\dim(H_B) = n$,

$$|\psi\rangle = |\psi\rangle_A \otimes |\psi\rangle_B \,. \tag{4.1}$$

On the other hand, if a pure state is not separable, i.e. $|\psi\rangle \neq |\psi\rangle_A \otimes |\psi\rangle_B$, is called entangled (inseparable).

There is a very simple necessary and sufficient condition for bipartite pure states. The Schmidt decomposition [8, 7] is the useful tool in this context.We know from section 3.1 that any pure state $|\psi\rangle \in H_A \otimes H_B$ can be decomposed as a biorthogonal sum

$$|\psi\rangle = \sum_{i=1}^{r} p_i \, |a_i'\rangle \otimes |b_i'\rangle \,. \tag{4.2}$$

The number of non-vanishing coefficients in this sum had been called the Schmidt rank r. A pure state is separable if $r = 1$, otherwise , $r > 1$, it is entangled. We can make an equivalent definition of maximally entangled states; if $r = n$ and all the nonzero coefficients are equal then the state is maximally entangled.

4.2 Mixed State

A mixed state ρ_{AB} is called separable [17, 10] if and only if it can be written as a product linear convex combination

$$\rho_{AB} = \sum_k p_k \rho_A^k \otimes \rho_B^k \tag{4.3}$$

where $p_k \geq 0$, $\sum_k p_k = 1$, $\{\rho_A^k\}$ and $\{\rho_B^k\}$ are mixed states of the respective subsystems.

Separable states are also known as classically correlated states, since they can be obtained through a LOCC preparation course, two parties can agree on the local preparation of states via classical communication. A mixed state that is not separable is called entangled.

For a given mixed state it is not an easy task to decompose it as in (4.3). To illustrate this difficulty, let us give a density matrix for a composite system of two qubits :

$$\begin{aligned} \sigma \quad &= \frac{3}{20}\left|00\right\rangle\left\langle 00\right| + i\frac{\sqrt{6}}{40}\left|00\right\rangle\left\langle 01\right| + i\frac{\sqrt{6}}{20}\left|00\right\rangle\left\langle 11\right| - i\frac{\sqrt{6}}{40}\left|01\right\rangle\left\langle 00\right| \qquad (4.4)\\ &+\frac{9}{40}\left|01\right\rangle\left\langle 01\right| - i\frac{\sqrt{6}}{20}\left|01\right\rangle\left\langle 10\right| + i\frac{\sqrt{6}}{20}\left|10\right\rangle\left\langle 01\right| + \frac{1}{4}\left|10\right\rangle\left\langle 10\right| \\ &-i\frac{\sqrt{6}}{40}\left|10\right\rangle\left\langle 11\right| - i\frac{\sqrt{6}}{20}\left|11\right\rangle\left\langle 00\right| + i\frac{\sqrt{6}}{40}\left|11\right\rangle\left\langle 10\right| + \frac{15}{40}\left|11\right\rangle\left\langle 11\right| \end{aligned}$$

and we can write it as a matrix in the basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$

$$\sigma = \begin{pmatrix} \frac{3}{20} & i\frac{\sqrt{6}}{40} & 0 & i\frac{\sqrt{6}}{20} \\ -i\frac{\sqrt{6}}{40} & \frac{9}{40} & -i\frac{\sqrt{6}}{20} & 0 \\ 0 & i\frac{\sqrt{6}}{20} & \frac{1}{4} & -i\frac{\sqrt{6}}{40} \\ -i\frac{\sqrt{6}}{20} & 0 & i\frac{\sqrt{6}}{40} & \frac{15}{40} \end{pmatrix}. \tag{4.5}$$

Though it seems so complicated, we can build it as an admixture of two product states ;

$$\begin{aligned} \sigma \quad &= \frac{1}{2}\rho_A^1 \otimes \rho_B^1 + \frac{1}{2}\rho_A^2 \otimes \rho_B^2 \qquad (4.6)\\ &= \frac{1}{2}\left[\left(\frac{1}{2}\left|a^-\right\rangle\left\langle a^-\right| + \frac{1}{2}\left|1\right\rangle\left\langle 1\right|\right) \otimes \left|d^+\right\rangle\left\langle d^+\right|\right] \\ &\quad +\frac{1}{2}\left[\left(\frac{1}{2}\left|a^+\right\rangle\left\langle a^+\right| + \frac{1}{4}I\right) \otimes \left|d^-\right\rangle\left\langle d^-\right|\right] \end{aligned}$$

where

$$\left|a^\pm\right\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle \pm |1\rangle\right) \tag{4.7}$$

$$\left|d^{\pm}\right\rangle = \sqrt{\frac{2}{5}}\left|0\right\rangle \pm i\sqrt{\frac{3}{5}}\left|1\right\rangle \tag{4.8}$$

This is just one of its infinitely many realizations. As it is seen from the above example that mixed state situation is more complex than the pure state case. We will see later that mixed state entanglement can be classified as free and bound with respect to its distillability, so it has more fruitful structure.

In practice to look over all possible realizations of a density matrix is out of the question since it has infinitely many decompositions. We need practical criteria allowing us to detect the existence of entangled states. Before giving the examples of operational separability criteria we will split the criteria into two groups and first investigate the structural ones.

4.2.1 Structural Separability Criteria

These separability criteria give us necessary and sufficient conditions. They provide us constructional insight into the separability problem and mathematics behind being capable of detecting states as separable or entangled.

i) Entanglement Witnesses :

The concept of entanglement witness (EW), [10], arises from the convex structure of the set of states. Let S be the set of all states. If $\rho_1, \rho_2 \in S$ then $a_1\rho_1 + a_2\rho_2 \in S$, where $a_1, a_2 \geq 0$ and $a_1 + a_2 = 1$. The set S_1 of all separable states is convex : $\rho_1, \rho_2 \in S_1 \Rightarrow \lambda\rho_1 + (1-\lambda)\rho_2 \in S_1$, where $\lambda \in [0,1]$. Also we know that mixing can conceal the entanglement in effect, for instance the state

$$\rho = \frac{1}{2}(\left|\phi^+\right\rangle\left\langle\phi^+\right| + \left|\phi^-\right\rangle\left\langle\phi^-\right|) = \frac{1}{2}(\left|0\right\rangle\left\langle 0\right| \otimes \left|0\right\rangle\left\langle 0\right| + \left|1\right\rangle\left\langle 1\right| \otimes \left|1\right\rangle\left\langle 1\right|) \tag{4.9}$$

is separable even though it is a mixture of two maximally entangled states. So entanglement detection problem is the same as determining if a point belongs to a convex set or not. Here we can ask the question : Can we find an observable whose measurement reveals that a given quantum state is entangled? It can be indicated by the following definition and theorem that there are such observables.

An operator $W \in A_A \otimes A_B$ is an entanglement witness if it is a non-positive operator, $W \ngeq 0$, but it is a positive operator on product states $Tr(WP \otimes Q) \geqslant 0$, where A_A and A_B denotes the set of operators and P and Q's are any pure states acting on H_A and H_B respectively.

Theorem: Given two subsets $S_1, S \subseteq A_A \otimes A_B$ such that $S_1 \subset S$ and S_1 is convex and closed, for every $\rho \in S \setminus S_1$ there is an observable $W \in A_A \otimes A_B$ such that

$$Tr(W\sigma) \geq 0 \quad \forall \sigma \in S_1 \tag{4.10}$$

and

$$Tr(W\rho) < 0. \tag{4.11}$$

This theorem can be proved by using the Hahn-Banach theorem and recalling that the set of operators acting on some Hilbert space builds a Hilbert space itself with Hilbert-Schmidt scalar product $\langle A|B\rangle = Tr(A^\dagger B)$.

Corollary: For every entangled state $\rho \in S \setminus S_1$ there is a $W \in A_A \otimes A_B$ such that $\langle W|\rho\rangle < 0$ while $\langle W|\sigma\rangle \geq 0$ for any separable state $\sigma \in S_1$.

Proof: The set of all states S and the set of all separable states S_1 are both convex and closed. So from the above theorem for any $\rho \in S \setminus S_1$ there is a $W \in A_A \otimes A_B$ such that

$$Tr(W\rho) = \langle W|\rho\rangle < 0 \tag{4.12}$$

while

$$Tr(W\sigma) = \langle W|\sigma\rangle \geq 0 \quad \forall \sigma \in S_1. \tag{4.13}$$

The Hermitian operator W defines a hyper-plane $\{\varrho | Tr(W\varrho) = 0\}$ in real Euclidean space which separates the entangled state ρ from the convex set of all separable states S_1. Because W detects that ρ is not in S_1 , i.e. it is entangled, it is called *entanglement witness.*

ii) Positive Maps:

In this part, we will introduce the concept of linear maps from A_A to A_B. Since we consider the finite dimensional cases we can represent the space A of operators acting on a Hilbert space H with a space of matrices M_n , $n = dim(H)$. After giving some definitions about linear maps we will connect positive maps [15, 9, 18] to entanglement witnesses with an isomorphism.

The space of linear maps between matrix algebras M_m and M_n is denoted by $L(M_m, M_n)$. For a given orthonormal basis $\{F_k\} \subset M_m^n$, where M_m^n is the set of $m \times n$ matrices, we can write the maps $\Phi_{kl} : M_m \to M_n$ as

$$\Phi_{kl}[\varrho] = F_k^\dagger \varrho F_k. \tag{4.14}$$

With respect to the scalar product between two maps $\Lambda_1, \Lambda_2 \in L(M_m, M_n)$, for a given matricial orthonormal basis, $\{E_i\} \subset M_m$ defined as

$$\langle\langle \Lambda_1 | \Lambda_2 \rangle\rangle = \sum_i \langle \Lambda_1 [E_i] | \Lambda_2 [E_i] \rangle = \sum_i Tr((\Lambda_1 [E_i])^\dagger \Lambda_2 [E_i]) \tag{4.15}$$

the maps Φ_{kl} defined in (4.14) form an orthonormal basis in $L(M_m, M_n)$, then we can write any $\Lambda \in L(M_m, M_n)$ as

$$\Lambda = \sum_{kl} \lambda_{kl} \Phi_{kl} \tag{4.16}$$

where $\lambda_{kl} = \langle\langle \Phi_{kl} | \Lambda \rangle\rangle$. λ_{kl} constitute a coefficient matrix $C = [\lambda_{kl}]$ depending on both Λ and the chosen basis, such that

$$\langle\langle \Lambda_1 | \Lambda_2 \rangle\rangle = \langle C_1 | C_2 \rangle . \tag{4.17}$$

A map $\Lambda \in L(M_m, M_n)$ is *positive* (P) if for all $0 \leq \varrho \in M_m$, $\Lambda[\varrho] \geq 0$ and it is called *k-positive* if its extension

$$\Lambda_k = id_k \otimes \Lambda \in L(M_{k.m}, M_{k.n}) \tag{4.18}$$

is positive. With the help of this definitions we say that a *map* is *completely positive* (CP) if it is *k-positive* for all k, i.e. if all its extensions are positive [9]. Since linear maps of the form $\Lambda_A : \varrho \to A^\dagger \varrho A$ with an arbitrary operator K are CP, we can obtain the general form of CP maps as

$$\Lambda[\varrho] = \sum_i A_i^\dagger \varrho A_i, \tag{4.19}$$

such expression of any CP maps is named Kraus decomposition and with completeness relation

$$\sum_i A_i^\dagger A_i = I \tag{4.20}$$

we have a trace preserving CP map. Actually, any physical action corresponds to a CP maps that does not increase trace $Tr(\Lambda[\varrho]) \leq Tr(\varrho)$. CP maps are important because they correspond to the general physical operations, but there is no such a compact representation for P maps except for some low dimensional cases. For $dim(H_A)dim(H_B) \leq 6$, all P maps can be represented in the following way [18]

$$\Lambda = \Lambda^1_{CP} + \Lambda^2_{CP} \circ T \tag{4.21}$$

that is, they are *decomposable.* $\Lambda^1_{CP}, \Lambda^2_{CP}$ are CP maps, $\circ$ is composition and T is the transposition.

After the introductory information about maps now we can handle the separability problem in context of positive maps. Since product density operators are mapped into positive operators by the extension of a positive map : $(id \otimes \Lambda)(\rho_A \otimes \rho_B) = \rho_A \otimes \Lambda(\rho_B) \geq 0$, the convex sum of product states, that is, separable states are always mapped into positive ones. This fundamental property of separable states brings us to the fact that if we have an entangled state ρ, then there is a positive map Λ such that $(id \otimes \Lambda)\rho \ngeq 0$. From this discussion we can deduce that the presence of the entangled states can be detected by the positive maps. But we have to be careful because CP maps map all density operators into positive operators, so we need positive but not completely positive ($PnCP$) maps to look for entanglement.

We can now relate the ($PnCP$) maps to the separability problem via Jamiołkowski isomorphism [2] that provides a one-to-one correspondence between entanglement witnesses and ($PnCP$) maps. The isomorphism says that for any map $\Lambda \in L(M_m, M_n)$ we define

$$A = (id_m \otimes \Lambda)[mP^m_+] \in M_{m.n} \tag{4.22}$$

where

$$P^d_+ = \frac{1}{m} \sum_{i,j=1}^{m} |i\rangle\langle j| \otimes |i\rangle\langle j| \tag{4.23}$$

projector onto the maximally entangled state in M_{m^2}. From the definition of entanglement witness we know that a map Λ_W is P if and only if $W = (id_m \otimes \Lambda_W)[mP^m_+]$ is EW.

Theorem: Let ρ act on Hilbert space $H_A \otimes H_B, dim(H_A) = m$ and $dim(H_B) = n$. Then ρ is entangled if and only if for any P map $\Lambda : M_n \to M_m$, $(id \otimes \Lambda)\rho < 0$.

Proof: To any EW, W there related a P map Λ_W such that

$$\begin{aligned} Tr(W\rho) &= Tr((id \otimes \Lambda_W)\left[mP_+^m\right]\rho) \\ &= Tr(mP_+^m(id \otimes \Lambda_W^T)[\rho]) \\ &= m\left\langle \psi_+^m\right|(id \otimes \Lambda_W^T)[\rho]\left|\psi_+^m\right\rangle < 0 \end{aligned} \tag{4.24}$$

where $\left|\psi_+^m\right\rangle = \frac{1}{\sqrt{m}}\sum_{i=1}^m |ii\rangle$ is the maximally entangled state.

Example: Consider the $PnCP$ map the transposition T, the EW corresponding to T is the swap operator V. Let $\{|i\rangle\}_{i=1}^m$ be the orthonormal basis of the Hilbert space of $dim(H) = m$.

$$(id \otimes T)\left[mP_+^m\right] = \sum_{i,j=1}^m |i\rangle\langle j| \otimes T\left[|i\rangle\langle j|\right] = \sum_{i,j=1}^m |i\rangle\langle j| \otimes |j\rangle\langle i| \tag{4.25}$$

Indeed, this operator acts as the following

$$(\sum_{i,j=1}^m |i\rangle\langle j| \otimes |j\rangle\langle i|)(|\psi\rangle \otimes |\phi\rangle) = \sum_{i,j=1}^m |i\rangle\langle i|\phi\rangle \otimes |j\rangle\langle j|\psi\rangle = |\phi\rangle \otimes |\psi\rangle \tag{4.26}$$

Since $V^2(|\psi\rangle \otimes |\phi\rangle) = |\psi\rangle \otimes |\phi\rangle$ its eigenvalues are ± 1. Its eigenvectors are

$$\left|v_+^{ij}\right\rangle = \frac{|ij\rangle + |ji\rangle}{\sqrt{2}} \tag{4.27}$$

and

$$\left|v_-^{ij}\right\rangle = \frac{|ij\rangle - |ji\rangle}{\sqrt{2}}. \tag{4.28}$$

We can verify that V is an EW detecting the entanglement of its eigenvectors $\left|v_-^{ij}\right\rangle$. However, the map T can detect more entangled states than the EW V due to the fact that it summarizes different scalar relations, but the condition $Tr(W\rho) < 0$ is just a scalar relation.

$EW's$ and P maps do not provide us simple methods to examine the separability properties of a given state, since we have to check infinitely many $EW's$ to completely characterize the set S_1 of all separable states and there exist no classification of P maps except for some simple cases, it is still an open problem. So we need criteria that can be tested in easier ways.

4.2.2 Practical Separability Criteria

They are also called operational criteria because they can be applied easily to an explicit density matrix. They provide us necessary conditions, namely all separable states satisfy them, if they are violated by a state, then it must be entangled.

i)Peres - Horodecki separability criterion [4, 10] :

We know that any P map supplies us a separability criterion. Here we will deal with the transposition T. It is a P map since it preserves the spectrum of any operator. It can be noted that if a separable state is partially transposed over one of the subsystems it goes on to be positive :

$$(I \otimes T)(\rho) = \rho^{T_B} = \sum_k p_k \rho_k^A \otimes (\rho_k^B)^T \tag{4.29}$$

where ρ acts on the Hilbert space $H_A \otimes H_B$, $dim(H_A) = m$ and $dim(H_B) = n$. Total state remains positive because $(\rho_k^B)^T$ is positive. Given orthonormal bases $\{|i\rangle\} \subset H_A$ and $\{|\mu\rangle\} \subset H_A$ we can define the partial transposition as

$$\rho^{T_B} = \sum_{ij\mu\nu} \langle i\nu|\rho|j\mu\rangle |i\mu\rangle\langle j\nu| \tag{4.30}$$

if the density matrix is

$$\rho = \sum_{ij\mu\nu} \langle i\mu|\rho|j\nu\rangle |i\mu\rangle\langle j\nu|. \tag{4.31}$$

It may be illustrative to use the matrix representation :

$$\rho = \begin{pmatrix} A_{1,1} & A_{1,2} & ... & A_{1,m} \\ A_{2,1} & A_{2,2} & ... & A_{2,m} \\ ... & ... & ... & ... \\ A_{m,1} & A_{m,2} & ... & A_{m,m} \end{pmatrix} \tag{4.32}$$

where $A_{i,j}$ are the $n \times n$ matrices acting on H_B then its partial transpose with respect to H_B is

$$\rho^{T_B} = \begin{pmatrix} A^T_{1,1} & A^T_{1,2} & ... & A^T_{1,m} \\ A^T_{2,1} & A^T_{2,2} & ... & A^T_{2,m} \\ ... & ... & ... & ... \\ A^T_{m,1} & A^T_{m,2} & ... & A^T_{m,m} \end{pmatrix}. \tag{4.33}$$

We can surely perform the partial transposition with respect to the first system :

$$\rho^{T_A} = \begin{pmatrix} A_{1,1} & A_{1,2} & ... & A_{1,m} \\ A_{1,2} & A_{2,2} & ... & A_{m,2} \\ ... & ... & ... & ... \\ A_{1,m} & A_{2,m} & ... & A_{m,m} \end{pmatrix}. \tag{4.34}$$

By an example we can show that the P map T is not a CP map or equivalently partial transposition is not a P map. Let us take our density operator as $\rho = |\phi^+\rangle\langle\phi^+|$, then its matrix representation in the basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ is

$$\rho = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \tag{4.35}$$

and if we perform partial transposition, we get

$$\rho^{T_B} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{4.36}$$

One of the eigenvalues of this matrix is negative, it is $-\frac{1}{2}$.

Now we will give an important result as a theorem due to the fact that all P maps are decomposable in low dimensions.

Theorem : A state ρ acting on a Hilbert space whose dimension $dim(H) = dim(H_A)dim(H_B) \leq 6$ is separable if and only if $\rho^{T_B} \geq 0$.

Proof : If ρ is separable, then its partial transposition is positive from the fact that transposition is a P map. The sufficiency part can be proved by considering that P maps are decomposable for $dim(H) = dim(H_A)dim(H_B) \leq 6$. Thus,

$$\left(I \otimes \Lambda\right)(\rho) = \left(I \otimes \Lambda^1_{CP}\right)(\rho) + \left(I \otimes \Lambda^2_{CP} \circ T\right)(\rho), \tag{4.37}$$

in this expression the first term is positive and the whole is positive if $\rho^{T_B} \geq 0$ because second term is positive in this condition. Hence, partial transposition provides us a necessary and sufficient condition for separability in low dimensions.

A density matrix verifying Peres - Horodecki criterion will be termed PPT for positive partial transpose, otherwise we call it NPT for non-positive partial transpose. In general, there are PPT states in higher dimensions ($dim(H_A) = 2, dim(H_B) \geq 4$ *or* $dim(H_A) \geq 3$), which not separable. The PPT entangled states are named 'bound entangled states' to set them apart 'free entangled states'. These terms are associated with the concept of distillability to be discussed in next section.

ii) Reduction Criterion [14]: Let us consider the map

$$\Lambda_r(\rho) = Tr(\rho)I - \rho, \tag{4.38}$$

here ρ is a positive operator with eigenvalues $\Lambda_i \geq 0$, if we look at the eigenvalues of the mapped operator $\Lambda_r(\rho)$:

$$\lambda'_i = \sum_i \lambda_i - \lambda_j = \sum_{i \neq j} \lambda_i \geq 0, \tag{4.39}$$

then it is a positive map.We can now express the criterion : If ρ_{AB} is separable, then

$$(id \otimes \Lambda_r)[\rho] = \rho_A \otimes I - \rho_{AB} \geq 0 \tag{4.40}$$

$$(id \otimes \Lambda_r)[\rho] = I \otimes \rho_B - \rho_{AB} \geq 0. \tag{4.41}$$

We can obtain a very important and useful result from the reduction criterion that the states ρ acting on the Hilbert space $H_A \otimes H_B$, $dim(H_A) = dim(H_B) = m$, with $F(\rho) > \frac{1}{m}$, where $F(\rho) = \langle \psi_+^m | \rho | \psi_+^m \rangle = Tr(\rho \left| \psi_+^m \right\rangle \left\langle \psi_+^m \right|)$ is the fidelity of ρ, must be entangled. Actually, we know that $\left\langle \psi_+^m \right| \varrho_A \otimes I - \varrho \left| \psi_+^m \right\rangle \geq 0$ is true for any separable state ϱ and maximally entangled state $|\psi_+^m\rangle$. Since $Tr_B(\left| \psi_+^m \right\rangle \left\langle \psi_+^m \right|) = \frac{1}{m} I$, $\left\langle \psi_+^m \right| \varrho_A \otimes I \left| \psi_+^m \right\rangle = Tr(\varrho_A \otimes I \left| \psi_+^m \right\rangle \left\langle \psi_+^m \right|) = \frac{1}{m}$. Thus we obtain $F(\rho) \leq \frac{1}{m}$ for separability. This result also provides us a separability criterion. Additionally, reduction criterion is a necessary and sufficient condition for the cases $dim(H) = dim(H_A) dim(H_B) \leq 6$ like the PPT criterion.

5 BOUND ENTANGLEMENT

For many applications in quantum information theory, as in the example of quantum teleportation and superdense coding mentioned in the third section, maximally entangled states are needed. In the real world; however, interaction with the environment is an inevitable process. Because of that interaction, the pure states transform into mixed states, with which quantum protocols cannot be performed reliably. Fortunately, to overcome the noise due to interaction, there is a process called "distillation." The idea is to concentrate the entanglement of many copies of a state, which is not useful for quantum information processing, into fewer copies of the maximally entangled states by means of local operations and classical communication (LOCC). Besides the separability problem, there exists the question of distillability: Which states can be distilled? In this section, it will be shown that PPT states cannot be distilled, so we cannot extract any entanglement from separable states by LOCC. The so-called Range criterion will be mentioned to give examples of entangled states that cannot be distilled, that is the PPT entangled states. The entangled states, which cannot be distilled, will be termed "bound entangled states," in contrast to free entangled states. There is also another problem: Do NPT bound entangled states exist? In the context of this question, we have worked on a generalized Werner state with three parameters and found a region where it is NPT 1-undistillable. Bound entangled states alone are useless for the applications of quantum information, but they can be "activated," that is, they can be useful by combining them with a free entangled state in a protocol called quasi-distillation. With the help of activation, we have made a full characterization of a general state with two parameters in $3 \otimes 3$ space.

5.1 BBPSSW Distillation Protocol

Let us consider that Alice and Bob share a large number n of qubit pairs each with nonmaximally entangled state $\rho_w(F)$, which is known as the two-qubit Werner state [17]:

$$\rho_w(F) = F\left|\psi^-\right\rangle\left\langle\psi^-\right| + \frac{1-F}{3}\left(\left|\psi^+\right\rangle\left\langle\psi^+\right| + \left|\phi^+\right\rangle\left\langle\phi^+\right| + \left|\phi^-\right\rangle\left\langle\phi^-\right|\right). \quad (5.1)$$

In this protocol [5], Alice and Bob begins with two pairs of Werner states of fidelity F and by using LOCC operations, they obtain one pair in a Werner state of fidelity $F' > F$. Here are the steps:

Step 1) By performing a unilateral σ_y on each of the two pairs, they transform mostly $|\psi^-\rangle$ Werner states to mostly $|\phi^+\rangle$ Werner states,

$$W_F^{'} = F\left|\phi^+\right\rangle\left\langle\phi^+\right| + \frac{1-F}{3}\left|\phi^-\right\rangle\left\langle\phi^-\right| + \frac{1-F}{3}\left|\psi^-\right\rangle\left\langle\psi^-\right| + \frac{1-F}{3}\left|\psi^+\right\rangle\left\langle\psi^+\right| \quad (5.2)$$

Step 2) Alice and Bob operate CNOT transformation locally on the pairs in their possesion

$$U_{CNOT}(|i\rangle \otimes |j\rangle) = |i\rangle \otimes |i + j(mod2)\rangle\,. \quad (5.3)$$

The first qubit is called control qubit, and the second one is target qubit. After this operation, they measure the target pair locally along the z axis. If their outcomes are parallel, they keep source pair, otherwise they discard it. In the case of success, we get the following unnormalized state.

$$\begin{aligned} W_F^{''} &= \left(F^2 + \frac{(1-F)^2}{9}\right)\left|\phi^+\right\rangle\left\langle\phi^+\right| + \frac{2}{3}F(1-F)\left|\phi^-\right\rangle\left\langle\phi^-\right| \\ &+ \frac{2}{9}(1-F)^2\left|\psi^-\right\rangle\left\langle\psi^-\right| + \frac{2}{9}(1-F)^2\left|\psi^+\right\rangle\left\langle\psi^+\right|. \end{aligned} \quad (5.4)$$

Step 3) If the source pair has been kept, it is converted back to a mostly $|\psi^-\rangle$ state by performing a unilateral σ_y again. We obtain the unnormalized state

$$\begin{aligned} W_F^{'''} &= \left(F^2 + \frac{(1-F)^2}{9}\right)\left|\psi^-\right\rangle\left\langle\psi^-\right| + \frac{2}{3}F(1-F)\left|\psi^+\right\rangle\left\langle\psi^+\right| \\ &+ \frac{2}{9}(1-F)^2\left|\phi^-\right\rangle\left\langle\phi^-\right| + \frac{2}{9}(1-F)^2\left|\phi^+\right\rangle\left\langle\phi^+\right|. \end{aligned} \quad (5.5)$$

The normalization constant is

$$tr(W_F^{'''}) = F^2 + \frac{2}{3}F(1-F) + \frac{5}{9}(1-F)^2. \quad (5.6)$$

We get the normalized state from the relation

$$W_{F'} = \frac{W_F^{'''}}{Tr(W_F^{'''})}. \quad (5.7)$$

The fidelity of the new state is

$$
\begin{aligned}
F' &= \langle\psi^-|\, W_{F'}\, |\psi^-\rangle \\
&= Tr(W_{F'}\, |\psi^-\rangle\langle\psi^-|) \qquad (5.8)\\
&= \frac{F^2+\frac{1}{9}(1-F)^2}{F^2+\frac{2}{3}F(1-F)+\frac{5}{9}(1-F)^2}, \qquad (5.9)
\end{aligned}
$$

for which we have

$$F' > F \quad for \quad \frac{1}{2} < F < 1. \tag{5.10}$$

By repeating this procedure with the pairs of fidelity F', we can obtain states arbitrarily close to singlet state.

There are entangled two-qubit states with the singlet fraction $\langle\psi_0^-|\,\rho\,|\psi_0^-\rangle < \frac{1}{2}$, whose F cannot be made larger than $\frac{1}{2}$ by any product unitary transformation. The family of states

$$\rho(F) = F\,|\psi^-\rangle\langle\psi^-| + (1-F)\,|11\rangle\langle 11| \tag{5.11}$$

are entangled for $F > 0$. In order to increase the fidelity , Alice and Bob perform the so-called 'filtering' operation that contains generalized measurement. If they perform the following POVM operators

$$A_0 = a\,|0\rangle\langle 0| + \sqrt{1-a^2}\,|1\rangle\langle 1|\,, \tag{5.12a}$$

$$A_1 = a\,|1\rangle\langle 1| + \sqrt{1-a^2}\,|0\rangle\langle 0\,| \tag{5.12b}$$

where $0 < a < 1$, and get 0 as a result, then they keep the state. Otherwise, they discard it. The probability P_0 of getting 0 for both Alice and Bob is

$$P_0 = (1-a^2)\left[a^2F + (1-a^2)(1-F)\right], \tag{5.13}$$

and the state after obtaining 0 is, $\rho' = \rho(F')$, where

$$F' = \frac{a^2F}{a^2F + (1-a^2)(1-F)}. \tag{5.14}$$

Here we can make the new fidelity F' larger than $\frac{1}{2}$, also arbitrarily close to one. By this discussion it has been shown that all entangled two-qubit states are distillable [11]. The filtering protocol works for the states of $2 \otimes 3$ systems and NPT states of $2 \otimes n$ systems .

5.2 Distillability

In the previous subsection it was concluded that all entangled two qubit states are distillable. Thus there occurs a suspicion whether all entangled states are distillable. However we will see that there are entangled states which cannot be distilled. The following theorem will provide us a definition of distillability of mixed states. It expresses that distillability of a given state can be determined by handling the projections on 2×2 dimensional subspace of the Hilbert space on which ρ acts.

Theorem 1 [13] : A state ρ is distillable if and only if the state

$$\sigma = P_A \otimes P_B \rho^{\otimes n} P_A \otimes P_B \tag{5.15}$$

is entangled (or NPT) for some two dimensional local projectors P_A, $P_B$and for some number n. The projector $P_A \otimes P_B$ is onto a $2 \otimes 2$ subspace spanned by

$$|e_i\rangle \in H_A^{\otimes n}, \quad |f_j\rangle \in H_B^{\otimes n}, \quad i = 1, 2. \tag{5.16}$$

We can reformulate the theorem alternatively : ρ is distillable if and only if there exists a state $|\psi\rangle = a_1 |e_1\rangle \otimes |f_1^*\rangle + a_2 |e_2\rangle \otimes |f_2^*\rangle$ from a 2×2 dimensional subspace such that

$$\langle\psi| (\rho^{T_B})^{\otimes n} |\psi\rangle < 0. \tag{5.17}$$

If the above condition is satisfied by the k copies of any state it will be called k-distillable [19]. Presently, we can relate the Peres-Horodecki criterion to the distillability problem with the following two theorems.

Theorem 2 [13, 15]: If a state ρ is PPT, i.e., $\rho^{T_B} \geq 0$, then it is not distillable.

Proof: The general $LOOC$ operation acting on $\rho^{\otimes n}$ can be written as

$$\rho' = \sum_i A_i \otimes B_i \rho^{\otimes n} A_i^\dagger \otimes B_i^\dagger \tag{5.18}$$

this is also the general form of the way that any CP map acts. Now if we take the partial transposition of ρ' we obtain

$$(\rho')^{T_B} = \sum_i A_i \otimes (B_i^\dagger)^T (\rho^{T_B})^{\otimes n} A_i^\dagger \otimes (B_i)^T \tag{5.19}$$

This is also action of a CP map on $(\rho^{T_B})^{\otimes n}$. If $\rho^{T_B} \geq 0$ then is $(\rho')^{T_B}$. Here it is concluded that by any $LOOC$ it is not possible to get a maximally entangled state, which NPT, from a PPT one.

Theorem 3: If $\rho^{T_B} \ngeq 0$, i.e. is NPT in dimensions satisfying $dim(H_A)dim(H_B) \leq 6$ then ρ is distillable. The idea of the proof of the above theorem is simple: We know from section4 that all NPT states of $2 \otimes 2$, $2 \otimes 3$ systems are entangled and from subsection 5.1 we also know that all entangled states of systems having the same dimensions are distillable. Combining these two statements gives us the proof [15].

5.3 PPT Entangled States

The first examples of PPT entangled states [16], that can not be distilled due to Theorem 2 in previous section, will be mentioned in this part. Since they are undistillable they are termed 'bound': PPT bound entangled states (PPTBES's). A separability criterion providing the method of construction of PPTBES's, which is called Range criterion, will be given.

Range criterion [16]: If a state ρ acting on the Hilbert space $H_{AB} = H_A \otimes H_B$ is separable then there exists a set of product vectors $\{|\psi_i\rangle \otimes |\phi_j\rangle\}$ and probabilities p_{ij} such that

(i) ρ and ρ^{T_B} can be written as

$$\rho = \sum_{ij} p_{ij} |\psi_i\rangle \langle\psi_i| \otimes |\phi_j\rangle \langle\phi_j| \tag{5.20}$$

$$\rho^{T_B} = \sum_{ij} p_{ij} |\psi_i\rangle \langle\psi_i| \otimes \left|\phi_j^*\right\rangle \left\langle\phi_j^*\right| . \tag{5.21}$$

(ii) The product vectors $\{|\psi_i\rangle \otimes |\phi_j\rangle\}$ and their partial complex conjugate $\left\{|\psi_i\rangle \otimes \left|\phi_j^*\right\rangle\right\}$ span the range of ρ and ρ^{T_B} respectively,

$$R(\rho) = span\{|\psi_i\rangle \otimes |\phi_j\rangle\} \tag{5.22}$$

$$R\left(\rho^{T_B}\right) = span\left\{|\psi_i\rangle \otimes \left|\phi_j^*\right\rangle\right\} . \tag{5.23}$$

Range criterion provides us a method to determine the entangled states, which are not detected by Peres-Horodecki criterion, in dimensions higher than $2 \otimes 2$ and $2 \otimes 3$.

Example : Consider the following state of a $3 \otimes 3$ system.

$$\rho_a = \frac{1}{8a+1} \begin{pmatrix} a & 0 & 0 & 0 & a & 0 & 0 & 0 & a \\ 0 & a & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a & 0 & 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 & a & 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 & 0 & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(1+a) & 0 & \frac{1}{2}\sqrt{1-a^2} \\ 0 & 0 & 0 & 0 & 0 & a & 0 & a & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2}\sqrt{1-a^2} & 0 & \frac{1}{2}(1+a) \end{pmatrix} \quad (5.24)$$

with $0 \leq a \leq 1$. It can be obtained easily by taking the partial transpose and then finding the eigenvalues, that it is PPT. Besides if we take the partial conjugate of the product vectors belonging to $R\left(\rho^{T_B}\right)$ we see that all such product vectors are orthogonal to a product vector belonging to $R(\rho)$. Because of the violation of the second statement of the above criterion for any $a \neq 0, 1$. It is then a PPTBES.

5.4 NPT Bound Entangled States

It has been seen until now that every density operator with PPT property cannot be distilled, and that all NPT states are distillable for dimensions $2 \otimes 2$, $2 \otimes 3$ and $2 \otimes n$. Due to Theorem2 in subsection 5.2 non-positive partial transposition forms a necessary condition, but it seems not to be a sufficient condition for distillability because of a conjecture expressing that there exist entangled states with non-positive partial transpose, which are, however undistillable. Here is the conjecture [19, 6] : Given is the one parameter family of Werner states of $3 \otimes 3$ system,

$$\rho_W(\lambda) = \frac{1}{8\lambda - 1}(\lambda I - \frac{\lambda + 1}{3} V). \quad (5.25)$$

V is the swap operator, i.e., $V|ij\rangle = |ji\rangle$ for all states $|ij\rangle$ where $i, j = 1, 2, 3$. The state $\lim_{\lambda \to \infty} \rho_W(\lambda)$ is separable and for any finite $\lambda \geq \frac{1}{2}$ $\rho_W(\lambda)$ is entangled and violates the Peres-Horodecki criterion. It is conjectured that for all $\lambda \geq 2$ the state $\rho_W(\lambda)$ is undistillable.

This conjecture eases the problem of finding NPT bound entangled states, because it deals only with entangled Werner states. However, there is no problem with handling only Werner states since the statement 'Any NPT state is distillable.' is equivalent to the statement 'Any entangled Werner states is distillable.' We can support the conjecture by the following results. It has been proved that $\rho_W(\lambda)$

is 1-undistillable for all $\lambda \geq 2$ and numerical evidence has been found for 2 and 3-undistillability [19].

We have generalized the 1-undistillable case by working on the Werner states with three parameters :

$$\rho_b = b_1(|\psi_+\rangle \langle\psi_+|)^{T_B} + b_2\sigma_+ + b_3\sigma_- + b_4\sigma_0 \tag{5.26}$$

where

$$|\psi_+\rangle = \frac{1}{\sqrt{3}}\sum_{i=1}^{3}|ii\rangle\,, \tag{5.27}$$

$$\sigma_+ = \frac{1}{3}(|0\rangle\langle 0|\otimes|1\rangle\langle 1| + |1\rangle\langle 1|\otimes|2\rangle\langle 2| + |2\rangle\langle 2|\otimes|0\rangle\langle 0|), \tag{5.28}$$

$$\sigma_- = \frac{1}{3}(|1\rangle\langle 1|\otimes|0\rangle\langle 0| + |2\rangle\langle 2|\otimes|1\rangle\langle 1| + |0\rangle\langle 0|\otimes|2\rangle\langle 2|), \tag{5.29}$$

$$\sigma_0 = \frac{1}{3}(|0\rangle\langle 0|\otimes|0\rangle\langle 0| + |1\rangle\langle 1|\otimes|1\rangle\langle 1| + |2\rangle\langle 2|\otimes|2\rangle\langle 2|). \tag{5.30}$$

In ρ_b the partial transposition of the maximally entangled state occurs since

$$\frac{1}{3}V = (I\otimes T)[P_+^3] = (|\psi_+\rangle\langle\psi_+|)^{T_B}. \tag{5.31}$$

Now we will investigate the relations for which our generalized Werner state ρ_b is PPT or NPT. But first we notice that there must be some other relations for ρ_b to be a legal density matrix. It must have a trace equal to one: $Tr(\rho_b) = b_1 + b_2 + b_3 + b_4 = 1$. And it must be a positive operator, so its eigenvalues are positive for the following conditions:

$$b_1 + b_4 > 0, \quad b_2 + b_3 > 0, \quad b_2 b_3 > 0. \tag{5.32}$$

If we take its partial transposition the state will be as

$$\rho_b^{T_B} = b_1|\psi_+\rangle\langle\psi_+| + b_2\sigma_+ + b_3\sigma_- + b_4\sigma_0 \tag{5.33}$$

Only one of the eigenvalues of $\rho_b^{T_B}$ can be negative if

$$3b_1 + b_4 < 0 \quad\Rightarrow\quad b_1 < -\frac{b_4}{3} \tag{5.34}$$

Then, we can conclude that ρ_b is NPT for this condition. Now we will investigate the distillability of ρ_b by applying the local projectors P_A and P_B:

$$P_A\otimes P_B = (|0\rangle\langle 0| + |1\rangle\langle 1|)\otimes(|0\rangle\langle 0| + |1\rangle\langle 1|) \tag{5.35}$$

onto 2×2 dimensional subspace. We obtain the following normalized state after performing the local projectors,

$$\rho_b^{'} = \frac{P_A \otimes P_B \rho_b P_A \otimes P_B}{Tr(P_A \otimes P_B \rho_b P_A \otimes P_B)} \tag{5.36}$$

that is,

$$\begin{aligned} \rho_b^{'} &= \frac{b_1 + b_4}{c}(|00\rangle\langle 00| + |11\rangle\langle 11|) \\ &+ \frac{b_1}{c}(|01\rangle\langle 10| + |10\rangle\langle 01|) + \frac{b_2}{c}|01\rangle\langle 01| + \frac{b_3}{c}|10\rangle\langle 10| \end{aligned} \tag{5.37}$$

where $c = Tr(P_A \otimes P_B \rho_b P_A \otimes P_B) = 2(b_1 + b_4) + b_2 + b_3$ is the normalization constant. If partial transposition is applied to $\rho_b^{'}$ we get

$$\begin{aligned} \left(\rho_b^{'}\right)^{T_B} &= \frac{b_1 + b_4}{c}(|00\rangle\langle 00| + |11\rangle\langle 11|) \\ &+ \frac{b_1}{c}(|00\rangle\langle 11| + |11\rangle\langle 00|) + \frac{b_2}{c}|01\rangle\langle 01| + \frac{b_3}{c}|10\rangle\langle 10|. \end{aligned} \tag{5.38}$$

This state is NPT for

$$2b_1 + b_4 < 0 \Rightarrow b_1 < -\frac{b_4}{2}, \tag{5.39}$$

and is PPT for

$$2b_1 + b_4 > 0 \Rightarrow b_1 > -\frac{b_4}{2}. \tag{5.40}$$

The last inequality is also the non-distillabity condition for ρ_b. Thus, if we bring the NPT condition together with the non-distillabity one we obtain a condition for ρ_b to be NPT bound entangled:

$$-\frac{b_4}{2} < b_1 < -\frac{b_4}{3}. \tag{5.41}$$

5.5 Activation of Bound Entanglement

Bound entangled states cannot be distilled to a maximally entangled state. Thus, we cannot use them alone for quantum information purposes, for example they are useless for teleportation, if used alone. In this situation, they just behave like separable states; their entanglement does not show itself. Surprisingly, they can be useful in teleportation process indirectly by producing a non-classical effect. Suppose Alice and Bob share a single pair of spin-1 particles in a free entangled mixed state

$$\rho_F = F\left|\psi^+\right\rangle\left\langle\psi^+\right| + (1 - F)\sigma_+, \tag{5.42}$$

where σ_+ is the separable state given by (5.28), and $0 < F < 1$. Here the entanglement of ρ_F is so weak in the case of a single pair that no LOCC operation can enhance its fidelity above some threshold value $F_0 < 1$. In other words, it is not possible for Alice and Bob to get a state ρ' having fidelity $F(\rho') > F_0$. If, in addition, Alice and Bob are supplied with a very large number of pairs in the following state

$$\sigma_\alpha = \frac{2}{7} \left|\psi^+\right\rangle \left\langle\psi^+\right| + \frac{\alpha}{7}\sigma_+ + \frac{5-\alpha}{7}\sigma_-, \tag{5.43}$$

which is PPT entangled, i. e. bound entangled, for any $3 < \alpha \leq 4$, σ_- is the separable state given by (5.29). Then, it can be shown that the bound F_0 on the fidelity can be exceeded by a protocol which is a direct $3 \otimes 3$ analogue of the BBPSSW protocol, and the fidelity of the original free entangled state can be made arbitrarily close to one. This procedure is called activation of bound entanglement or quasidistillation [12]. This process is not achievable by separable states instead of bound entangled states. However, the free entangled pair as well as the set of all bound entangled pairs cannot be quasidistilled themselves. Here is the activation protocol:

(i) Alice and Bob take the free entangled pair in the state $\rho_{free}(F)$ and one of the the pairs being in the state σ_α. They perform the bilateral XOR operation $U_{BXOR} \equiv U_{XOR} \otimes U_{XOR}$, each of them treating the member of a free (bound) entangled pair as a source (target). The unitary XOR gate is defined as

$$U_{XOR}|a\rangle|b\rangle = |a\rangle|b \oplus a\rangle, \quad b \oplus a = (b+a)mod3 \tag{5.44}$$

where the initial state $|a\rangle(|b\rangle)$ corresponds to a source (target) state of 3-level system.

(ii) Alice and Bob measure the members of target pair in basis $|0\rangle, |1\rangle, |2\rangle$. They compare their results via classical communication. If the compared results differ from each other they have to discard both pairs and then the trial of improvement of F fails. If the results agree then the trial succeeds and they discard only the target pair, coming back with improved source pair to the first step (i).

After some calculations it can be seen that the success in step (ii) occurs with a finite probability

$$p(F \to F') = \frac{2F + (1-F)(5-\alpha)}{7} \tag{5.45}$$

leading then to the the transformation $\rho(F) \to \rho(F')$ with the improved fidelity

$$F' = \frac{2F}{2F + (1-F)(5-\alpha)} \tag{5.46}$$

If only $\alpha >$, the above function of F exceeds the value of F on the whole region $(0,1)$. All states σ_α with $3 < \alpha \le 5$. In particular, the effect holds for the region $3 < \alpha \le 4$ conforming that the target state σ_α is inseparable, thus bound entangled in this region.

We have completely characterized the general two-parameter state σ_a

$$\sigma_a = a_1 \left|\psi_+\right\rangle \left\langle\psi_+\right| + a_2\sigma_+ + a_3\sigma_- \tag{5.47}$$

with the help of the quasidistillation protocol. The protocol is useful while distinguishing the bound entangled states and the separable ones from each other. We have used two different free entangled pairs ρ_1 and ρ_2 as source to fully classify our target pair σ_a,

$$\rho_1(F) = F\left|\psi_+\right\rangle \left\langle\psi_+\right| + (1-F)\sigma_+ \ , \quad 0 < F < 1 \tag{5.48}$$

$$\rho_2(F) = F\left|\psi_+\right\rangle \left\langle\psi_+\right| + (1-F)\sigma_- \ , \quad 0 < F < 1 \tag{5.49}$$

Before proceeding the activation protocol, let us find the PPT and NPT conditions by applying partial transposition to σ_a and investigating the eigenvalues of $\sigma_a^{T_B}$. After some algebra we find the NPT condition as

$$a_2 a_3 < a_1^2. \tag{5.50}$$

Then, the PPT condition is

$$a_2 a_3 > a_1^2. \tag{5.51}$$

To find the conditions under which our general state is free entangled (distillable) or bound entangled (undistillable), we project our state to 2×2 subspace with local projectors given in equation (5.35) and we obtain the following normalized state

$$\sigma_A' = \frac{a_1}{d}(\left|00\right\rangle\left\langle00\right| + \left|00\right\rangle\left\langle11\right| + \left|11\right\rangle\left\langle00\right| + \left|11\right\rangle\left\langle11\right|) + \frac{a_2}{d}\left|01\right\rangle\left\langle01\right| + \frac{a_3}{d}\left|10\right\rangle\left\langle10\right|, \tag{5.52}$$

where $d = Tr(P_A \otimes P_B \sigma_a P_A \otimes P_B) = 2a_1 + a_2 + a_3$. The partially transposed state $(\sigma_A')^{T_B}$ has a negative eigenvalue in its spectrum for

$$a_2 a_3 < a_1^2. \tag{5.53}$$

This is the distillability condition for σ_a, which is the same as the one for NPT. That is, NPT condition is sufficient for σ_a to be distillable. Now it is time to categorize the PPT states as bound entangled and separable via activation protocol. Let us take first, ρ_1 as a source pair, so the state to be quasidistilled is

$$\begin{aligned}\rho_1(F)\otimes\sigma_a &= a_1F\left|\psi_+\right\rangle\left\langle\psi_+\right|\otimes\left|\psi_+\right\rangle\left\langle\psi_+\right| + a_2F\left|\psi_+\right\rangle\left\langle\psi_+\right|\otimes\sigma_+ \\ &+ a_3F\left|\psi_+\right\rangle\left\langle\psi_+\right|\otimes\sigma_- + a_1(1-F)\sigma_+\otimes\left|\psi_+\right\rangle\left\langle\psi_+\right| \\ &+ a_2(1-F)\sigma_+\otimes\sigma_+ + a_3(1-F)\sigma_+\otimes\sigma_-.\end{aligned} \tag{5.54}$$

By applying the two steps of the quasidistillation we see that only the first and the last terms contributes, the state

$$\begin{aligned}\sigma' = M_z(A_2,B_2) \quad & [U_{XOR}(A_1,A_2)\otimes U_{XOR}(B_1,B_2)(\rho_1(F)\otimes\sigma_a) \\ & U^\dagger_{XOR}(A_1,A_2)\otimes U^\dagger_{XOR}(B_1,B_2)]M^\dagger_z(A_2,B_2) \\ & = a_1F\left|\psi_+\right\rangle\left\langle\psi_+\right|\otimes\left|\psi_+\right\rangle\left\langle\psi_+\right| + a_3(1-F)\sigma_+\otimes\sigma_-\end{aligned} \tag{5.55}$$

is obtained with the probability

$$p = Tr(\sigma') = a_1F + a_3(1-F). \tag{5.56}$$

Then, our new normalized state is

$$\sigma'' = \frac{\sigma'}{Tr(\sigma')} = \frac{a_1F}{a_1F + a_3(1-F)}\left|\psi_+\right\rangle\left\langle\psi_+\right| + \frac{a_3(1-F)}{a_1F + a_3(1-F)}\sigma_+. \tag{5.57}$$

The new fidelity is

$$F' = \frac{a_1F}{a_1F + a_3(1-F)}, \tag{5.58}$$

and for

$$a_1 > a_3 \tag{5.59}$$

F' is larger than F, $F' > F$. Namely, our state σ_a is bound entangled for this condition together with the PPT condition. If we do the same calculations with ρ_2 we get that

$$a_1 > a_2 \tag{5.60}$$

for the improvement of the new fidelity. We conclude that together with the PPT condition our general state σ_a is bound entangled either for $a_1 > a_3$ or $a_1 > a_2$. And we can easily show that σ_a is separable for

$$a_2 > a_1 \quad \wedge \quad a_2 > a_3, \tag{5.61}$$

together with the PTT criterion. Because it can be written as

$$\sigma_a = a_1 \rho_s + (a_2 - a_1)\sigma_+ + (a_3 - a_1)\sigma_-. \tag{5.62}$$

Here, the coefficients are positive and ρ_s is defined as

$$\rho_s = |\psi_+\rangle \langle\psi_+| + \sigma_+ + \sigma_-, \tag{5.63}$$

which is separable since it can be obtained as

$$\rho_s = \frac{1}{8} \int_0^{2\pi} |\psi(\phi)\rangle \langle\psi(\phi)| \otimes |\psi(-\phi)\rangle \langle\psi(-\phi)| \tag{5.64}$$

where $|\psi(\phi)\rangle = \frac{1}{\sqrt{3}}(|0\rangle + e^{i\phi}|1\rangle + e^{-i2\phi}|2\rangle)$.

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BIOGRAPHY

Serkan Karaçuha was born in İstanbul in 1978. He graduated from Pertevniyal High School in 1996. He obtained his BSc. degree from İstanbul Technical University, Department of Physics. He started MSc education at the same department in 2004. He has been working in İstanbul Technical University, Department of Mathematics as a research assistant since March, 2006.