COHOMOLOGY GROUPS OF MAPPING CLASS GROUPS

by

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B.S., Mathematics, Boğaziçi University, 2004

Submitted to the Institute for Graduate Studies in
Science and Engineering in partial fulfillment of
the requirements for the degree of
Master of Science

Graduate Program in Mathematics
Boğaziçi University
2006

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DATE OF APPROVAL: 15.06.2006 [section]

ACKNOWLEDGEMENTS

I would like to express my sincere gratitude to my thesis supervisor, Assist. Prof. Ferit Öztürk, for his endless support, patience and guidance at all stages of this work. I am extremely thankful to him for the valuable discussions we had and the motivation he has provided during the preparation of this study.

I would like to thank Assist. Prof. Tolga Etgü for his participation in my thesis committee.

I would like to thank Prof. Ercüment Ortaçgil for his participation in my thesis committee and his valuable instructions in Boğaziçi University.

I am extremaly thankful to teaching assistants of department of mathematics in Boğaziçi University for their valuable friendship. I also would like to thank İlke Çanakçi, my dear friend, for her support during the preparation of this thesis.

Also, I greatly appreciate Tübitak, the Scientific and Technological Research Council of Turkey.

My special thanks are to my family for their support, confidence and patience throughout my education.

ABSTRACT

COHOMOLOGY GROUPS OF MAPPING CLASS GROUPS

The mapping class group of an orientable surface of genus g is the group of all orientation preserving piecewise linear homeomorphisms of the surface up to isotopy. In this thesis it is shown that the mapping class group of an orientable surface of genus g is generated by Dehn twists about nonseparating simple closed curves [15]. Then the notion of cohomology groups of a group is introduced following [18]. The first cohomology groups of the mapping class groups of orientable surfaces of genus g greater than one are shown to be trivial [16].

ÖZET

GÖNDERİM SINIFI GRUPLARININ KOHOMOLOJİ GRUPLARI

Oryantasyon verilebilir g delikli bir yüzeyin gönderim sınıfı grubu oryantasyonu koruyan parçalı doğrusal homeomorfizmaların oluşturduğu grubun, birim fonksiyona izotopik oryantasyonu koruyan parçalı doğrusal homeomorfizmaların oluşturduğu normal alt gruba bölünmesiyle elde edilen bölüm grubudur. Bu tezde oryantasyon verilebilir g delikli bir yüzeyin gönderim sınıfı grubunun yüzeyi ayırmayan basit kapalı eğriler etrafında oluşturulan Dehn bükümleri tarafından üretildiği gösterilmiştir [15]. Daha sonra bir grubun kohomoloji grupları kavramı tanıtılmıştır [18]. Oryantasyon verilebilir g delikli yüzeylerin gönderim sınıfı gruplarının birinci kohomoloji gruplarının tek elemanlı bir gruba izomorfik olduğu gösterilmiştir [16].

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LIST OF SYMBOLS

Aut G	The group of automorphisms of a group G
$B^n(\Gamma, G)$	The group of n-coboundaries of Γ over G
c, m, p, $\ldots$	Simple closed curves on a surface S
$C^n(\Gamma, G)$	The group of n-cochains of Γ over G
$D_{g,n}$	The group of orientation preserving piecewise linear homeomorphisms of $T_{g,n}$ which are isotopic to the identity
$G_{g,n}$	The group of orientation preserving piecewise linear homeomorphisms of $T_{g,n}$
$GL(2, \mathbb{Z})$	The general linear group of 2×2 matrices with integer entries
$H_{g,n}$	The subgroup of $G_{g,n}$ which fixes n distinguished points individually
$\mathbb{H}^n$	The half space
$H^n(\Gamma, G)$	The n^{th} cohomology group of Γ over G
$H_n(X, \mathbb{Z})$	The n^{th} homology group o a space X with coefficients in $\mathbb{Z}$
Hom(Γ, G)	The group of homomorphisms from Γ to G
Inn G	The group of inner automorphisms of a group G
$M_{g,n,r}$	The mapping class group of $T_{g,n,r}$
Out G	The quotient group of Aut G by Inn G
$SL(2, \mathbb{Z})$	The special linear group of 2×2 matrices with integer entries
$St(x, K)$	The star of x in K
$\overline{St}(x, K)$	The closure of the star of x in K
$T_{g,n,r}$	An oriebtable surface of genus g with n distinguished points and r boundary components
$Z^n(\Gamma, G)$	The group of n-cocycles of Γ over G
δ	Coboundary operator
$\pi_1(S)$	The fundamental group of a surface S
τ_c	Dehn twist about a simple closed curve c

1. INTRODUCTION

Let $T_{g,0}$ be an orientable surface of genus g with no boundary components and no distinguished points. The mapping class group $M_{g,0}$ of $T_{g,0}$ is the group of all orientation preserving piecewise linear homeomorphisms of $T_{g,0}$ modulo the subgroup of those homeomorphisms which are isotopic to the identity. These groups play an important role in low dimensional topology and are related with hyperbolic geometry and arithmetic groups [19].

The study of the mapping class groups of surfaces started with M. Dehn and J. Nielsen in 1920's. M. Dehn introduced Dehn twists . He was interested in questions like the existence of a finite set of generators for the mapping class groups. The theorem asserting that the mapping class group of an orientable surface is generated by Dehn twists about nonseparating simple closed curves is due to M. Dehn [3]. It was rediscovered by W. B. R. Lickorish at the beginning of 1960's [4]. J. Nielsen was interested in the individual elements of the mapping class groups . The work of Nielsen was forgotten for a while and the ideas of Nielsen were partially rediscovered, extended and brought to a complete form by W. Thurston in his theory of surface diffeomorphisms.

The mapping class group $T_{g,0}$ is very important in the construction of 3-manifolds. This construction involves the notion of *Heegaard splitting*. Let X_g be a *handlebody* of genus g in $\mathbb{R}^3$ and $[\alpha] \in M_{g,0}$ be represented by $\alpha : \partial X_g \to \partial X_g$. Then we define a 3-manifold M_α as $X_g \cup_\alpha X_g$. The manifold M_α is represented as a Heegaard splitting. It is closed and orientable. If one assumes that 3-manifolds can be triangulated , every closed orientable 3-manifold may be obtained in this way. The topological class of 3-manifolds coincides with the isotopy class of orientation preserving piecewise linear homeomorphisms of ∂X_g. In [4] Lickorish rediscovers that any piecewise linear, orientation preserving homeomorphism h of a closed, connected, orientable, combinatorial 2-manifold Y is isotopic to the product of a sequence of Dehn twists (the name he uses for Dehn twists is c-homeomorphisms). By using this theorem he proves any closed, connected, orientable, combinatorial 3-manifold M, is piecewise linearly homeomorphic to S^3 from which has been removed a finite set of disjoint polyhedral solid tori (i.e., discs$\times S^1$) which are sewn back in a different way.

The mapping class group $M_{g,0}$ of a surface $T_{g,0}$ is also important for algebraists. They have been trying to use geometric methods to understand automorphism groups of groups. The mapping class group $M_{g,0}$ of $T_{g,0}$ is an example of such a geometric method due to the following fact. If we consider the group of all piecewise linear homeomorphisms of $T_{g,0}$ modulo the subgroup of those homeomorphisms which are isotopic to the identity, then the quotient group is isomorphic to Out $\pi_1(T_{g,0})$ where Out $\pi_1(T_{g,0})$ is the group of automorphisms of $\pi_1(T_{g,0})$ [1], [2]. Then $M_{g,0}$ is a subgroup of index 2 in Out $\pi_1(T_{g,0})$.

After giving this brief introduction, we state what is done in this thesis. In *Chapter* 2, the equivalency of homeomorphic and diffeomorphic classifications of 2-manifolds is presented and then some notations and preliminary results are given.

In *Chapter* 3, Dehn twists and some relations between Dehn twists are introduced. It is proven that the mapping class group of an orientable closed surface is generated by Dehn twists[15].

The final chapter, *Chapter* 4 starts with the definition of cohomology groups of a group. It is shown that the first cohomology groups of the mapping class groups of orientable surfaces of genus 2 and 3 are trivial.

2. PRELIMINARIES

2.1. Homeomorphic and Diffeomorphic Classifications of 2-Manifolds

In this section, our aim is to sketch the proof of the theorem showing the equivalency of homeomorphic and diffeomorphic classifications of 2-manifolds. It is obvious that diffeomorphic manifolds are homeomorphic. We shall take homeomorphic 2-manifolds, show the existence of smooth structures on them and prove that they are in fact diffeomorphic. The existence of C^r triangulations of C^r manifolds and some important facts regarding triangulations of 2-manifolds are crucial steps of the proof.

Firstly, recall that a topological space M is an n-manifold if
(1) M is a Hausdorff space.
(2) M satisfies the second axiom of countability.
(3) Given $x \in M$, there is a neighbourhood U of x in M and a homeomorphism h from U onto an open set in $\mathbb{R}^n$ or $\mathbb{H}^n$, the half space.

We explain what we mean by a differentiable manifold of class C^r. A differentiable n-manifold of class C^r is an n-manifold M and a differentiable structure $\mathcal{D}$ of class C^r on M where $\mathcal{D}$ is a collection of coordinate neighbourhoods (U_λ, h_λ), $\lambda \in \Lambda$, where Λ is an indexed set, on M which satisfies the following properties:
(1) $\{U_\lambda\}_{\lambda\in\Lambda}$ is a covering of M.
(2) If (U, h_1) and (U, h_2) belong to $\mathcal{D}$, then $h_1 h_2^{-1} : h_2(U_1 \cap U_2) \to \mathbb{R}^n$ is differentiable of class C^r.
(3) The collection $\mathcal{D}$ is maximal with respect to property (2); i.e., if any coordinate neighbourhood not in $\mathcal{D}$ is adjoined to the collection $\mathcal{D}$, then property (2) fails.

It is a well known fact that every topological manifold can be embedded in a Euclidean space [5]. Moreover, if M is a C^r manifold of dimension n then M has a C^r embedding as a closed subset of a Euclidean space [6]. Hence, we can regard manifolds lying in some Euclidean space, their differentiable structure induced by that of the ambient space.

Now we give the definition of a simplicial complex.

Definition 2.1.1. *If* $v_0, v_1, \ldots, v_m$ *are independent points of* $\mathbb{R}^n$*, the simplex* σ *they span is the set of points* x *in* $\mathbb{R}^n$ *such that* $x = \sum_{i=0}^{m} b_i v_i$*, where* $b_i \geq 0$ *and* $\sum_{i=0}^{m} b_i = 1$*. A face of a simplex* σ *is the simplex spanned by a subset of the vertices of* σ*. A simplicial complex* K *is a collection of simplices in* $\mathbb{R}^n$ *which satisfies the following properties:*
(1) Every face of a simplex of K *is in* K*.*
(2) The intersection of two simplices of K *is a face of each of them.*
(3) Each point of $|K|$ *has a neighbourhood intersecting only finitely many simplices of* K *where* $|K|$ *denotes the union of the simplices of* K *and is called the polytope of* K*. Here* $|K|$ *has the topology induced from* $\mathbb{R}^n$*.*

We shall define a C^r triangulation of a C^r manifold. In order to understand this definition, firstly a few things related to simplicial complexes will be defined.

Let x be a point of $|K|$. The *star of* x in K, denoted by $St(x, K)$ is the union of the interiors of all simplices σ such that x lies in σ.

Let K be a simplicial complex. The map $f : |K| \to M$ is said to be differentiable of class C^r relative to K if $f|\sigma$ is of class C^r.

Definition 2.1.2. *Let* $f : |K| \to M$ *be a* C^r *map and* M *be a* C^r *submanifold of* $\mathbb{R}^n$*. Then* f *is called a* C^r *triangulation of* M *if* f *is a homeomorphism and*

$$df_b : \overline{St}(b, K) \to \mathbb{R}^n$$

is one to one for each b*, where* $\overline{St}(b, K)$ *is the closure of the star of* b*.*

The existence of C^r triangulations of C^r manifolds is guaranteed by the following lemma which can be found in [6].

Lemma 2.1.3. *Let M be a C^r manifold without boundary, r possibly ∞. Then M has a C^r triangulation. If M is a manifold with boundary, then any C^r triangulation of the boundary may be extended to a C^r triangulation of M. (If $f : |J| \to \partial M$ is a C^r triangulation of ∂M, an extension of f is a C^r triangulation $g : |L| \to M$ of M such that $g^{-1}f$ is a linear isomorphism of J with a subcomplex of L.)*

From now on we will work on simplicial complexes corresponding to given C^r manifolds since it is easy to work on them instead of manifolds. We shall introduce combinatorial equivalence of simplicial complexes and manifolds.

Let K, L be simplicial complexes in $\mathbb{R}^n$. They are said to be *combinatorially equivalent* if only if they have (simplicial) subdivisions which are isomorphic to each other. We note that combinatorial equivalence is an equivalence relation on simplicial complexes since two subdivisions of of a simplicial complex have a common subdivision.

Definition 2.1.4. *Let M and L be C^r n-manifolds with C^r triangulations $f : |K| \to M$ and $g : |L| \to N$. Then M and N are said to be combinatorially equivalent if and only if K and L are combinatorially equivalent.*

The following lemma explains the relationship between C^r triangulations of a C^r manifold and the proof can be found in [7].

Lemma 2.1.5. *Let M be a C^r n-manifold with two C^r triangulations $f : |K| \to M$ and $g : |L| \to M$. Then K and L are combinatorially equivalent.*

Now we consider the following question. Does the existence of a homeomorphism between $|K|$ and $|L|$ imply the existence of a piecewise linear homeomorphism of $|K|$ onto $|L|$? If the answer is affirmative, then this will imply that homeomorphic differentiable manifolds have isomorphic simplicial complexes. The following lemma gives the answer of the above question[8].

Lemma 2.1.6. *Let K and L be simplicial complexes and $\dim K = 2$. Suppose $|K|$ and $|L|$ are homeomorphic. Then there exists a piecewise linear homeomorphism of $|K|$ onto $|L|$.*

Finally we shall state a very important result, which can be found in [9], about the existence of a diffeomorphism between differentiable manifolds, which will be used in the proof of Theorem 2.1.8.

Lemma 2.1.7. *If M and N are differentiable n-manifolds which are combinatorially equivalent with $n \leq 3$, then M and N are diffeomorphic.*

Now we shall prove homeomorphic 2-manifolds are in fact diffeomorphic.

Theorem 2.1.8. *Let $(M, \mathcal{A})$ and $(N, \mathcal{B})$ be homeomorphic differentiable 2-manifolds. Then they are diffeomorphic.*

Proof. By Lemma 2.1.3 there are C^∞ triangulations $f : |K| \to M$, $g : |L| \to N$ where K and L are two dimensional simplicial complexes. Then $|K|$ and $|L|$ are homeomorphic since M and N are homeomorphic and f and g are homeomorphisms. By Lemma 2.1.6 there exists a piecewise linear homeomorphism of $|K|$ onto $|L|$. We can find subdivisions K' and L' of K and L respectively such that $h : |K'| \to |L'|$ is a linear isomorphism. So K and L are combinatorially equivalent, i.e. M and N are combinatorially equivalent. Hence we get M and N are diffeomorphic by Lemma 2.1.7. □

In the above theorem we start with differentiable homeomorphic manifolds. So we must show that a 2-manifold has a smooth structure and this can be found in [10].

We present some definitions and results which will be used in the sequel.

2.2. Orientation and Homology

Let M be an n-dimensional topological manifold and x be a point of M. Let U' be an open set of M containing x and $h' : \mathbb{R}^n \to U'$ be a homeomorphism. Also, let h

be the restriction of h' to D^n and $U = h(V)$, $V \subsetneq \text{int}D^n$. We show that

$$H_k(M, M - U) \simeq H_k(D^n, S^{n-1}) \simeq \begin{cases} \mathbb{Z}, & k = n; \\ 0, & \text{otherwise.} \end{cases} \tag{2.1}$$

and

$$H_k(M, M - \{x\}) \simeq H_k(D^n, D^n - \{y\}) \simeq \begin{cases} \mathbb{Z}, & k = n; \\ 0, & \text{otherwise.} \end{cases} \tag{2.2}$$

Let $A = M - U$ and $B = h(D^n)$. Then by Excision Theorem,

$$H_k(M, A) \simeq H_k(B, A \cap B) \tag{2.3}$$

Since h is a homeomorphism from D^n onto a subset of M and $h^{-1}(A \cap B)$ deformation retracts onto $h^{-1}(\partial A)$ which is homeomorphic to S^{n-1}, we get

$$H_k(B, A \cap B) \simeq H_k(D^n, h^{-1}(A \cap B)) \simeq H_k(D^n, S^{n-1}).$$

Also we know that

$$H_k(D^n, S^{n-1}) \simeq \begin{cases} \mathbb{Z}, & k = n; \\ 0, & \text{otherwise.} \end{cases} \tag{2.4}$$

Hence this result and 2.3 gives 2.1. Similarly we prove 2.2. Since $D^n - \{y\}$ deformation retracts onto S^{n-1} we have $H_k(D^n, D^n - \{y\}) \simeq H_k(D^n, S^{n-1})$. Then from 2.1 and 2.2 we get

$$H_k(M, M - U) \simeq H_k(M, M - \{x\}).$$

Now suppose that M is a connected n-manifold without boundary. It is known that the map $i_* : H_n(M, M - U) \simeq \mathbb{Z} \to H_n(M, M - \{x\}) \simeq \mathbb{Z}$ induced by the inclusion $i : (M, M - U) \to (M, M - \{x\})$ is an isomorphism. If μ_U is a generator of $H_n(M, M - U)$, then the corresponding generator $\mu_x \in H_n(M, M - \{x\})$ is determined as $i_*(\mu_U) = \mu_x$. Hence we can give the following definition.

Definition 2.2.1. *An n dimensional manifold M is called orientable if there is a choice of $\mu_x \in H_n(M, M - \{x\})$ for each $x \in M$ such that*
(i) μ_x is a generator of $H_n(M, M - \{x\})$ for each $x \in M$.
(ii) These choices are locally consistent in the sense that for each $x \in M$, there is an open set U containing x which is homeomorphic to a proper subset of the interior of an embedded disc in $\mathbb{R}^n$ and a generator $\mu_U \in H_n(M, M - U) \simeq \mathbb{Z}$ such that $i_*(\mu_U) = \mu_x = \mu_y$ for all $y \in U$.

A choice of μ_x for all x is called a *homology orientation* or just an *orientation* for M. If an orientation exists for M then M is called *orientable*.

Remark 2.2.2. *If M is a connected, orientable n-manifold, then the choice of homology orientation at a fixed x_0 determines the choice everywhere.*

Now we give a theorem which explains the relationship between the orientability of a closed manifold and the structure of its homology [11].

Theorem 2.2.3. *Let M be a compact, connected n-manifold without boundary. Then,*
(1) If M is orientable, the map $H_n(M) \to H_n(M, M - \{x\})$ is an isomorphism for all $x \in M$.
(2) If M is nonorientable, then $H_n(M) = 0$.

Let M be n-dimensional orientable manifold and $H_n(M) \simeq \mathbb{Z}$. Let f be a homeomorphism of M. Then f induces an isomorphism, denoted by f_* on $H_k(M)$ for every k. Consider $f_* : H_n(M) \to H_n(M)$ where $H_n(M)$ is isomorphic to $\mathbb{Z}$. Then f_* is either id or $-id$ isomorphisms. This leads to the following definition.

Definition 2.2.4. *A homeomorphism f of an n-manifold is said to be orientation preserving if the induced isomorphism $f_* : H_n(M) \to H_n(M)$ is the identity isomorphism.*

2.3. Surfaces

A surface S means a connected 2-manifold. A closed surface means a compact surface without boundary. Starting with this section we assume that S is compact. Now we present some known facts about surfaces.

We start with the classification of surfaces. It is a well known fact that any closed orientable surface is homeomorphic to a sphere or a connected sum of tori. Similarly, any closed, nonorientable surface is homeomorphic to the connected sum of projective planes. The proof uses the fact that compact surfaces can be triangulated and provides us with a model of the surface. In fact, a compact surface S of genus g is homeomorphic to a polygon with the vertices all identified and the edges identified in a well known manner[12].

If S has l boundary components, then our model is obtained via removing l disjoint discs from the interior of the polygon having $4g$ sides if S is orientable and $2g$ sides if S is nonorientable. Then the fundamental group of S has a representation as follows where a_i's and b_i's denote the edges of S and c_j's denote the boundary components of S:

(1) If S is orientable

$$\pi_1(S) = \langle a_1, b_1, \ldots, a_g, b_g, c_1, \ldots, c_l \mid a_1 b_1 a_1^{-1} b_1^{-1} \ldots a_g b_g a_g^{-1} b_g^{-1} c_1 \ldots c_l = 1 \rangle$$

(2) If S is nonorientable

$$\pi_1(S) = \langle a_1, \ldots, a_g, c_1, \ldots, c_l \mid a_1^2 \ldots a_g^2 c_1 \ldots c_l = 1 \rangle$$

Also, $\pi_1(S^2) = 0$. Moreover, the homology groups of S are as follows:

(1) If S is orientable

$$H_k(S) = \begin{cases} \mathbb{Z}, & k = 0 \text{ or } 2; \\ \mathbb{Z}^{2g}, & k = 1; \\ 0, & \text{otherwise.} \end{cases}$$

(2) If S is nonorientable

$$H_k(S) = \begin{cases} \mathbb{Z}, & k = 0; \\ \mathbb{Z}^{g-1} \oplus \mathbb{Z}_2, & k = 1; \\ 0, & \text{otherwise.} \end{cases}$$

Also,

$$H_k(S^2) = \begin{cases} \mathbb{Z}, & k = 0 \text{ or } 2; \\ 0, & \text{otherwise.} \end{cases}$$

2.4. Isotopies and Curves on Surfaces

Definition 2.4.1. *We say that the homeomorphisms* $f_0, f_1 : X \to X$ *are isotopic, if there is a level preserving homeomorphism* $F : X \times I \to X \times I$ *with* $F(x,t) = (F_t(x), t)$ *and* $F_0 = f_0,\ F_1 = f_1$. *Then* F *is called an isotopy between* f_0 *and* f_1. *We say that embeddings* $g_0, g_1 : X \to Y$ *are ambient isotopic if there is an isotopy* $F : Y \times I \to Y \times I$ *such that* $F_0 =$*id and* $F_1 g_0 = g_1$.

Let S be a surface. From now on we assume that S is orientable.

A *simple closed curve* on a surface S is an embedding $c : S^1 \to S$ where S^1 is the unit circle in $\mathbb{R}^2$. For our purposes, we will call the image $c(S^1)$ as a simple closed curve on S as well, i.e., a simple closed curve c on a surface S is a submanifold of S homeomorphic to the standard circle in $\mathbb{R}^2$. An *arc* on a surface S is an embedding from the unit interval $[0,1]$ to S. Similarly we call the image of the embedding as an arc on S.

Proposition 2.4.2. *Let c be a simple closed curve on S. Then c has a neighbourhood N homeomorphic to a cylinder.*

Proof. Consider the polygon model of S. If c lies in the interior of the polygon or c is ambient isotopic to a boundary component then the result is obvious. If c consists of arcs intersecting the edges of the polygon, we make c intersect the edges of the polygon transversally and the two boundary points of each arc lie on distinct edges. Also, these arcs do not intersect each other since c is a simple closed curve. We prove that the number of these arcs is finite. Suppose that the number of the arcs is infinite and let $\mathcal{A} = \{A_j : j \in J\}$ be the set of these arcs where J is an index set. Consider a countable subset $\mathcal{B}$ of $\mathcal{A}$ and enumerate the elements of $\mathcal{B}$ to get a sequence $\mathcal{B} = \{B_1, B_2, \ldots, B_n, \ldots\}$ of arcs. Now we form a new sequence $(x_n)_{n\in\mathbb{N}}$ where for each n, $x_n \in B_n$ and x_n is not a boundary point of B_n. Since $(x_n)_{n\in\mathbb{N}}$ is a bounded sequence in $\mathbb{R}^2$ it has a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ converging to, say, x_0. But then $c^{-1}((x_{n_k}))_{k\in\mathbb{N}}$ converges to $c^{-1}(x_0)$ because $c^{-1} : c(S^1) \to S^1$ is a homeomorphism being the inverse of c. This gives a contradiction since c is an embedding and we chose each x_n from a different arc. So we have a finite number of arcs, say $\{A_i : i = 0, \ldots, k.\}$. For each arc A_i, we choose an ε_i-neighbourhood. By taking the minimum of ε_i's we can make ε_i-neighbourhoods disjoint. Then we take the union N of the closures of the $\frac{\varepsilon_i}{2}$ -neighbourhoods, so we get either a cylinder or a Möbius strip. Since S is orientable, N is a cylinder. □

We say that a simple closed curve c on S is *separating* if $S-\text{int}N$ is connected and *nonseparating* otherwise. We note that $S-\text{int}N$ is an orientable surface with two more boundary components. For simplicity we use the notation $S - c$ for $S-\text{int}N$. Also, a simple closed curve c on S is called *trivial* if it is either homotopic to zero in $\pi_1(S)$ or homotopic to some boundary component of S.

Proposition 2.4.3. *Suppose that S is a surface of genus g and c is a nonseparating simple closed curve. Then $S - c$ is a surface of genus $g - 1$.*

Proof. S is the disjoint union of $S - c$ and N. Hence,

$$\chi(S) = \chi(S - c) + \chi(N). \tag{2.5}$$

We know that $\chi(S) = 2 - 2g - r$, and $\chi(S - c) = 2 - 2\widetilde{g} - (r + 2)$, where r is the number of boundary components of S and $\widetilde{g}$ is the genus of $S - c$. Also, $\chi(N) = 0$. Thus we get $\widetilde{g} = g - 1$ from Equation 2.5. $\square$

3. MAPPING CLASS GROUPS OF ORIENTABLE SURFACES

In this chapter, we define the mapping class groups of orientable surfaces. Then we introduce the most important examples of nontrivial elements of mapping class groups, namely Dehn twists. Our aim is to show that the mapping class group of an orientable surface is generated by Dehn twists about non-separating simple closed curves.

3.1. Definitions

Let $T_{g,0}$ denote a closed orientable surface of genus g, and let $T_{g,n}$ denote the same surface after n distinct points $z_1, z_2, \ldots, z_n$ are removed. Let $G_{g,n}$ be the group with operation the composition of all orientation preserving piecewise linear homeomorphisms of $T_{g,n} \to T_{g,n}$. Then $G_{g,n}$ contains a subset $D_{g,n}$ consisting of homeomorphisms which are isotopic to the identity.

Proposition 3.1.1. *$\mathrm{D}_{g,n}$ is a normal subgroup of $\mathrm{G}_{g,n}$.*

Proof. Since every map is isotopic to itself $id \in D_{g,n}$ so $D_{g,n} \neq \emptyset$. Let f and g be isotopic to the identity via isotopies $F(x,t) = (F_t(x), t)$ where $F_0 = id,\ \ F_1 = f$ and $G(x,t) = (G_t(x), t)$ where $G_0 = id,\ \ G_1 = g$ respectively. Then $H(x,t) = (H_t(x), t)$ where $H_t(x) = F_t G_t(x)$ is the required isotopy between fg and id implying that $fg \in D_{g,n}$. Also, $\widetilde{F}(x,t) = (F_t^{-1}(x), t)$ is the isotopy between id and f^{-1}. Hence, $D_{g,n}$ is a subgroup of $G_{g,n}$. Let $g \in G_{g,n}$. Then $\overline{F}(x,t) = (gF_t g^{-1}(x), t)$ is the isotopy between gfg^{-1} and id. So $gfg^{-1} \in D_{g,n}$. Thus, $D_{g,n}$ is a normal subgroup of $G_{g,n}$. □

Definition 3.1.2. *The mapping class group $\mathrm{M}_{g,n}$ of $\mathrm{G}_{g,n}$ is defined to be the quotient group $\mathrm{G}_{g,n} \ / \ \mathrm{D}_{g,n}$, i.e., the mapping class group $\mathrm{M}_{g,n}$ of $\mathrm{T}_{g,n}$ is the group of isotopy classes of all orientation preserving piecewise linear homeomorphisms of $\mathrm{T}_{g,n} \to \mathrm{T}_{g,n}$.*

In the above definition, $T_{g,n}$ is obtained from $T_{g,0}$ by removing n points. Since a neighbourhood of z_i is mapped to a neighbourhood of $z_j, 1 \leq i, j \leq n$, by a homeomorphism, we can regard $G_{g,n}$ as the group of all orientation preserving piecewise linear homeomorphisms fixing $\{z_1, z_2, \ldots, z_n\}$ setwise. Now, let $H_{g,n}$ be the subgroup of $G_{g,n}$ which fixes $z_1, z_2, \ldots, z_n$ individually. Then a subgroup $P_{g,n}$ of $M_{g,n}$ is defined to be the quotient group $H_{g,n}/D_{g,n}$. Notice that since $D_{g,n}$ consists of homeomorphisms isotopic to the identity then all element of $D_{g,n}$ must fix $\{z_1, z_2, \ldots, z_n\}$ pointwise.

Proposition 3.1.3. $\mathrm{P}_{g,n}$ *has index* $n!$ *in* $\mathrm{M}_{g,n}$.

Proof. Let $g \in G_{g,n}$ and $h \in H_{g,n}$. Since h fixes $\{z_1, z_2, \ldots, z_n\}$ pointwise then ghg^{-1} fixes $\{z_1, z_2, \ldots, z_n\}$ pointwise so $H_{g,n}$ is a normal subgroup of $G_{g,n}$. Then by the third isomorphism theorem $(G_{g,n}/D_{g,n})/(H_{g,n}/D_{g,n}) \simeq G_{g,n}/H_{g,n}$. Let $g_1, g_2 \in G_{g,n}$. Then $g_1 H_{g,n} = g_2 H_{g,n}$ if and only if $g_1 g_2^{-1} \in H_{g,n}$ i.e., $g_1 g_2^{-1}$ fixes $\{z_1, z_2, \ldots, z_n\}$ pointwise. So, $G_{g,n}/H_{g,n}$ has $n!$ distinct cosets. □

Remark 3.1.4. $\mathrm{T}_{g,n}$ *may be regarded as being obtained from* $\mathrm{T}_{g,0}$ *by removing the interior of* n *closed discs* $D_1, D_2, \ldots, D_n$ *where* D_i *contains* z_i *as an interior point,* $1 \leq i \leq n$. *Here, homeomorphisms and isotopies are those which keep the union* $\partial D_1 \cup \partial D_2 \cup \cdots \cup \partial D_n$ *fixed setwise. Also,* $\mathrm{M}_{g,n}$ *may be interpreted as the group of mapping classes of* $\mathrm{T}_{g,0} - (D_1 \cup D_2 \cup \cdots \cup D_n)$.

3.2. Dehn Twists

In this section, we introduce the main nontrivial examples of elements of mapping class groups, namely Dehn twists, which will be shown to be the generators of the mapping class groups. We shall give the definition and then present some properties of Dehn twists.

Let A be the annulus in $\mathbb{R}^2$ given by the inequality $1 \leq r \leq 2$ in the standard polar coordinates (r, θ) on $\mathbb{R}^2$. We define a map $h : A \to A$ given by

$$h(r, \theta) = (r, \theta - 2r\pi). \tag{3.1}$$

Observe that h is orientation preserving and h restricted to ∂A is the identity. Let c be a simple closed curve on $T_{g,n}$. By Proposition 2.4.2 there is a neighbourhood of c, denoted by N which is homeomorphic to the cylinder. So there is an embedding $e : A \rightarrow T_{g,n}$ with $e(A) = N$. In fact we can make e orientation preserving. If e is orientation reversing we compose e by an orientation reversing homeomorphism φ of annulus. Then $e\varphi : A \rightarrow T_{g,n}$ becomes orientation preserving. Hence we assume that $e : A \rightarrow T_{g,n}$ is orientation preserving. Moreover, we can choose e in such a way that $e(\frac{3}{2}, \theta) = c$. If $e^{-1}(c) = s$ in A, let $\psi : A \rightarrow A$ be an orientation preserving homeomorphism with $\psi(\frac{3}{2}, \theta) = s$. Then $e\psi : A \rightarrow T_{g,n}$ is the required embedding. Hence we may assume that there is an orientation preserving embedding

$$e : A \rightarrow T_{g,n} \;\text{ with }\; e(\frac{3}{2}, \theta) = c. \tag{3.2}$$

Then $ehe^{-1} : N \rightarrow N$ is an orientation preserving homeomorphism which is the identity on the two boundary curves of N. Hence, we can extend ehe^{-1} to a self homeomorphism of $T_{g,n}$, denoted by h_e, by defining it as the identity map on $T_{g,n} - N$. Now we show that h_e depends on the isotopy class of the embedding e.

Proposition 3.2.1. *Let* e_0 *and* e_1 *be embeddings of* A *into* $\mathrm{T}_{g,n}$ *of a simple closed curve* c *with a neighbourhood of* N. *If* e_0 *and* e_1 *are isotopic then* h_{e_0} *and* h_{e_1} *are isotopic.*

Proof. Let $F : T_{g,n} \times I \rightarrow T_{g,n} \times I$ be the isotopy between e_0 and e_1 such that $F(s,t) = (F_t(s), t)$ where $F_0 = id$ and $F_1 e_0 = e_1$. Then define $\widetilde{F} : T_{g,n} \times I \rightarrow T_{g,n} \times I$ as FG where $G : T_{g,n} \times I \rightarrow T_{g,n} \times I$ is the map given by $G(s,t) = (G_t(s), t)$ where $G_t = h_{e_0}$ for every t. Then $\widetilde{F} = FG$ is the required isotopy between h_{e_0} and h_{e_1}. □

Remark 3.2.2. *Let* N_1 *and* N_2 *be two neighbourhoods of* c *which are homeomorphic to the cylinder and* e_1 *and* e_2 *be the corresponding orientation preserving embeddings. If* e_1 *and* e_2 *are isotopic then* h_{e_1} *and* h_{e_2} *are isotopic. In fact, if* e_1 *and* e_2 *are orientation preserving embeddings of* A *into* $\mathrm{T}_{g,n}$ *then they are isotopic so* h_{e_1} *and* h_{e_2} *are isotopic. Hence we can give the following definition.*

Definition 3.2.3. *The unique isotopy class of* h_e, *denoted by* τ_c, *is called (right)Dehn twist about* c.

We use τ_c to denote both isotopy class of τ_c and a representative of the isotopy class. Similarly, a simple closed curve and its isotopy class will be denoted by the same letter. The reason why we call τ_c as a right Dehn twist is the following: Let p be a path which crosses the curve c and let $\Delta p \subset p$ be a small segment of p. Then, Δp must move to the right(respectively left) when τ_c (respectively τ_c^{-1}) is applied.

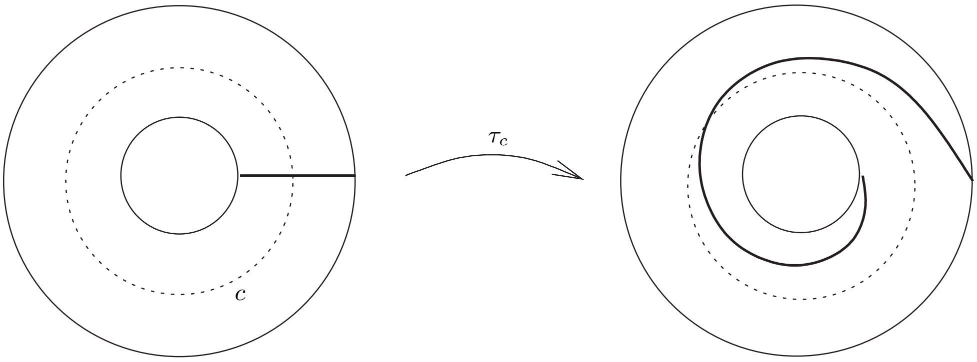

Figure 3.1. the effect of τ_c on an arc

Proposition 3.2.4. *(i) If* $f : \mathrm{T}_{g,0} \to \mathrm{T}_{g,0}$ *is a homeomorphism, then* $\tau_{f(c)} = f\tau_c\, f^{-1}$.
(ii) If c *and* p *are isotopic simple closed curves on* $\mathrm{T}_{g,n}$*, then* τ_c *is isotopic to* τ_p.

Proof. (i) Let N be a neighbourhood of c which is homeomorphic to the cylinder. Then we form τ_c by using h and an orientation preserving embedding e of annulus into $T_{g,0}$. Then $f(N)$ is a neighbourhood of $f(c)$ which is homeomorphic to the cylinder since f is a homeomorphism. If f is orientation preserving then $fehe^{-1}f^{-1} : f(N) \to f(N)$ is an orientation preserving homeomorphism from $f(N)$ to $f(N)$ which is identity on the boundary components of $f(N)$. By using $fehe^{-1}f^{-1}$ we obtain $\tau_{f(c)}$. If f is not an orientation preserving embedding then we may make it orientation preserving by using an orientation reserving homeomorphism from annulus to annulus. Hence the result follows.

(ii) Let $F : T_{g,n} \times I \to T_{g,n} \times I$ be an isotopy $F(x,t) = (F_t(x), t)$ with $F_0 = id$ and $F_1 p = c$. Define $\widetilde{F} : T_{g,n} \times I \to T_{g,n} \times I$ as $\widetilde{F}(x,t) = (F_t \tau_p F_t^{-1}(x), t)$. We see that $\widetilde{F}$ is an isotopy with $\widetilde{F}(x,0) = (F_0\tau_p F_0^{-1}(x), 0) = (id\tau_p id(x), 0) = (\tau_p(x), 0)$ and $\widetilde{F}(x,1) = (F_1\tau_p F_1^{-1}(x), 1) = (\tau_c F_1 F_1^{-1}(x), 1)$ by Proposition 3.2.4, part (i). Hence, $\widetilde{F}(x,1) = (\tau_c(x), 1)$. □

Remark 3.2.5. *Let* $\{z_1, z_2, \ldots, z_n\}$ *be arbitrary set of* n *distinguished points of* $\mathrm{T}_{g,0}$.

Then any twist τ_c *is isotopic to a twist which fixes* $\{z_1, z_2, \ldots, z_n\}$ *setwise. Thus any twist on* $\mathrm{T}_{g,0}$ *is isotopic to a twist on* $\mathrm{T}_{g,n}$.

3.3. Relations among Dehn Twists

3.3.1. The Braid Relations

Suppose we have two simple closed curves on a surface. Then the following relations hold.

Lemma 3.3.1. *(i) If* p *and* c *are disjoint, then* $\tau_p\tau_c \cong \tau_c\tau_p$.
(ii) If p *and* c *are intersect transversally at only one point, then* $\tau_p\tau_c\tau_p \cong \tau_c\tau_p\tau_c$.

Proof. (i) We know that p and c have neighbourhoods homeomorphic to the cylinder. Since p and c are disjoint we can choose these neighbourhoods disjoint. Hence, $\tau_p(c) = c$. Thus by Proposition 3.2.4, part (i) we get

$$\tau_p\tau_c \cong \tau_p\tau_c\tau_p^{-1}\tau_p \cong \tau_{\tau_p(c)}\tau_p \cong \tau_c\tau_p. \tag{3.3}$$

(ii) Since p and c intersect transversally at only one point, $\tau_p\tau_c(p) \approx c$ by Lemma 3.5.2 Hence, by Proposition 3.2.4, part (i), and the first part of this lemma we have

$$\tau_p\tau_c\tau_p \cong \tau_p\tau_c\tau_p\tau_c^{-1}\tau_p^{-1}\tau_p\tau_c \cong \tau_{\tau_p\tau_c(p)}\tau_p\tau_c \cong \tau_c\tau_p\tau_c. \tag{3.4}$$

□

The above relations in Equation 3.3 and Equation 3.4 are called braid relations.

3.3.2. The Two Holed Torus Relation

Let X be a torus with the interiors of two disjoint discs removed. Consider the simple closed curves c, m, p, r and s where c is disjoint from p, and m intersects c

and p transversally at one point as shown in Figure 3.2 and Figure 3.3. Then embed X into a surface of genus $g \geq 2$ such that all five curves become nonseparating in the surface. We show that Dehn twists about these simple closed curves satisfy the following relation

$$(\tau_c \tau_p \tau_m)^4 \cong \tau_r \tau_s. \tag{3.5}$$

This relation is called the two holed torus relation.

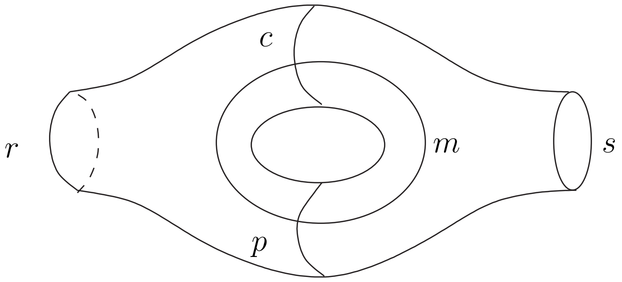

Figure 3.2. the circles of the two holed torus relation

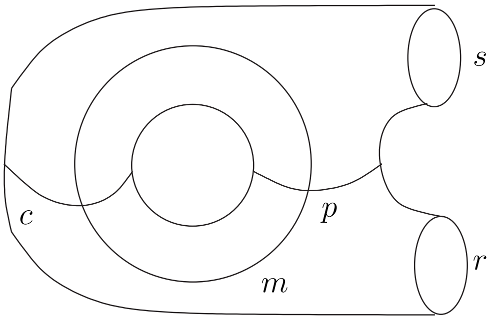

Figure 3.3. the circles of the two holed torus relation

Lemma 3.3.2. *(i) Let f be the homeomorphism given by*

$$f = \tau_c \tau_p \tau_m \tau_c \tau_p \tau_m^2 \tau_c \tau_p \tau_m \tau_c \tau_p,$$

then $\tau_r \cong f \tau_s^{-1}$.
(ii) $(\tau_c \tau_p \tau_m)^4 \cong \tau_r \tau_s$.

Proof. (i) [13].
(ii) By the first part, $\tau_r \tau_s \cong (\tau_c \tau_p \tau_m)^2 (\tau_m \tau_c \tau_p)^2 \cong (\tau_c \tau_p \tau_m)^4$

$(\tau_m\tau_c\tau_p)^2 = (\tau_c\tau_p\tau_m)^2$ follows from braid relations in Equation 3.3 and Equation 3.4.

$$\begin{aligned}
(\tau_m\tau_c\tau_p)^2 &= \tau_m\tau_c\tau_p\tau_m\tau_c\tau_p = \tau_m\tau_c\tau_p\tau_m\tau_p\tau_c \\
&= \tau_m\tau_c\tau_m\tau_p\tau_m\tau_c = \tau_c\tau_m\tau_c\tau_p\tau_m\tau_c \\
&= \tau_c\tau_m\tau_p\tau_c\tau_m\tau_c = \tau_c\tau_m\tau_p\tau_m\tau_c\tau_m \\
&= \tau_c\tau_p\tau_m\tau_p\tau_c\tau_m = \tau_c\tau_p\tau_m\tau_c\tau_p\tau_m \\
&= (\tau_c\tau_p\tau_m)^2.
\end{aligned}$$

□

3.3.3. The Lantern Relation

Let X be a sphere with four holes. Then embed X into a surface S such that the boundary circles of X are the simple closed curves c_0, c_1, c_2, c_3 on S as seen in Figure 3.4. There are three circles denoted by c_{12}, c_{13} and c_{23} on the four holed sphere satisfying the following relation which can be found [14].

$$\tau_{c_0}\tau_{c_1}\tau_{c_2}\tau_{c_3} \cong \tau_{c_{12}}\tau_{c_{13}}\tau_{c_{23}}$$

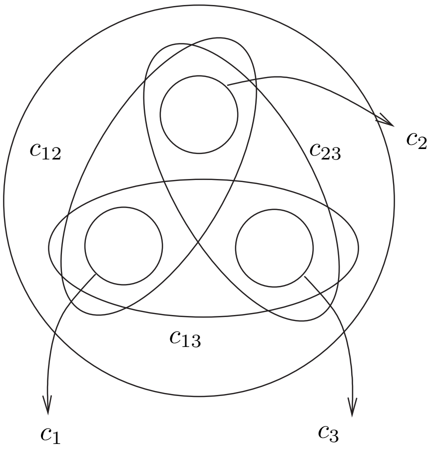

Figure 3.4. the circles of the lantern relation

3.4. Examples

We will find the mapping class groups of the sphere and the torus.

(1) We show that the mapping class group of the sphere is trivial by proving that any piecewise linear orientation preserving homeomorphism f of S^2 is isotopic to the identity map. Let $h : S^2 \to S^2$ be a homeomorphism. Then deg $f = \pm 1$. We know that deg $id = 1$ and deg $ant = -1$, where ant denote the antipodal map. Also, deg $f =$ deg h if only if f and h are homotopic for any map of S^2. So, any homeomorphism of S^2 is homotopic to either the identity map or the antipodal map. Since f is orientation preserving f is homotopic to the identity.

(2) We prove that the mapping class group of the torus $T^2$is the special linear group $SL(2, \mathbb{Z})$, i.e. the group of 2×2 matrices with integer entries and with determinant 1.

Consider a path-connected topological space X. We know that every homeomorphism of X induces an isomorphism on the fundamental group $\pi_1(X)$. However the converse is also true for orientable surfaces of genus g with no distinguished points [1]. Also orientation preserving homeomorphisms of $T_{g,0}$ induce a well-defined subgroup Aut$^+$ $\pi_1(T_{g,0})$ of Aut $\pi_1(T_{g,0})$ and $M_{g,0}$ is isomorphic to Aut $^+$ $\pi_1(T_{g,0})/$Inn $\pi_1(T_{g,0})$ where $\pi_1(T_{g,0})$ is the group of inner automorphisms of $\pi_1(T_{g,0})$ [1]. The fundamental group of T^2, the torus, is $\pi(T_1) = \mathbb{Z} \oplus \mathbb{Z}$. Since $\mathbb{Z} \oplus \mathbb{Z}$ is abelian, Inn $\pi_1(T_1)$ is trivial. So it is enough to find Aut $^+$ $\pi_1(T^2)$. Now we will find Aut $\pi_1(T_1)$. $\mathbb{Z} \oplus \mathbb{Z}$ is a free abalian group of rank 2. So, $\{(a, b), (c, d)\} \subseteq \mathbb{Z} \oplus \mathbb{Z}$ is a set of generators if and only if every element of $\mathbb{Z} \oplus \mathbb{Z}$ is written uniquely as a linear combination of (a, b) and (c, d). Let $(x, y) \in \mathbb{Z} \oplus \mathbb{Z}$.

$$(x, y) = \alpha(a, b) + \beta(c, d)$$

If we want a unique pair (α, β) as a solution, this means that the matrix

$\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ is invertible.

We define a map ψ from the group Aut $\mathbb{Z} \oplus \mathbb{Z}$ with the operation composition to the group $GL(2, \mathbb{Z})$ with the operation multiplication: the map ψ takes an automorphism h of $\mathbb{Z} \oplus \mathbb{Z}$ to a matrix in the following way. We write $h((1,0))$ as the first column and $h((0,1))$ as the second column of the matrix. One can show that ψ is an isomorphism between $\mathbb{Z} \oplus \mathbb{Z}$ and $GL(2, \mathbb{Z})$. For $A \in GL(2, \mathbb{Z})$ $\det(AA^{-1}) = 1$. Then $(\det A)$ $(\det A^{-1}) = 1$ so $GL(2, \mathbb{Z})$ consists of 2×2 invertible matrices with integer entries and determinants 1 or -1. Then the mapping class group of the torus is $SL(2, \mathbb{Z})$ since we take the orientation preserving homeomorphisms of the torus [19].

3.5. Generators for $M_{g,0}$

In this section we prove the following important theorem about mapping class groups. This theorem is due to M. Dehn and rediscovered by W. B. R. Lickorish. The following proof is given by J. Birman [15] who simplified the proof of Lickorish.

Theorem 3.5.1. *Every piecewise linear orientation preserving homeomorphism of a closed orientable surface of genus* $g \geq 0$ *is isotopic to a product of Dehn twists about non-separating simple closed curves.*

Proof. We will do induction on the genus of the surface. Our plan is the following: We will take an orientation preserving homeomorphism $\alpha : T_{g,0} \to T_{g,0}$ and a non-separating curve m on $T_{g,0}$. Then we will investigate the image $p = \alpha(m)$. We will find a sequence of twists $\{\tau_{c_i} : i = 1, \ldots, r\}$ about non-separating curves $c_1, c_2, \ldots, c_r$ and a homeomorphism δ which is isotopic to the identity where $\alpha' = \delta\tau_{c_r}^{\epsilon^r} \ldots \tau_{c_1}^{\epsilon^1 \alpha}$, $\varepsilon_i = \mp 1$ keeps m fixed pointwise. Then the restriction of α' to $T_{g,0} - m$ is an orientation preserving homeomorphism of a surface of genus $g - 1$ with two boundary components. By induction $M_{g-1,0}$ is generated by the isotopy classes of twists about non-separating curves. Also, $M_{g-1,0} = P_{g-1,0}$. We will show that the kernel of the natural homomorphism from $P_{g-1,2} \to P_{g-1,0}$ is also generated by compositions of twists so the result will follow.

For $g = 0$ the result is trivial.

Assume that the theorem is true for closed surfaces of genus $g - 1$.

We prove four lemmas. In what follows, elements of the group $G_{g,0}$ of orientation preserving homeomorphisms of $T_{g,0}$ and elements of the subgroup $D_{g,0}$ of $T_{g,0}$ are denoted by the Greek letters $\alpha, \beta, \tau, \ldots$. Lower case roman letters $m, p, c \ldots$ will be used to denote simple closed curves on $T_{g,0}$. The symbols $\tau_m, \tau_p, \tau_c, \ldots$ denote twists about $m, p, c, \ldots$. Also we write:

$\alpha \cong \beta$ if there exists $\delta \in D_{g,0}$ such that $\alpha = \delta\beta$, that means α and β are isotopic maps.

$p \approx m$ if there exists $\delta \in D_{g,0}$ such that $\delta(p) = m$, that means p and m are isotopic simple closed curves.

$p \sim_\tau m$ if there exist non-separating curves $c_1, c_2, \ldots, c_r$ such that $\tau_{c_r}^{\epsilon^r} \ldots \tau_{c_1}^{\epsilon^1}(p) \approx m$ where $\epsilon_i = \pm 1, 1 \leq i \leq r$.

$|p \cap m|$ = cardinality of $p \cap m$, when p and m intersect transversally.

Lemma 3.5.2. *Let r, s be simple closed curves on* $\mathrm{T}_{g,0}$. *Suppose that r and s intersect transversally at one point. Then $r \sim_\tau s$.*

Proof. Firstly we prove that r and s are non-separating. Assume without loss of generality that r is separating. Then $T_{g,0} - r$ has two connected components, say C_1 and C_2. Since s is connected, it lies completely in C_1 or it lies completely in C_2. But $C_1 \cap s \neq \emptyset$ and $C_2 \cap s \neq \emptyset$ since r and s intersect transversally. Since we get a contradiction, we conclude that r and s are non-separating.

Now consider neighbourhoods of r and s which are homeomorphic images of annulus. In the first picture of Figure 3.5 we see segments of s and r in the intersection of these neighbourhoods including the point at which they intersect transversally. A part of $\tau_r(s)$ is seen in the second picture. We know that s moves the right when τ_r is applied and the effect of τ_r on s is to break up s and insert a copy of r. In the last picture we observe that $\tau_s\tau_r(s)$ which is isotopic to r. Hence, $r \sim_\tau s$. $\square$

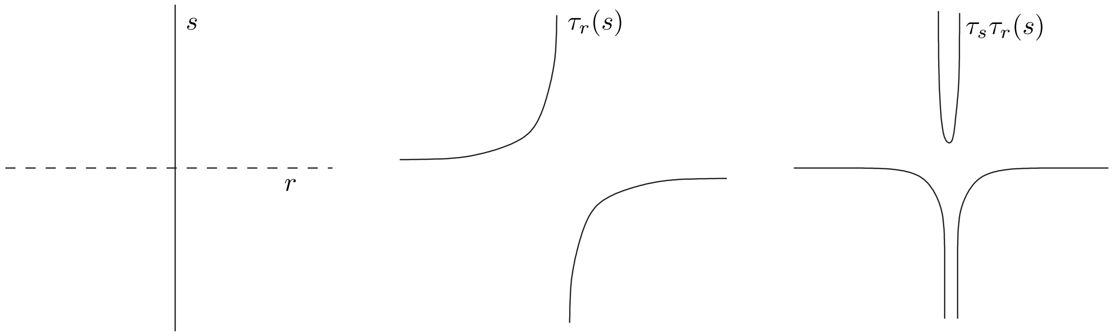

Figure 3.5. the effect of $\tau_s\tau_r$ on s in a neighbourhood of the point at which r and s intersect transversally

Lemma 3.5.3. *Let m be a non-separating oriented simple closed curve on $\mathrm{T}_{g,0}$. Then there exists a sequence of twists about non-separating curves, with product α, such that $\alpha(m) \approx m$, but the induced orientation of $\alpha(m)$ is opposite to that of m.*

Proof. First we find a non-separating simple closed curve p intersecting m transversally at one point. Consider the annulus with the map h given in 3.1. The circle corresponding to $r = \frac{3}{2}$ and the arc corresponding to $\theta = 0$ intersect transversally. The image of the circle corresponding to $r = \frac{3}{2}$ under an embedding e, defined in (2), is m and the image of the arc corresponding to $\theta = 0$ is an arc. Since e is an embedding, m and the image of the arc intersect transversally at exactly one point. Now consider the points $e(1,0)$ and $e(2,0)$ in $T_{g,0}$. Since m is a non-separating curve $T_{g,0} - m$ is connected so is path connected since a manifold is connected if and only if it is path connected. Then there is a path joining these two points. Then this path with the image of the arc $\theta = 0$ gives a simple closed curve, denote it by p, intersecting transversally at one point with m. Also p is non-separating since they intersect transversally at one point.

Now, by the proof of Lemma 3.5.2, $\tau_m\tau_p(m) \approx p$ and $\tau_p\tau_m(p) \approx m$, hence

$\tau_p\tau_m^2\tau_p(m) \approx m$. Assume m is oriented. It is easy to verify that after applying the above twists to m the induced orientation of $\alpha(m)$ is opposite to that of m.

□

Lemma 3.5.4. *Let r and s be disjoint non-separating simple closed curves on $\mathrm{T}_{g,0}$. Then $r \sim_\tau s$.*

Proof. The result is trivial if $r \approx s$. Suppose $r \not\approx s$. We claim that there is a simple closed curve c with $|c \cap r| = |c \cap s| = 1$. If such a c exists it must be non-separating since $|c \cap r| = 1$. Then by Lemma 3.5.2 $r \sim_\tau c$ and $s \sim_\tau c$. Hence, $r \sim_\tau s$.

Now we show the existence of such a curve c. Choose points A_r and A_s on r and s respectively. Consider $r \cup s$ and suppose first that $r \cup s$ separates. Then $T_{g,0} - \{r \cup s\}$ has two components. Thus we may join A_r and A_s by arcs c_1 and c_2 in each component to obtain $c = c_1 \cup c_2$. If $r \cup s$ does not separate, we choose arcs $c_r = \overline{A_{r_1}A_{r_2}}$ and $c_s = \overline{A_{s_1}A_{s_2}}$ which cut r and s transversally at A_r and A_s respectively. Then we join A_{r_1} and A_{s_1} by an arc c_1 on $T_{g,0} - \{r \cup s \cup c_r \cup c_s\}$. Then $T_{g,0} - \{r \cup s \cup c_r \cup c_s \cup c_1\}$ is still connected, so we may find an arc c_2 in it joining A_{r_2} to A_{s_2} to obtain $c = c_1 \cup c_r \cup c_2 \cup c_s$. □

Lemma 3.5.5. *Let p, m be arbitrary non-separating curves on $\mathrm{T}_{g,0}$. Then $p \sim_\tau m$.*

Proof. We do induction on $|p \cap m|$. By Lemmas 3.5.3 and 3.5.4, the claim is true if $|p \cap m| = 0$ or 1. Suppose $|p \cap m| > 1$. We consider two cases.

Case 1 : Assume that the set $p \cap m$ contains two adjacent points on m, say A and B, which are such that p is oriented on the same direction at A as it is at B with respect to the assigned orientation on m. (Figure 3.6). Then we choose a point A' close to A, and not on p or m. Let c be a simple closed curve which starts at A' and proceeds, close to p but without crossing p until it reaches B' near $B \in p \cap m$, and then crosses each of the curves p and m once to return to A'. Then the curve must be separating since $|c \cap p| = 1$. Hence by Lemma 3.3.2, we have $p \sim_\tau c$. We note that $|c \cap m| < |p \cap p|$, so by induction we have $c \sim_\tau m$. Therefore $p \sim_\tau m$.

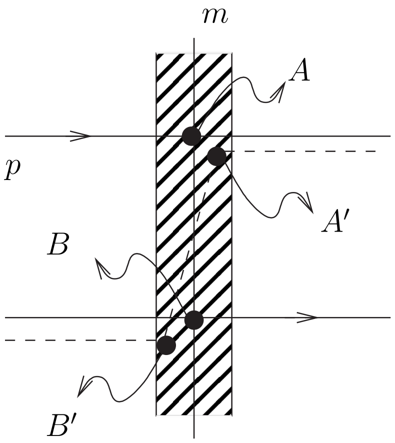

Figure 3.6. the simple closed curves m and p in Case 1

Case 2 : Assume that the set $|p \cap m|$ contains no pair of adjacent points on m which satisfy the condition of Case 1. Then the intersections of p with m alternate as in Figure 3.7. Let N_p be a neighbourhood of p which is homeomorphic to the cylinder on $T_{g,0}$, and let c be one of the boundary curves of N_p. Then c is a non-separating simple closed curve on $T_{g,0}$. Also, points of $c \cap m$ alternate with those of $p \cap m$ in the manner indicated in Figure 7 since if p is oriented , then c will always appear to the right of p to an observer walking along p in the assigned direction. Now we choose two points A and B in $c \cap m$ which are adjacent on m and also one of the arcs AB of m contains no points of $p \cap m$. Let N_m be a neighbourhood of m which is homeomorphic to a cylinder. Now we examine the intersection of N_m with c in the vicinity of A and B, see Figure 3.8. The boundary of N_m intersects c at four points, namely A_1, A_2, B_1, B_2 and so divide c in into four arcs. We denote these arcs in the following manner: $b = \overline{B_2B_1}$(containing B), $d_1 = \overline{B_1A_1}$, $a = \overline{A_1A_2}$(containing A) and $d_2 = \overline{A_2B_2}$. Also let $e_1 = \overline{A_1B_1}$ and $e_2 = \overline{A_2B_2}$ be boundaries of N_m. Now we construct simple closed curves $c_1 = d_1 \cup e_1$, $c_2 = d_2 \cup e_2$. We note that c_1 and c_2 are disjoint from p and each of them intersects m fewer times than p.

Now we prove that one of these curves c_1 or c_2 must be separating. Assume for contrary both c_1 and c_2 were separating. Then $T_{g,0} - c_1$ has two connected components.

Since c_2 is connected, it lies in one of the connected components of $T_{g,0} - c_1$. Similarly, $T_{g,0} - c_2$ has two connected components and c_1 lies in one of them. Hence $T_{g,0} - c_1 - c_2$ has three connected components. Since $a + e_2 + b + e_1$ bounds a disc then $T_{g,0} - c_1 - c_2 - a - b$ has four components. Then we can obtain $T_{g,0} - c_1 - c_2 - a - b + e_2$ from $T_{g,0} - c_1 - c_2 - a - b$ by gluing together two of the four components along e_2 hence $T_{g,0} - c_1 - c_2 - a - b - c_2$ has three components now. Similarly, we get $T_{g,0} - c_1 - c_2 - a - b + e_2 + e_3 = T_{g,0} - c$ which has two components. But this gives a contradiction because c is non-separating.

Now assume that c_i is non-separating where $i = 1$ or $i = 2$. By construction, $|c_i \cap p| = 0$ and $|c_i \cap m| < |p \cap m|$. Then by Lemma 3.3.4, $p \sim_\tau c_i$ and by induction on $|p \cap m|$ we have $c_i \sim_\tau m$. Thus, we get $p \sim_\tau m$.

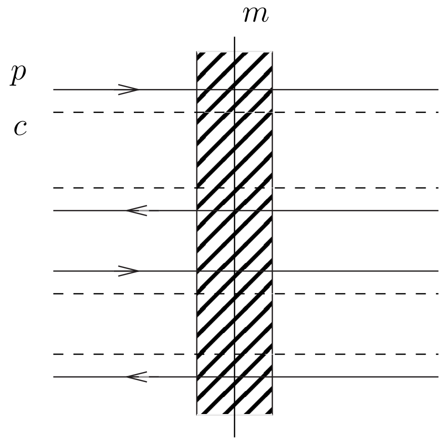

Figure 3.7. the simple closed curves m and p in Case 2

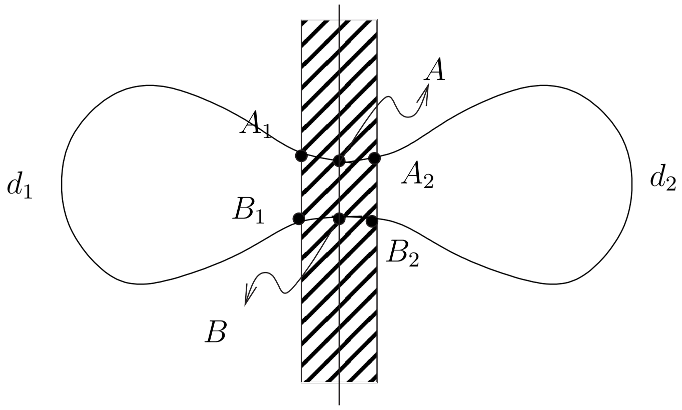

Figure 3.8. the simple closed curves m and p in Case 2

□

Now we continue the proof of the theorem. Assume that $\alpha : T_{g,0} \to T_{g,0}$ is an orientation preserving homeomorphism and m is a non-separating simple closed curve. Then $p = \alpha(m)$ is a non-separating simple closed curve since $T_{g,0} - p = \alpha(T_{g,0} - m)$ and image of a connected set is connected under a contituous map. Then by Lemma 3.5.2 there exists a sequence of twists $\tau_1, \tau_2, \ldots, \tau_r$ about non-separating simple closed curves $c_1, c_2, \ldots, c_r$ such that $\tau_{c_r}^{\epsilon^r} \ldots \tau_{c_1}^{\epsilon^1} \alpha(m) \approx m$. We can do this in the following way due to Lemma 3.5.3. If m is oriented then $\tau_{c_r}^{\epsilon^r} \ldots \tau_{c_1}^{\epsilon^1} \alpha(m)$ has an induced orientation which agree with the orientation of m. So there is $\delta \in D_{g,0}$ with $\alpha' = \delta \tau_{c_r}^{\epsilon^r} \ldots \tau_{c_1}^{\epsilon^1} \alpha$ leaves m fixed pointwise. If we can write α' as a product of as a product of Dehn twists then the result follows. Consider $T_{g,0} - m$. We may regard α' as a homeomorphism of $T_{g,0}$ from which interior of two discs D_1 and D_2 are removed and α' is identity when restricted to the boundaries of the discs. Let x_1 and x_2 be interior points of D_1 and D_2 respectively. By defining the identity map in the interior of discs we extend α' to a map $\alpha'' : T_{g-1,2} \to T_{g-1,2}$ fixing x_1 and x_2. By Remark 3.1.4 it is enough to write α'' as a product of Dehn twists. By induction hypothesis on genus, the group $M_{g-1,0}$ is generated by Dehn twists about simple closed curves. $M_{g-1,0} = P_{g-1,0}$ since there is no distinguished points. The kernel of the homomorphism from $P_{g-1,2} \to P_{g-1,0}$ is generated twists about non-separating simple closed curves [15]. Hence α'' can be written as a product of Dehn twists about simple closed curves. □

4. COHOMOLOGY GROUPS OF MAPPING CLASS GROUPS

4.1. Cohomology Groups of a Group

Let Γ be a discrete multiplicative group and G be a discrete abelian group written additively. Suppose Γ acts on G on the left. This means that there is a map from $\Gamma \times G$ to G, $(x, g) \rightarrow xg$, satisfying the following conditions:
(i) $x(g_1 + g_2) = xg_1 + xg_2$, for all $x \in \Gamma$ and for all $g \in G$,
(ii) $x_2(x_1 g) = (x_2 x_1)g$, for all x_1, $x_2 \in \Gamma$ and for all $g \in G$,
(iii) $1g = g$, for all $g \in G$.

If for every $x \in \Gamma$ and $g \in G$ we have $xg = g$ then we say that Γ acts on G simply.

Consider the following set of functions from $\prod_{i=1}^{n+1} \Gamma_i$ to G where $\Gamma_i = \Gamma$ for all $i = 1, \ldots, n+1$. Such a function F is called an n-dimensional cochain $(n = 0, 1, \ldots)$ of Γ over G if it satisfies the following *homogeneity* condition:

$$F(xx_0, \ldots, xx_n) = xF(x_0, \ldots, x_n). \tag{4.1}$$

Given two n-dimensional cochains F_1 and F_2, their sum $F_1 + F_2$ defined as

$$(F_1 + F_2)(x_0, \ldots, x_n) = F_1(x_0, \ldots, x_n) + F_2(x_0, \ldots, x_n)$$

is also a cochain. Then with the above operation of addition, the n-dimensional cochains form an additive abelian group. We denote this group by $C^n(\Gamma, G)$ where $n = 1, 2, \ldots$

Now we define a function $\delta : C^n(\Gamma, G) \to C^{n+1}(\Gamma, G)$ as follows:

$$(\delta F)(x_0, \ldots, x_{n+1}) = \sum_{i=0}^{n+1} (-1)^i F(x_0, \ldots, \widehat{x_i}, \ldots, x_{n+1}),$$

where $\widehat{x_i}$ means that the variable x_i is omitted.

Proposition 4.1.1. *The operation δ satisfies the following properties:*
(i) δF is an $(n+1)$-cochain,
(ii) $\delta(F_1 + F_2) = \delta F_1 + \delta F_2$,
(iii) $\delta(\delta F) = 0$.

Proof. The proofs of (i) and (ii) are obvious.

$$\begin{aligned}
(\delta(\delta F))(x_0, \ldots, x_{n+2}) &= \sum_{i=0}^{n+2} (-1)^i (\delta F)(x_0, \ldots, \widehat{x_i}, \ldots, x_{n+2}) \\
&= \sum_{j<i} (-1)^i (-1)^j F(x_0, \ldots, \widehat{x_j}, \ldots, \widehat{x_i}, \ldots, x_{n+2}) \\
&+ \sum_{j>i} (-1)^i (-1)^{j-1} F(x_0, \ldots, \widehat{x_i}, \ldots, \widehat{x_j}, \ldots, x_{n+2}) \\
&= \sum_{j<i} (-1)^i (-1)^j F(x_0, \ldots, \widehat{x_j}, \ldots, \widehat{x_i}, \ldots, x_{n+2}) \\
&+ \sum_{i>j} (-1)^j (-1)^{i-1} F(x_0, \ldots, \widehat{x_j}, \ldots, \widehat{x_i}, \ldots, x_{n+2}) \\
&= 0.
\end{aligned}$$

□

The operator δ is called the *coboundary* operator and the cochain δF is called the coboundary of F. Those n-cochains with $\delta F = 0$ are called *n-cocycles* and they form a subgroup $Z^n(\Gamma, G)$ of $C^n(\Gamma, G)$. If $n > 0$, the n-cochains F such that $F = \delta F'$ for some $F' \in C^{n-1}(\Gamma, G)$ are called *coboundaries* and they form a subgroup $B^n(\Gamma, G)$ of $C^n(\Gamma, G)$. If $n = 0$, we set $B^0(\Gamma, G) = 0$. Since $\delta(\delta F) = 0$, $B^n(\Gamma, G)$ is a subgroup of $Z^n(\Gamma, G)$. Hence we can give the following definition. The n^{th} *cohomology* group

$H^n(\Gamma, G)$ of Γ over G is defined as the qoutient group

$$H^n(\Gamma, G) = Z^n(\Gamma, G)/B^n(\Gamma, G).$$

The elements of $H^n(\Gamma, G)$ are called cohomology classes and two cycles in the same cohomology class is called cohomologous.

4.2. The Nonhomogeneous Approach

The cochains F in 4.1 are called homogeneous. They may be replaced by equivalent non-homogeneous cochains. An n-dimensional cochain is a function f from Γ^n to G. Also, by a function of 0 variable on Γ to G we mean any element $g \in G$. There is a one-to-one correspondence, which can be found in [18], between homogeneous and nonhomogeneous n-cochains by using the following formulas:

$$F(x_0, x_1, \ldots, x_n) = x_0 f(x_0^{-1}x_1, x_1^{-1}x_2, \ldots, x_{n-1}^{-1}x_n)$$

$$f(x_1, \ldots, x_n) = F(1, x_1, x_1x_2, \ldots, x_1x_2 \ldots, x_n)$$

, where 1 is the identity element of Γ. Hence, we use the same symbol $C^n(\Gamma, G)$ to denote the group of the non-homogeneous n-cochains. We define the coboundary of f as follows:

$$\begin{aligned}(\delta f)(x_1, \ldots, x_{n+1}) &= x_1 f(x_2, \ldots, x_{n+1}) \\ &\quad + \sum_{i=1}^{n} (-1)^i f(x_1, \ldots, x_i x_{i+1}, \ldots, x_{n+1}) + (-1)^{n+1} f(x_1, \ldots, x_n).\end{aligned}$$

It can be verified directly from the above formula that for a non-homogeneous n-cochain f, $\delta\delta f = 0$. Hence, the n^{th} cohomology group $H^n(\Gamma, G)$ of Γ over G for non-homogeneous case is defined the quotient group

$$H^n(\Gamma, G) = Z^n(\Gamma, G)/B^n(\Gamma, G).$$

as in the homogenous case.

4.3. $H^0(\Gamma, G)$ and $H^1(\Gamma, G)$

We use the non-homogeneous approach in order to describe the cohomology groups $H^n(\Gamma, G)$ for $n = 0, 1$.

Proposition 4.3.1. *(i) The group* $H^0(\Gamma, G)$ *is the subgroup of those elemnts* g *of* G *on which* Γ *operates simply.*
(ii) The first cohomology group $H^1(\Gamma, G)$ *is the group of crossed homomorphisms of* Γ *into* G *modulo the principal homomorphisms.*

Proof. (i) A 0-cochain $f \in C^0(\Gamma, G)$ is by definition an element $g \in G$. Then f is a cocycle if and only if $xg = g$ for all $x \in \Gamma$ since $(\delta f)(x) = xg - g$. Since by definition $B^0(\Gamma, G) = 0$, the result follows. Also, if Γ acts on G simply, then $H^0(\Gamma, G) \simeq G$.
(ii) A 1-cochain $f \in C^1(\Gamma, G)$ is a function from Γ to G satisfying

$$(\delta f)(x_1, x_2) = x_1 f(x_2) - f(x_1 x_2) + f(x_1)$$

for all x_1, $x_2 \in \Gamma$. Hence, f is a 1-cocycle if and only if

$$f(x_1 x_2) = f(x_1) + x_1 f(x_2) \tag{4.2}$$

Functions which satisfy the property 4.2 are called *crossed homomorphisms* in algebra. Hence, $Z^1(\Gamma, G)$ is the group of all crossed homomorphisms from Γ to G. Also, $f \in B^1(\Gamma, G)$ if and only if

$$f(x) = xg - g \tag{4.3}$$

for some constant $g \in G$. Homomorphisms which satisfy 4.3 are called *principal homomorphisms* and form a subgroup $B^1(\Gamma, G)$ of the crossed homomorphisms. Hence, we get the result. □

Proposition 4.3.2. *If* Γ *operates simply on* G*, then* $H^1(\Gamma, G) \simeq Hom(\Gamma/[\Gamma,\Gamma], G)$ *where* $[\Gamma,\Gamma]$ *and* $Hom(\Gamma/[\Gamma,\Gamma], G)$ *denote the commutator subgroup of* Γ *and the group of homomorphisms from* $\Gamma/[\Gamma,\Gamma]$ *to* G *respectively.*

Proof. Γ operates simply on G means $xg = g$ for all $x \in \Gamma$. If Γ operates simply on G, then the crossed homomorphisms become ordinary homomorphisms since

$$f(x_1x_2) = f(x_1) + x_1f(x_2) = f(x_1) + f(x_2)$$

for all x_1, x_2 in Γ. Also, the principal homomorphisms are all zero since

$$f(x) = xg - g = g - g = 0$$

for all $x \in \Gamma$. Hence, $H^1(\Gamma, G)$ =Hom(Γ, G) where Hom(Γ, G) denotes the group of all homomorphisms from Γ to G. Now we prove that

$$\mathrm{Hom}(\Gamma, G) \simeq \mathrm{Hom}(\Gamma/[\Gamma,\Gamma], G).$$

Let $H : \mathrm{Hom}(\Gamma, G) \to \mathrm{Hom}(\Gamma/[\Gamma,\Gamma], G)$ be defined as $H(f)([x]) = f(x)$ where $x \in \Gamma$, $f \in \mathrm{Hom}(\Gamma, G)$ and $[x] \in \Gamma/[\Gamma,\Gamma]$. Let $\widetilde{f}$ denote $H(f)$. So, $\widetilde{f}([x]) = f(x)$.
Firstly we show that $\widetilde{f}$ is well-defined.
Let $[x_1] = [x_2] \in \Gamma/[\Gamma,\Gamma]$. Then $x_1x_2^{-1} \in [\Gamma,\Gamma]$. We know that $[\Gamma,\Gamma]$ is generated by commutators of Γ where a commutator is $[y, z] = yzy^{-1}z^{-1}$ for y, $z \in \Gamma$. So, it is enough to know the image of commutators under a homomorphism h from Γ to G in order to understand $h([\Gamma,\Gamma])$. Let $[y, z]$ be an arbitrary commutator of Γ. Then $h([y, z]) = h(yzy^{-1}z^{-1}) = h(y) + h(z) - h(y) - h(z) = 0$. Hence, $h([\Gamma,\Gamma]) = 0$.
Since $x_1x_2^{-1} \in [\Gamma,\Gamma]$ then $f(x_1x_2^{-1}) = 0$, so $f(x_1) = f(x_2)$, i.e. $\widetilde{f}([x_1]) = \widetilde{f}([x]_2)$.

$\widetilde{f} : \Gamma/[\Gamma;\Gamma] \to G$ is a homomorphism since

$$\widetilde{f}([x_1]+[x_2]) = f(x_1)+f(x_2) = f(x_1+x_2) = \widetilde{f}([x_1+x_2]).$$

Now we show that $H : \mathrm{Hom}(\Gamma, G) \to \mathrm{Hom}(\Gamma/[\Gamma,\Gamma], G)$ is well-defined.
If $f_1 = f_2 \in \mathrm{Hom}(\Gamma, G)$ then by the definition of H, $H(f_1) = H(f_2)$.
H is a homomorphism since

$$\begin{aligned} H(f_1+f_2)(x) &= (\widetilde{f_1+f_2})([x]) = (f_1+f_2)(x) \\ &= f_1(x)+f_2(x) = \widetilde{f_1}([x]) + \widetilde{f_2}([x]) \\ &= H(f_1)(x) + H(f_2)(x). \end{aligned}$$

H is one-to-one: Let f_1, f_2 be in $\mathrm{Hom}(\Gamma, G)$ and $f_1 \neq f_2$ i.e. there is $x \in \Gamma$ such that $f_1(x) \neq f_2(x)$. Since $\widetilde{f}([x]) = f(x)$ this gives $\widetilde{f_1}([x]) \neq \widetilde{f_2}([x])$. H is onto: Let $h : \Gamma/[\Gamma,\Gamma] \to G$ be a homomorphism. Then define $\overline{h} : \Gamma \to G$ as $h(x) = \overline{h}([x])$.
h is well-defined: If x_1, x_2 in Γ such that $x_1 = x_2$ then $[x_1] = [x_2]$ so $\overline{h}([x_1]) = \overline{h}([x_2])$. Hence, $h(x_1) = h(x_2)$.
$h : \Gamma \to G$ is a homomorphism: Let x_1 and x_2 be elements of Γ.

$$h(x_1+x_2) = \overline{h}([x_1+x_2]) = \overline{h}([x_1]) + \overline{h}([x_2]) = h(x_1)+h(x_2).$$

Hence, H is an isomorphism between $\mathrm{Hom}(\Gamma/[\Gamma,\Gamma], G)$ and $\mathrm{Hom}(\Gamma, G)$.

□

4.4. The First Cohomology Group of Mapping Class Groups

Let $T_{g,n,r}$ be a connected orientable surface of genus g with r boundary components and n punctures. We will compute the first cohomology group of the mapping class group of $T_{g,n,r}$ for some cases. In Chapter 3 we proved that every piecewise linear orientation preserving homeomorphism of a closed orientable surface of genus $g \geq 0$ is isotopic to a product of Dehn twists about non-separating simple closed curves. From now on we assume that the homeomorphisms and the isotopies fix each puncture and the points in the boundary. We denote the mapping class group of $T_{g,n,r}$ by $M_{g,n,r}$. We have the following theorem for $M_{g,n,r}$.

Theorem 4.4.1. *If* $g \geq 2$ *then the mapping class group* $\mathrm{M}_{g,n,r}$ *is generated by Dehn twists about finitely many nonseparating simple closed curves.*

Proof. See [16]. □

This theorem does not hold for $g = 1$ and $r \geq 2$.

Theorem 4.4.2. *Let* $r \geq 2$. *The mapping class group* $\mathrm{M}_{1,n,r}$ *can not be generated by Dehn twists about non-separating simple closed curves.*

Proof. See [17]. □

Let $M_{g,n,r}$ be the mapping class group of $T_{g,n,r}$. We will compute $H^1(M_{g,n,r}, \mathbb{Z})$ for $g = 2$ and $g = 3$ where $T_{g,n,r}$ acts trivially on $\mathbb{Z}$.

Theorem 4.4.3. *The first cohomology group* $H^1(\mathrm{M}_{g,r,n}, \mathbb{Z})$ *of the mapping class group* $\mathrm{M}_{g,n,r}$ *is trivial for* $g = 2$ *and* $g \geq 3$.

Proof. Suppose c and m are two non-separating simple closed curves on $T_{g,n,r}$. Then there is a homeomorphism $f : T_{g,n,r} \to T_{g,n,r}$ such that $f(c) = m$ [19]. Hence, by Proposition 3.2.4, part (i) we have $\tau_m \cong f\tau_c f^{-1}$. Then

$$\tau_m \cong f\tau_c f^{-1} \cong f\tau_c f^{-1}\tau_c^{-1}\tau_c \cong [f, \tau_c]\tau_c$$

where $[f, \tau_c]$ is the commutator of f and τ_c. This implies that τ_c and τ_m represent the same class say σ in $M_{g,n,r}/[M_{g,n,r}, M_{g,n,r}]$. Since the mapping class group $M_{g,n,r}$ is generated by Dehn twists about non-separating simple closed curves for $g \geq 2$, we get $M_{g,n,r}/[M_{g,n,r}, M_{g,n,r}]$ is cyclic and generated by σ. Now suppose $g = 2$. We embed the two-holed torus relation in $T_{2,n,r}$ so that the five simple closed curves in the two-holed torus relation becomes nonseparating on $T_{2,n,r}$. From Equation 3.3 we get $12\sigma = 2\sigma$, i.e. $10\sigma = 0$. Since there is no other relation which σ satisfies, we get $M_{2,n,r}/[M_{2,n,r}, M_{2,n,r}] \simeq \mathbb{Z}_{10}$. Then by Proposition 4.3.2, $H^1(M_{2,n,r}, \mathbb{Z}) \simeq 0$ since $\mathrm{Hom}(\mathbb{Z}_{10}, \mathbb{Z}) \simeq 0$. Suppose that $g \geq 3$. We embed the four-holed sphere X of the lantern relation in $M_{g,n,r}$ so that all seven curves in the lantern relation become nonseparating as in Figure 4.1 and Figure 4.2. So this gives us $4\sigma = 3\sigma$ from which we get $M_{g,n,r}/[M_{g,n,r}, M_{g,n,r}] \simeq 0$. Thus we get $H^1(M_{2,n,r}, \mathbb{Z}) \simeq 0$. □

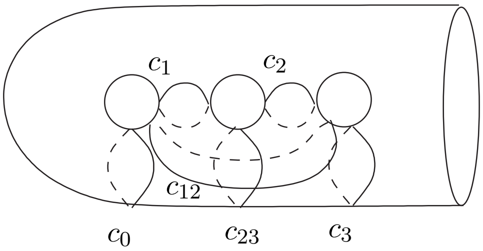

Figure 4.1. the circles of the lantern relation

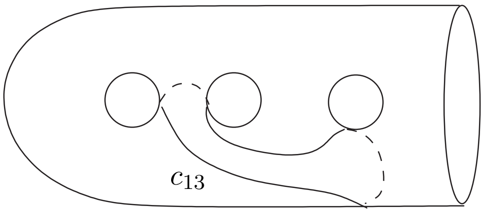

Figure 4.2. the circles of the lantern relation

5. CONCLUSION

The mapping class group of an orientable surface of genus g is the group of all orientation preserving piecewise linear homeomorphisms of the surface up to isotopy. In this thesis, firstly some known facts and results about surfaces were presented. The aim was to prove the theorem stating that every piecewise linear orientation preserving homeomorphism of a closed orientable surface of genus $g \geq 0$ is isotopic to a product of Dehn twists about non-separating simple closed curves. This theorem is due to M. Dehn. Then W. B. R. Lickorish rediscovered this theorem. We followed the proof of J. Birman who simplified the proof of W. B. R. Lickorish. Then the cohomology groups of a group has been introduced following [18]. The first cohomology groups of the mapping class groups of orientable surfaces of genus g greater than one were shown to be trivial [16].

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