

**MODULAR REPRESENTATIONS AND  
MONOMIAL BURNSIDE RINGS**

By  
Olcay Coşkun  
August, 2004

I certify that I have read this thesis and that in my opinion it is fully adequate, in scope and in quality, as a thesis for the degree of Master of Science.

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# ABSTRACT

## MODULAR REPRESENTATIONS AND MONOMIAL BURNSIDE RINGS

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M.S. in Mathematics

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We introduce canonical induction formulae for some character rings of a finite group, some of which follows from the formula for the complex character ring constructed by Boltje. The rings we will investigate are the ring of modular characters, the ring of characters over a number field, in particular, the field of real numbers and the ring of rational characters of a finite  $p$ -group. We also find the image of primitive idempotents of the algebra of the complex and modular character rings under the corresponding canonical induction formulae. The thesis also contains a summary of the theory of the canonical induction formula and a review of the induction theorems that are used to construct the formulae mentioned above.

*Keywords:* Canonical induction, modular characters, real characters, rational characters of  $p$ -groups, primitive idempotents.

## ÖZET

# MODÜLER TEMSİLLER VE TEK TERİMLİ BURNSIDE HALKALARI

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Sonlu grupların bazı karakter halkaları için standart genişletme formülleri ortaya koyduk. Bu formüllerden bazıları Boltje tarafından karmaşık karakter halkaları için verilen formülden elde edilebilir. Modüler karakterlerin halkası, sayı cisimleri üzerindeki karakterlerin halkası, özellikle gerçel sayı cismi ve sonlu  $p$ -gruplarının rasyonel karakter halkası araştırdığımız halkalardır. Ayrıca, karmaşık ve modüler karakter halka cebirlerinin ilkel denkgüçlülerinin standart genişletme formülleri altındaki görüntülerini bulduk. Bu tez ayrıca, standart genişletme formülü kuramının bir özetini ve yukarıda adı geçen formülleri elde etmekte kullanılan genişletme teoremlerinin bir gözden geçirmesini içermektedir.

*Anahtar sözcükler:* Standart genişletme, modüler karakterler, gerçel karakterler,  $p$ -grupların rasyonel karakterleri, ilkel denkgüçlüler.

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# Chapter 1

## Introduction

Artin's induction theorem states that any rational character of a finite group can be expressed as a rational combination of characters induced from trivial characters of cyclic subgroups of the group. Brauer used Mobius inversion to find the coefficient at such induced characters.

Another theorem in this direction is Brauer's induction theorem which states that each complex character of a finite group can be expressed as integral linear combination of characters induced from characters of elementary subgroups. An elementary group is a direct product of a  $p$ -group with a cyclic  $p'$ -group for a prime  $p$ . Knowing that the characters of such groups are monomial, that is, induced from one-dimensional characters of subgroups, one can conclude that any complex character of a finite group is integral combination of monomial characters.

The theorem was stated by Brauer [Br] as an existence theorem and a formula for the coefficients was not known until the end of 1980s when two different formulae appeared. The idea is to use the monomial version of the theorem instead of the original version which allows one to use all subgroups instead of elementary subgroups. The formulae introduce a distinguished way of getting the combination which keeps us away from some arbitrary choices. The first formula by Snaith ([Sn]) is introduced by using topological methods. But, unfortunately the formula is not additive which makes it difficult to apply as we have an additive

structure on the characters. On the other hand, Boltje ([B.89]) used algebraic methods to construct his additive formula and although the construction uses some properties of complex characters it has partial generalizations to some other representation rings. Another formula is constructed using geometric methods by Symonds ([Sy]) later in 1991 but it turns out that this formula is equivalent to the Boltje's formula. It is not clear from these formulae that the coefficients are integers and the authors have different proofs of integrality. But for different character rings, either the construction fails to be integral or the methods are not sufficient to prove the integrality, for example in the case of the ring of Brauer character.

In [B.98], Boltje refines the definition of a canonical induction formula and gives a construction of canonical induction formula for Mackey functors and a sufficient condition for the formula to be integral, which is equivalent to the image of the formula being contained in a certain Mackey functor. In particular, this construction covers the formula for the character ring in [B.89] and introduces an integral formula for the ring of Brauer characters, as well as for some other representation rings. However, the sufficient condition is still not enough to prove integrality in some special cases, some of which we shall consider in later chapters.

In chapter 2, we review some induction theorems, which we shall use in later chapters.

In chapters 3 and 4, we explain the construction in [B.98], referring [B.98] for most of the proofs.

Our main results are in chapter 5, where we introduce some induction formulae. We will use the canonical induction formula for the complex character ring from [B.98] to get canonical induction formula for the modular character ring and for characters over arbitrary number fields. The method we apply to get modular formula is applicable to any complex canonical induction formula and the result is integral if the complex one is. Unfortunately, the second formula is not integral, in general. Also, we get a non-canonical integral induction formula for the real characters. We construct two more canonical induction formula, for real characters and for rational characters of  $p$ -groups. These are example for the

cases where the sufficient does not give information on integrality of the formulae. We just conjecture the integrality in these cases.

In chapter 6, we collect together some miscellaneous results on the primitive idempotents of the algebra of character rings. We introduce a formula for a special type of primitive idempotents of the monomial Burnside algebra. We also find the images of the idempotents under the canonical induction formula.

# Chapter 2

## Some Induction Theorems

In this chapter, we shall review some induction theorems using which we introduce explicit formulae and review known formulae. One can find the theorems, with proofs in classical books, for example [CR], except for Theorem 2.6, which can be found in [Bo].

### 2.1 Artin's induction theorem

**Theorem 2.1.** (Artin) *Let  $k$  be a field and  $G$  be a finite group and  $\mathcal{R}_k(G)$  denote the ring of  $k$ -characters of  $G$ . Then*

$$\mathbb{C} \otimes_{\mathbb{Z}} \mathcal{R}_k(G) = \sum_{\langle g \rangle \leq G} \text{Im}(\text{ind}_{\langle g \rangle}^G : \langle g \rangle \rightarrow G)$$

*where  $\langle g \rangle$  runs over the cyclic subgroups of  $G$  if  $k$  is of characteristic zero and the cyclic  $p'$ -subgroups if the characteristic is  $p$ .*

An explicit formula for this theorem was given by Brauer, who used the M\"obius inversion to get the formula but we shall recover it as a part of a general construction, in the next chapter.

## 2.2 Brauer's induction theorem

**Theorem 2.2.** (Brauer) *Every complex character  $\chi$  of a finite group  $G$  can be expressed in the form*

$$\chi = \sum a_i \psi_i^*$$

where each  $\psi_i$  is an irreducible character of some elementary subgroup  $G_i$  of  $G$ ,  $\psi_i^*$  is an irreducible character of  $G$  induced by  $\psi_i$ , and where the  $a_i$  belong to the ring  $\mathbb{Z}$  of rational integers.

The theorem still holds when we replace complex characters with Brauer characters. Moreover, there is a generalization of the theorem to characters over arbitrary subfield of complex numbers.

**Theorem 2.3.** (Witt-Berman) *Let  $G$  be a finite group and  $K$  be a subfield of the complex field  $\mathbb{C}$ . Then we have a surjection*

$$\bigoplus_H \mathcal{R}_K(H) \rightarrow \mathcal{R}_K(G)$$

where  $H$  runs over the set of  $K$ -elementary subgroups of  $G$ .

**Definition 2.4.** A group  $G$  is called  $K$ -elementary for a prime  $p$  if  $G$  is a semi-direct product of a cyclic  $p'$ -group  $\langle x \rangle$  with a  $p$ -group  $P$  acting on  $\langle x \rangle$  such that for each  $u \in P$  we have  $uxu^{-1} = x^t$  for some  $t \in \text{Gal}(K_{|G|}/K)$ . Here  $K_{|G|}$  is splitting field for  $X^{|G|} = 1$  over  $K$ .

## 2.3 Serre's induction theorem

Putting  $K = \mathbb{R}$  in the above theorem simplifies the family of  $K$ -elementary subgroups as the Galois group is isomorphic to  $C_2$ . In that case, it is possible to classify the characters of  $\mathbb{R}$ -elementary subgroups.

**Theorem 2.5.** (Serre) *Every real character of a finite group  $G$  is a  $\mathbb{Z}$ -linear combination of induced characters  $\text{ind}_H^G \psi$ , where  $H \leq G$  and  $\psi \in \text{Irr}_{\mathbb{R}}(G)$  is one of the following types:*

- $\psi$  is a linear character, that is,  $\psi : H \rightarrow C_2$  for  $H \leq G$
- $\psi = \phi + \bar{\phi}$  for some linear character  $\phi : H \rightarrow \mathbb{C}$
- $\psi$  is a dihedral character, that is,  $\psi$  is inflated from a dihedral quotient.

## 2.4 Ritter-Segal-Bouc induction theorem

Ritter[R]-Segal[Se] theorem states that any rational character is a permutation character. The theorem is improved by Bouc [Bo] as follows:

**Theorem 2.6.** (Ritter-Segal-Bouc) *Let  $p$  be a prime number and  $P$  be finite  $p$ -group. If  $V$  is a non-trivial simple  $\mathbb{Q}P$ -module, then there exist subgroups  $R > Q$  of  $P$  with  $|R : P| = p$ , and an isomorphism of  $\mathbb{Q}P$ -modules*

$$V \cong \text{Ind}_R^P \text{Inf}_{R/Q}^R \Omega_{R/Q}$$

where  $\Omega_{R/Q}$  is the augmentation ideal of the group algebra  $\mathbb{Q}R/Q$ .

In other words, there is an exact sequence of  $\mathbb{Q}P$ -modules

$$0 \rightarrow V \rightarrow \mathbb{Q}(P/Q) \rightarrow \mathbb{Q}(P/R) \rightarrow 0$$

where the map  $\mathbb{Q}(P/Q) \rightarrow \mathbb{Q}(P/R)$  is the natural projection. In particular, in  $\mathcal{R}_{\mathbb{Q}}(P)$ ,

$$[V] = \mathbb{Q}(P/Q) - \mathbb{Q}(P/R).$$

Trying to construct an explicit formula for Brauer's induction theorem leads to a general theory of induction formulae. The theory introduced by R.Boltje is summarized in the next chapter.

# Chapter 3

## Boltje's Theory Part I : Definitions

In chapters 3 and 4, we are aiming to understand the theory of canonical induction formula for the rings of complex characters and modular characters of a finite group. Boltje's theory uses some Mackey functors. To summarize the theory using Mackey functors or concentrating on the special case for the character rings has the same difficulties. Hence, we shall deal with Mackey functors in these chapters although everything we will do in the next chapters are related to character rings. However, it would help if we think of these rings, throughout the two chapters.

### 3.1 Conjugation and restriction functors

Throughout, let  $G$  be a finite group and  $k$  be a commutative ring.

**Definition 3.1.** A  $k$ -conjugation functor on  $G$  is a pair  $(X, c)$  consisting of a family of  $k$ -modules  $X(H)$  for  $H \leq G$  and a family of  $k$ -module homomorphisms  $c_H^g : X(H) \rightarrow X(gH)$ , called the *conjugation maps* for  $H \leq G$  and  $g \in G$ , satisfying the axioms

1.  $c_H^h = \text{id}_{X(H)}$
2.  $c_H^{g'g} = c_{g'H}^{g'} \circ c_H^g$

for all  $h \in H$  and  $g, g' \in G$ .

**Definition 3.2.** A  $k$ -restriction functor on  $G$  is a triple  $(A, c, \text{res})$  consisting of a  $k$ -conjugation functor  $(A, c)$  on  $G$  together with a family of  $k$ -module homomorphisms  $\text{res}_K^H : A(H) \rightarrow A(K)$ , called the *restriction maps*, for  $K \leq H \leq G$ , satisfying the axioms

1.  $\text{res}_H^H = \text{id}_{A(H)}$
2.  $\text{res}_L^K \circ \text{res}_K^H = \text{res}_L^H$
3.  $c_K^g \circ \text{res}_K^H = \text{res}_{gK}^{gH} \circ c_H^g$

for all  $L \leq K \leq H \leq G$  and  $g \in G$ .

**Definition 3.3.** A  $k$ -Mackey functor on  $G$  is a quadruple  $(M, c, \text{res}, \text{ind})$  consisting of a  $k$ -restriction functor  $(M, c, \text{res})$  on  $G$  together with a family of  $k$ -module homomorphisms  $\text{ind}_K^H : M(K) \rightarrow M(H)$ , called the *induction maps*, for  $K \leq H \leq G$ , satisfying the axioms

1.  $\text{ind}_H^H = \text{id}_{M(H)}$
2.  $\text{ind}_K^H \circ \text{ind}_L^K = \text{ind}_L^H$
3.  $c_H^g \circ \text{ind}_K^H = \text{ind}_{gK}^{gH} \circ c_K^g$
4. Mackey Formula:  $\text{res}_U^H \circ \text{ind}_K^H = \sum_{h \in U \backslash H/K} \text{ind}_{U \cap^h K}^U \circ \text{res}_{U \cap^h K}^{hK} \circ c_K^h$

for all  $L \leq K \leq H \leq G$  and  $g \in G$ .

**Remark 3.4.** By these definitions, we obtain categories  $k\text{-Con}(G)$ ,  $k\text{-Res}(G)$ ,  $k\text{-Mack}(G)$  with morphisms defined as families of  $k$ -module homomorphisms commuting with the corresponding maps.

**Example 1.** The monomial Burnside ring  $B(C, H)$  of a finite group  $H$  with the fibre group  $C$ , studied in [LJB] and the cohomology groups  $H^n(H, V)$ ,  $H \leq G$  for fixed  $n \in \mathbb{N}$  and  $V \in \mathbb{Z}\text{-}G\text{-Mod}$  are all  $\mathbb{Z}$ -Mackey functors. We will see in the following sections that monomial Burnside rings  $B(C, H)$  can be constructed from a  $\mathbb{Z}$ -restriction functor applying a functor  $-_+$ .

## 3.2 (Co)primordials

For a  $k$ -Mackey functor  $M$  on  $G$ , and  $H \leq G$ , we define the  $k$ -submodule

$$\mathcal{I}(M)(H) := \sum_{K < H} \text{ind}_K^H(M(K))$$

of  $M(H)$ , which is, clearly, a  $k$ -conjugation subfunctor of  $M$  on  $G$ .

A subgroup  $H$  of  $G$  is called *primordial* for  $M$  if  $\mathcal{I}(M)(H) \neq M(H)$  and we denote the set of primordial subgroups for  $M$  by  $\mathcal{P}(M)$ .

For a  $k$ -restriction functor  $A$  on  $G$ , and  $H \leq G$ , we define the  $k$ -submodule

$$\mathcal{K}(A)(H) := \bigcap_{K < H} \ker(\text{res}_K^H : A(H) \rightarrow A(K))$$

of  $A(H)$  and again,  $\mathcal{K}(A)$  is a  $k$ -conjugation subfunctor of  $A$ .

We say that a subgroup  $H$  of  $G$  is *coprimordial* for  $A$  if  $\mathcal{K}(A)(H) \neq 0$ , and we denote by  $\mathcal{C}(A)$  the set of coprimordial subgroups for  $A$ .

**Example 2.** Let  $\mathcal{R}_{\mathbb{C}}(H)$  be the ring of  $\mathbb{C}$ -characters of  $H$ . Observe that  $\mathcal{R}_{\mathbb{C}}(H)$  is a  $\mathbb{Z}$ -Mackey functor with the usual conjugation, restriction, and induction maps. For  $H \leq G$  let  $\hat{\mathcal{R}}_{\mathbb{C}}(H) \subset \mathcal{R}_{\mathbb{C}}(H)$  denote the  $\mathbb{Z}$ -span of one dimensional characters of  $H$ . Plainly  $\hat{\mathcal{R}}_{\mathbb{C}}(H)$  is a  $\mathbb{Z}$ -restriction functor. The set  $\mathcal{C}(\mathcal{R}_{\mathbb{C}})$  of coprimordial subgroups for  $\mathcal{R}_{\mathbb{C}}$  consists of the cyclic subgroups and the set  $\mathcal{P}(\mathcal{R}_{\mathbb{C}})$  of primordial subgroups consists of the elementary subgroups of  $G$ . Note that a subgroup  $H$  of  $G$  is said to be *elementary* if it is  $p$ -elementary for some prime  $p$ , while  $H$  is a  *$p$ -elementary subgroup* if  $H = \langle x \rangle \times P$  where  $x \in G$  is a  $p'$ -element and  $P$  is a  $p$ -group.

**Example 3.** Let  $F$  be an algebraically closed field of characteristic  $p > 0$ . The Grothendieck ring  $\mathcal{R}_F(H)$  of  $FH$ -mod for  $H \leq G$  with respect to short exact sequences form a  $\mathbb{Z}$ -Mackey functor on  $G$ . We often identify  $\mathcal{R}_F(H)$  with the free abelian group on the set of isomorphism classes  $[V]$  of irreducible  $FH$ -modules  $V$  with the usual conjugation, restriction and induction maps. We denote by  $\hat{\mathcal{R}}_F(H)$  for  $H \leq G$  the  $\mathbb{Z}$ -restriction functor generated by the isomorphism classes  $[V]$  of  $FH$ -modules of  $F$ -dimension 1. For a homomorphism  $\phi : H \rightarrow F^*$  we denote by  $F_\phi$  the  $F$ -vector space  $F$  endowed with the  $H$ -action  $h.\alpha = \phi(h)\alpha$  for  $h \in H, \alpha \in F$ . The set  $\mathcal{C}(\mathcal{R}_F)$  of coprimordial subgroups for  $\mathcal{R}_F$  is the set of cyclic  $l'$ -subgroups.

**Remark 3.5.** *These families of subgroups plays an important role to prove necessary and sufficient condition for a morphism to be a section of the linearization morphism, defined in section 3.5.*

### 3.3 The plus-constructions

We are going to define two functors

$$-^+ : k\text{-Con}(G) \rightarrow k\text{-Mack}(G)$$

and

$$-_+ : k\text{-Res}(G) \rightarrow k\text{-Mack}(G)$$

with a natural transformation

$$\rho^A : A_+ \rightarrow A^+$$

for every  $A \in k\text{-Res}(G)$ .

These functors will be the keys for the construction of the “canonical induction formula” in the following sections.

Let  $X \in k\text{-Con}(G)$ . We define  $X^+ = (X^+, \text{con}^+, \text{res}^+, \text{ind}^+) \in k\text{-Mack}(G)$  by

$$X^+(H) := \left( \prod_{K \leq H} X(K) \right)^H$$

for  $H \leq G$  where  $H$  acts on the product by the conjugation maps  $(\text{con}_K^h)_{K \leq H}$  and the maps and morphisms are defined in the obvious way. It is easy to check that  $-^+$  is a functor from  $k\text{-Con}(G)$  to  $k\text{-Mack}(G)$ .

Let  $A \in k\text{-Res}(G)$ , we define  $A_+ = (A_+, \text{con}_+, \text{res}_+, \text{ind}_+) \in k\text{-Mack}(G)$  by

$$A_+(H) := \left( \bigoplus_{K \leq H} A(K) \right)_H$$

for  $H \leq G$  and we view  $\bigoplus_{K \leq H} A(K)$  as a  $kH$ -module via the sum of conjugation maps  $\text{con}_K^g$  and we write  $[K, a]_H$  in  $A_+(H)$  for  $K \leq H$  for the image of  $a \in A(K)$ . We define the maps as follows:

$$\begin{aligned} \text{con}_{+H}^g : A_+(H) &\rightarrow A_+({}^gH) \\ [V, a]_H &\mapsto [{}^gV, {}^g a]_{gH} \end{aligned}$$

and

$$\begin{aligned} \text{res}_{+K}^H : A_+(H) &\rightarrow A_+(K) \\ [V, a]_H &\mapsto \sum_{h \in K \backslash H/V} [K \cap {}^hV, \text{res}_{K \cap {}^hV}^{hV}({}^h a)]_K \end{aligned}$$

and

$$\begin{aligned} \text{ind}_{+H}^G : A_+(K) &\rightarrow A_+(H) \\ [U, b]_K &\mapsto [U, b]_H \end{aligned}$$

where  $V \leq H$  and  $U \leq K$  and  $a \in A(V)$  and  $b \in A(U)$ .

Let  $A \in k\text{-Res}(G)$  and  $H \leq G$ , we define

$$\rho_H^A := (\text{pr}_{+H}^A \circ \text{res}_{+K}^H) : A_+(H) \rightarrow A^+(H)$$

where

$$\begin{aligned} \text{pr}_{+H}^A : A_+(H) &\rightarrow A(H) \\ [K, a]_H &\mapsto \begin{cases} a & \text{if } K = H \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Straightforward calculations show that  $\rho^A : A_+ \rightarrow A^+$  is a morphism in  $k\text{-Mack}(G)$  and it is natural in  $A \in k\text{-Res}(G)$ .

The following map is "almost" inverse to  $\rho_H^A$  for each  $H \leq G$ .

$$\begin{aligned} \sigma_H^A : A^+(H) &\rightarrow A_+(H) \\ (a_K)_{K \leq H} &\mapsto \sum_{L \leq K \leq H} |L| \mu(L, K) [L, \text{res}_L^K(a_K)]_H \end{aligned}$$

where  $\mu(L, K)$  is the Mobius function of the poset of subgroups of  $H$  evaluated at  $(L, K)$ .

We mean by “almost” inverse that, one has

$$\sigma_H^A \circ \rho_H^A = |H|\text{id}_{A_+(H)} \quad \text{and} \quad \rho_H^A \circ \sigma_H^A = |H|\text{id}_{A_+(H)}.$$

**Remark 3.6.** *If we put  $A(H) = \mathbb{Z}\hat{H}$  where  $\hat{H} = \text{Hom}(H, C)$  for some finite abelian group  $C$  with the usual conjugation and restriction maps,  $A$  is a  $\mathbb{Z}$ -restriction functor. The construction  $-_+$  gives the monomial Burnside ring  $B(C, G)$  and  $-^+$  gives a ghost ring  $\beta(C, G)$  for the monomial Burnside ring  $B(C, G)$ . In particular, if  $C = 1$ , we get the ordinary Burnside ring  $B(G)$  and its ghost ring  $\beta(G)$  and the morphism  $\rho_G$  becomes the well-known mark morphism. For sufficiently large  $C$ , the ring  $B(C, G)$  is the source for the canonical induction formula for the complex character ring.*

### 3.4 Canonical induction formula

Let  $M$  be a  $k$ -Mackey functor on  $G$  and  $A \leq M$  a  $k$ -restriction subfunctor of  $M$ . For  $H \leq G$ , we define

$$\begin{aligned} b_H^{M,A} : A_+(H) &\rightarrow M(H) \\ [K, a]_H &\mapsto \text{ind}_K^H(a) \end{aligned}$$

where  $K \leq H$  and  $a \in A(K)$ . It is easy to see that  $b^{M,A} : A_+ \rightarrow M$  is a morphism of  $k$ -Mackey functors on  $G$ . We call  $b^{M,A}$  the *induction (or linearization) morphism* of  $M$  from  $A$ . We sometimes write  $b = b^{M,A}$ .

Now, we can state our aim. We want to construct a map  $a_H : M(H) \rightarrow A_+(H)$ , called a *canonical induction formula*, such that  $b \circ a = \text{id}_M$ . In general, there are many different choices for  $a_H$ . As  $b_H$  is a morphism of  $k$ -Mackey functors, we might reasonably demand the same for  $a_H$ . But, if we put  $M = \mathcal{R}$ , the character ring Mackey functor, and  $k = \mathbb{Z}$ , and  $A = \hat{\mathcal{R}}$ , the subfunctor spanned by the linear characters, we see that existence of such  $a_H$  contradicts

Artin's induction theorem, which states that

$$|H|\mathcal{R}(H) \subseteq \sum_{K \leq H, K \text{ cyclic}} \text{ind}_K^H(\mathcal{R}(K))$$

for  $H \leq G$ . This leads to a contradiction as applying  $a_H$  to the above inclusion we get

$$|H|a_H(\mathcal{R}(H)) \subseteq \sum_{K \leq H, K \text{ cyclic}} \text{ind}_{+K}^H(\mathcal{R}(K)).$$

On the other hand, by definition of  $\text{ind}_{+K}^H$ , the set  $|H|a_H(\mathcal{R}(H))$  is in the span of elements  $[L, \phi]_H$  where  $L$  is cyclic and  $\phi$  is a 1-dimensional character of  $L$ . But, the spanning set is a part of a basis of  $A_+(H)$ , hence  $a_H(\mathcal{R}(H))$  is a full sublattice. Then applying  $b_H$ , we get that  $\mathcal{R}(H)$  is a full sublattice of  $\sum_{K \leq H, K \text{ cyclic}} \text{ind}_K^H(\mathcal{R}(K))$  which is not possible.

The best we can hope for is:

**Definition 3.7.** A *canonical induction formula* for  $M$  from  $A$  is a morphism  $a \in k\text{-Res}(G)(M, A_+)$  with  $b^{M,A} \circ a = \text{id}_M$ .

**Remark 3.8.** One of the aims of this chapter is to construct two canonical induction formulae, one for the complex character ring  $\mathcal{R}_{\mathbb{C}}(H)$  from  $B(C, H)$  where  $C$  contains all roots of unity, and one for the Brauer character ring  $\mathcal{R}_{\mathfrak{K}}(H)$  from  $B(C', H)$  where  $C'$  contains all  $p'$ -roots of unity where  $p$  is the characteristic of the field  $\mathfrak{K}$ . One may choose different sub-restriction functor  $A$  of the Mackey functors  $\mathcal{R}_K$  for  $K = \mathbb{C}$  or  $\mathfrak{K}$  and construct canonical induction formulae, but our concern, at present, is to treat the scenario covered by Brauer's induction theorem.

### 3.5 The set $k\text{-Res}(G)(M, A_+)$

In this section, we will try to describe the set of morphisms  $a : M \rightarrow A_+$  of  $k$ -restriction functors on  $G$  as in Definition 3.7. In order to describe this set, we use another set of morphisms, namely  $k\text{-Con}(G)(M, A)$ . We shall see that these sets are almost isomorphic.

For  $A \in k\text{-Res}(G)$  and  $H \leq G$ , we define

$$\begin{aligned} \text{inc}_H^A : A(H) &\rightarrow A_+(H) \\ a &\mapsto [H, a]_H \end{aligned}$$

Note that  $\text{pr}_+^A \circ \text{inc}^A = \text{id}_A$ .

We also define, for  $X \in k\text{-Con}(G)$  and  $H \leq G$ ,

$$\begin{aligned} \text{pr}_H^{+X} : X^+(H) &\rightarrow X(H) \\ (x_K)_{K \leq H} &\mapsto x_H \end{aligned}$$

**Proposition 3.9.** *For  $B \in k\text{-Res}(G)$  and  $X \in k\text{-Con}(G)$ , we have  $k$ -linear inverse isomorphisms*

$$\begin{aligned} \lambda_{B,X} : \quad k\text{-Res}(G)(B, X^+) &\rightarrow k\text{-Con}(G)(B, X) \\ (r_H)_{H \leq G} &\mapsto (\text{pr}_H^{+X} \circ r_H)_{H \leq G} \\ (b \mapsto (p_K(\text{res}_K^H(b)))_{K \leq H})_{H \leq G} &\leftarrow (p_H)_{H \leq G} \end{aligned}$$

which are natural in  $B$  and  $X$ .

The proposition follows from the definitions of the categories and functors involved.

Now, we define a map which gives us an overview of the set  $k\text{-Res}(G)(M, A_+)$  for  $M \in k\text{-Mack}(G)$  and  $A \subseteq M$  a  $k$ -restriction subfunctor.

**Definition 3.10.** For  $M, A \in k\text{-Res}(G)$ , we define the composite

$$\text{Res}_A^M : k\text{-Res}(G)(M, A_+) \rightarrow k\text{-Res}(G)(M, A^+) \rightarrow k\text{-Con}(G)(M, A)$$

where the first map is  $\rho^A$  and the second one is  $\lambda_{M,A}$ .

As  $\text{pr}^{+A} \circ \rho^A = \text{pr}_+^A$ , we have

$$\text{Res}_A^M(a) = \text{pr}_+^A \circ a \quad \text{for } a \in k\text{-Res}(G)(M, A_+).$$

We write  $p = \text{res}(a) = \text{Res}_A^M(a)$  and call  $p$  the *residue* of  $a$ .

Applying the inverse of  $\lambda_{M,A}$  to  $\text{res}(a)$  and composing with  $\sigma_H^A$  we get

$$|H|a_H(m) = \sum_{L \leq K \leq H} |L|\mu(L, K)[L, \text{res}_L^K(\text{res}_K^H(\text{res}_K^H(a)))]_H$$

This leads us to

**Corollary 3.11.** *Same notation as above. For each  $H \leq G$ , the diagram*

$$\begin{array}{ccc}
 M(H) & \xrightarrow{a_H} & A_+(H) \\
 & \searrow & \downarrow \rho_H = (\text{pr}_{+K}^A \circ \text{res}_+^H)_{K \leq H} \\
 & & A^+(H)
 \end{array}$$

$(\text{Res}_K \circ \text{res}_K^H)_{K \leq H}$

*commutes.*

*If  $A_+(H)$  has trivial  $|H|$ -torsion for all  $H \leq G$ , then  $\text{Res}_A^M$  is injective.*

*If  $|G|$  is invertible in  $k$ , then  $\text{Res}_A^M$  is an isomorphism.*

Finally we can specify a morphism  $a \in k\text{-Res}(G)(M, A_+)$ , depending on  $p$ . Later, we shall see that, for suitable  $p$ , the morphism  $a$  is a canonical induction formula for  $M$  from  $A$ .

**Definition 3.12.** Let  $M, A \in k\text{-Res}(G)$  and assume that  $A_+(H)$  has trivial  $|H|$ -torsion for each  $H \leq G$ . Then the residue map  $\text{Res}_A^M$  of definition 3.10 is injective and for each  $p \in \text{im}(\text{Res}_A^M)$ , we define

$$a := a^{M, A, p} \in k\text{-Res}(G)(M, A_+)$$

as the unique preimage of  $p$  under  $\text{Res}_A^M$ .

**Remark 3.13.** *Note that in [B.90] and [B.89], the canonical induction formula for the character ring is proved to be unique subject to compatibility with the usual projection  $\mathcal{R}(G) \rightarrow \hat{\mathcal{R}}(G)$ . Uniqueness is preserved in this settings as follows: Associated with each choice of  $p_H$  in  $\text{im}(\text{Res}_A^M)$ , there is at most one canonical induction formula.*

# Chapter 4

## Boltje's Theory Part II : Theorems

In this chapter, we state a necessary and sufficient condition for a morphism defined in the previous chapter to be a canonical induction formula. We also state a sufficient condition for integrality of a canonical induction formula.

### 4.1 Induction Formula Revisited

Throughout this section, suppose  $k$  is a commutative ring such that  $|G|$  is invertible in  $k$ .

We are going to derive a condition for a morphism  $a \in k\text{-Res}(G)(M, A_+)$  for a given Mackey functor  $M$  on  $G$  and a given  $k$ -restriction subfunctor  $A$  of  $M$  to be a canonical induction formula for  $M$  from  $A$ . By Corollary 3.11, we have an isomorphism

$$\text{Res}_A^M : k\text{-Res}(G)(M, A_+) \rightarrow k\text{-Con}(G)(M, A).$$

For  $p \in k\text{-Con}(G)(M, A)$  and  $a^{M,A,p}$  as above, we have the following explicit

formula

$$a_H^{M,A,p}(m) = \frac{1}{|H|} \sum_{L \leq K \leq H} |L| \mu(L, K) [L, \text{res}_L^K(p_K(\text{res}_K^H(m)))]_H \quad (4.1)$$

for all  $H \leq G$  and  $m \in M(H)$  (by 3.5) and the following commutative diagram by Corollary 3.11:

$$\begin{array}{ccc} M(H) & \xrightarrow{a_H} & A_+(H) \\ & \searrow & \downarrow \rho_H \\ & (p_K \circ \text{res}_K^H)_{K \leq H} & A^+(H) \end{array}$$

Now, we want to determine for which  $p$ , the morphism  $a^{M,A,p}$  is a canonical induction formula, that is,  $b \circ a = \text{id}_M$ . To get the condition, we need the following proposition on a natural decomposition of a Mackey functor:

**Proposition 4.1.** *Let  $M \in k\text{-Mack}(G)$ . For each  $H \leq G$  we have a decomposition into  $k$ -submodules*

$$M(H) = e_H^{(H)} M(H) \oplus (1 - e_H^{(H)}) M(H)$$

and the summands are given by

$$e_H^{(H)} M(H) = \mathcal{K}(M)(H) \quad \text{and} \quad (1 - e_H^{(H)}) M(H) = \mathcal{I}(M)(H).$$

where  $\mathcal{K}$  and  $\mathcal{I}$  are defined in Section 1 and  $e_H^{(H)}$  is a primitive idempotent of the Burnside ring  $B(H)$ .

**Remark 4.2.** *Every  $k$ -Mackey functor  $M$  on  $G$  has a  $k \otimes B$ -module structure where the Burnside ring functor  $B$  acts via*

$$(k \otimes_{\mathbb{Z}} B(H)) \otimes_k M(H) \rightarrow M(H)$$

$$[H/K] \otimes_k m \mapsto \text{ind}_K^H(\text{res}_K^H(m))$$

for  $K \leq H \leq G$  and  $m \in M(H)$ .

We have the following corollary:

**Corollary 4.3.** *Let  $M \in k\text{-Mack}(G)$  for  $H \leq G$  and  $m \in M(H)$ . Then  $m = 0$  if and only if  $e_H^{(H)} m = 0$  and  $\text{res}_K^H(m) = 0$  for all  $K < H$ .*

Using this decomposition, we get the necessary and sufficient condition on  $p$  that ensures the corresponding morphism  $a$  is a canonical induction formula:

**Proposition 4.4.** *Let  $M \in k\text{-Mack}(M)$  and  $A \leq M$  be a  $k$ -restriction subfunctor of  $M$ . Let  $p \in k\text{-Con}(G)(M, A)$  and  $a^{M,A,p}$  be the corresponding morphism in  $k\text{-Res}(G)(M, A_+)$ . Then the following are equivalent:*

1. *The morphism  $a^{M,A,p}$  is a canonical induction formula.*
2. *For all  $H \leq G$  and  $m \in M(H)$  we have  $e_H^{(H)}(p_H(m) - m) = 0$ , i.e.*

$$p_H(m) - m \in \sum_{K < H} \text{ind}_K^H(M(K)).$$

The following two corollaries give more explicit conditions:

**Corollary 4.5.** *Let  $M, A$  be as above. If there exists a canonical induction formula for  $M$  from  $A$  then  $e_H^{(H)}M(H) \subseteq A(H)$  for all  $H \leq G$ .*

**Corollary 4.6.** *Let  $M, A$  and  $p$  be as above, and assume that  $A(H) = M(H)$  and  $p_H = \text{id}_{M(H)}$  for all  $H \in \mathcal{C}(M)$ . Then  $a^{M,A,p}$  is a canonical induction formula for  $M$  from  $A$ .*

The following proposition gives the connection between canonical induction formulae for different  $k$ -Mackey functors.

**Proposition 4.7.** *Let  $M, M'$  be  $k$ -Mackey functors on  $G$ . Let  $A \subseteq M$  and  $A' \subseteq M'$  be  $k$ -restriction subfunctors. Let  $p : M \rightarrow A$  and  $p' : M' \rightarrow A'$  be morphisms of  $k$ -conjugation functors on  $G$ , and let  $f : M \rightarrow M'$  be a morphism of  $k$ -restriction functors with  $f(A) \subseteq A'$ . Then the diagram*

$$\begin{array}{ccc} M & \xrightarrow{a^{M,A,p}} & A_+ \\ \downarrow f & & \downarrow f_+ \\ M' & \xrightarrow{a^{M',A',p'}} & A'_+ \end{array}$$

commutes if and only if the diagram

$$\begin{array}{ccc} M & \xrightarrow{p} & A \\ \downarrow f & & \downarrow f \\ M' & \xrightarrow{p'} & A' \end{array}$$

commutes.

Let's have two examples of residues before going further:

**Example 4.** Define  $p \in \mathbb{Z}\text{-Con}(G)(\mathcal{R}_\kappa, \hat{\mathcal{R}}_\kappa)$  by

$$p_H(\chi) = \begin{cases} \chi & \text{if } \chi(1) = 1 \\ 0 & \text{otherwise} \end{cases}$$

for  $H \leq G$  and  $\chi \in \text{Irr}(G)$ .

Since  $\mathcal{R}_\kappa(H) = \hat{\mathcal{R}}_\kappa(H)$  and  $p_H = \text{id}_{\mathcal{R}(H)}$  for all cyclic subgroups of  $G$ , Corollary 4.6 implies that  $a = a^{\mathbb{Q}\mathcal{R}, \mathbb{Q}\hat{\mathcal{R}}, \mathbb{Q}p}$  is a canonical induction formula.

**Example 5.** Define  $p \in \mathbb{Z}\text{-Con}(G)(\mathcal{R}_\mathfrak{K}, \hat{\mathcal{R}}_\mathfrak{K})$  by

$$p_H([V]) = \begin{cases} [V] & \text{if } \dim_{\mathfrak{K}}(V) = 1 \\ 0 & \text{otherwise} \end{cases}$$

for  $H \leq G$  and a simple  $\mathfrak{K}H$ -module  $V$  where  $\mathfrak{K}$  is a field of characteristic  $l$ .

We have  $\mathcal{R}_\mathfrak{K}(H) = \hat{\mathcal{R}}_\mathfrak{K}(H)$  for all cyclic  $l'$ -subgroups  $H$  of  $G$ , so  $a = a^{\mathbb{Q}\mathcal{R}, \mathbb{Q}\hat{\mathcal{R}}, \mathbb{Q}p}$  is a canonical induction formula.

## 4.2 Extension of scalars

Although we are interested in the base ring  $\mathbb{Z}$ , the results we found so far are for the case that  $|G|$  is invertible in the base ring. In this section, we are going to introduce some techniques that can be used to carry over the results to the case  $k = \mathbb{Z}$ . To see this we need the following notion:

**Definition 4.8.** Let  $A \in k\text{-Res}(G)$ . A stable  $k$ -basis of  $A$  is a family  $\mathcal{B} = (\mathcal{B}(H))_{H \leq G}$  of subsets  $\mathcal{B}(H) \subset A(H)$ ,  $H \leq G$  such that  $\mathcal{B}(H)$  is a  $k$ -basis of  $A(H)$  for each  $H \leq G$  and  $\text{con}_H^g(\mathcal{B}(H)) = \mathcal{B}(gH)$  for all  $g \in G$ ,  $H \leq G$ .

**Lemma 4.9.** Let  $A \in k\text{-Res}(G)$  and  $\mathcal{B}$  be a stable  $k$ -basis of  $A$ . For each  $H \leq G$ , the set  $\text{ch}(H) := \{(K, b) \mid K \leq H, b \in \mathcal{B}(K)\}$  is a left  $H$ -set via the conjugation maps and isomorphic to the disjoint union

$$\bigcup_{K \leq H} \mathcal{B}(K)$$

as an  $H$ -set. Moreover, the elements  $[K, b]_H \in A_+(H)$  where  $[K, b]$  runs over a set of representatives for the  $H$ -orbits of  $\text{ch}(H)$  form a  $k$ -basis of  $A_+(H)$ .

**Remark 4.10.** Stable  $k$ -basis of  $\hat{\mathcal{R}}$  is  $\hat{H}$ .

**Definition 4.11.** Let  $A$  be a ring and  $X$  be a left  $A$ -module. Then  $X$  is flat if  $- \otimes_A X$  is an exact functor.

In particular, any field of characteristic zero is flat as a  $\mathbb{Z}$ -module. For flat modules we have the following results for extending scalars.

**Lemma 4.12.** Let  $X \in k\text{-Con}(G)$  and  $A \in k\text{-Res}(G)$ . Let  $k \subset k'$  be commutative rings with the same unit element. Then,

1. If  $k'$  is flat over  $k$ , then the canonical morphism  $k' \otimes_k X^+ \rightarrow (k' \otimes_k X)^+$  is an isomorphism.
2. The canonical morphism  $k' \otimes_k A_+ \rightarrow (k' \otimes_k A)_+$  is an isomorphism.
3. If  $k'$  and  $X(H)$  for  $H \leq G$  are flat over  $k$  and if  $k \rightarrow k'$  is injective, then the canonical morphism  $X^+ \rightarrow k' \otimes_k X^+$  is injective.
4. If  $A_+(H)$  is flat over  $k$  for all  $H \leq G$  and if  $k \rightarrow k'$  is injective, then the canonical morphism  $A_+ \rightarrow k' \otimes_k A_+$  is injective.

Finally, we have the following commutative diagram and the corollary:

$$\begin{array}{ccccc}
 A_+ & \longrightarrow & k' \otimes_k A_+ & \longrightarrow & (k' \otimes_k A)^+ \\
 \rho^A \downarrow & & k' \otimes_k \rho^A \downarrow & & \rho^{k' \otimes_k A} \downarrow \\
 A^+ & \longrightarrow & k' \otimes_k A^+ & \longrightarrow & (k' \otimes_k A)^+
 \end{array}$$

**Corollary 4.13.** *Let  $A \in k\text{-Res}(G)$ . If  $A$  has a stable basis,  $k'$  is flat over  $k$ , and  $k \rightarrow k'$  is injective, then the left horizontal maps in the above diagram are injective, the right horizontal maps are isomorphisms and the vertical morphisms are injective.*

### 4.3 Integrality Condition

Now, we give conditions on  $M$ ,  $A$  and  $p$  such that  $a^{M,A,p}$  is integral. We will need the following

**HYPOTHESIS 1.** Let  $M$  be a  $\mathbb{Z}$ -Mackey functor on a finite group  $G$ ,  $A$  be a  $\mathbb{Z}$ -restriction subfunctor of  $M$  on  $G$ , and  $p \in \mathbb{Z}\text{-Con}(G)(M, A)$  such that

1.  $M(H)$  is a free abelian group for all  $H \leq G$ ,
2.  $A$  has a stable basis  $\mathcal{B}$  such that for all  $K \leq H \leq G$  and  $\phi \in \mathcal{B}(H)$  the element  $\text{res}_K^H \phi \in A(K)$  is a linear combination

$$\text{res}_K^H \phi = \sum_{\psi \in \mathcal{B}(K)} m(H, \phi; K, \psi) \psi$$

where  $m(H, \phi; K, \psi)$  is a nonnegative integer.

Note that this hypothesis is designed for the various representation rings of  $G$ . Assuming the hypothesis, we have the following observations:

For each  $H \leq G$ , the free  $\mathbb{Z}$ -module  $A_+(H)$  has a basis  $\{[K, \psi]_H\}$  (as in lemma 4.9). We identify this  $\mathbb{Z}$ -basis of  $A_+(H)$  with the  $\mathbb{Q}$ -basis  $\{1 \otimes [K, \psi]_H\}$

of  $\mathbb{Q}A_+(H)$  and write  $[K, \psi]_H \in \mathbb{Q}A_+$ . We identify  $\mathbb{Q}A_+$  with  $(\mathbb{Q}A)_+$  and identify  $\mathbb{Q}A^+$  with  $(\mathbb{Q}A)^+$ . Under these identifications, we also identify  $\text{pr}_+^{\mathbb{Q}A}$  with  $\mathbb{Q}\text{pr}_+^A$  and so on. Note that  $\rho^A : A_+ \rightarrow A^+$  is injective and  $\rho^A : \mathbb{Q}A_+ \rightarrow \mathbb{Q}A^+$  is an isomorphism.

With this identifications, we get the following commutative diagram

$$\begin{array}{ccc}
 \mathbb{Q}\text{-Res}(G)(\mathbb{Q}M, \mathbb{Q}A_+) & \xrightarrow{\text{Res}_{\mathbb{Q}A}^{\mathbb{Q}M}} & \mathbb{Q}\text{-Con}(G)(\mathbb{Q}M, \mathbb{Q}A) \\
 \uparrow \text{inc} & & \uparrow \text{inc} \\
 \mathbb{Z}\text{-Res}(G)(M, A_+) & \xrightarrow{\text{Res}_A^M} & \mathbb{Z}\text{-Con}(G)(M, A)
 \end{array}$$

Now, the set of canonical induction formula for  $M$  from  $A$  is a subset of  $\mathbb{Z}\text{-Res}(G)(M, A_+)$  and can be identified with a subset of  $\mathbb{Z}\text{-Con}(G)(M, A)$  via  $\text{Res}_A^M$ . We are going to drive sufficient condition for  $p \in \mathbb{Z}\text{-Con}(G)(M, A)$  to be the residue of a canonical induction formula for  $M$  from  $A$ , that is,  $a^{\mathbb{Q}M, \mathbb{Q}A, \mathbb{Q}p}(M) \subset A_+$ . We call  $a^{\mathbb{Q}M, \mathbb{Q}A, \mathbb{Q}p}$  *integral* if  $a^{\mathbb{Q}M, \mathbb{Q}A, \mathbb{Q}p}(M) \subset A_+$ . Note that, this is equivalent to the condition that  $p = \text{res}(a)$  for a unique  $a \in \mathbb{Z}\text{-Res}(G)(M, A_+)$ . Assuming the integrality of  $p$ , we get  $\mathbb{Q}a = a^{\mathbb{Q}M, \mathbb{Q}A, \mathbb{Q}p}$  under the above identifications. By the following commutative diagram, we conclude that  $a$  is a canonical induction formula if and only if  $\mathbb{Q}a$  is a canonical induction formula.

$$\begin{array}{ccccc}
 M & \xrightarrow{a^{M,A,p}} & A_+ & \xrightarrow{b^{M,A}} & M \\
 \downarrow r^p & & \downarrow \rho^p & & \downarrow \\
 & & A^+ & & \\
 & & \downarrow & & \\
 & & \mathbb{Q}A^+ & & \\
 \uparrow \mathbb{Q}r^p & & \uparrow \mathbb{Q}\rho^p & & \\
 \mathbb{Q}M & \xrightarrow{\mathbb{Q}a^{M,A,p}} & \mathbb{Q}A_+ & \xrightarrow{\mathbb{Q}b^{M,A}} & \mathbb{Q}M
 \end{array}$$

Hence we can apply the results about  $\mathbb{Q}a$  to obtain results about  $a$ . So, we identify  $a^{M,A,p}$  and  $a^{\mathbb{Q}M,\mathbb{Q}A,\mathbb{Q}p}$ . Now, we want to get rid of the coefficient  $\frac{1}{|H|}$  in the formula 4.1.

Suppose  $M, A, \mathcal{B}$  and  $p$  are as in the hypothesis 4.3. For  $H \leq G, \chi \in M(H)$ , we write

$$p_H(\chi) = \sum_{\phi \in \mathcal{B}(H)} m_\phi(\chi) \phi \in A(H)$$

and call  $m_\phi(\chi) \in \mathbb{Z}$  the *multiplicity* of  $\phi$  in  $\chi$ . Note that  $m_\phi(\chi)$  depends on  $p$ , and if  $\phi \in \mathcal{B}(H), \psi \in \mathcal{B}(K)$  and  $p_K|_{A(K)} = \text{id}_{A(K)}$  then  $m_\phi(\text{res}_K^H(\phi)) = m(H, \phi; K, \psi)$ .

Let  $\sigma = ((H_0, \phi_0) < \dots < (H_n, \phi_n))$  be a strictly ascending chain in  $\text{ch}(H)$ . Denote by  $sd(\text{ch}(H))$  the set of chains  $\sigma$ . We write  $|\sigma| = n$  for the *length* of  $\sigma$ . We define the *multiplicity*  $m_\sigma$  of  $\sigma$  by

$$m_\sigma := \prod_{i=1}^n m(H_i, \phi_i; H_{i-1}, \phi_{i-1}) \in \mathbb{N}_0.$$

**Lemma 4.14.** *We have*

$$a_H(\chi) := a_H^{M,A,p}(\chi) = \frac{1}{|H|} \sum_{\sigma \in sd(\text{ch}(H))} (-1)^n |H_0| m_\sigma m_{\phi_n}(\text{res}_{H_n}^H(\chi)) [H_0, \phi_0]_H$$

for all  $H \leq G$  and  $\chi \in M(H)$ .

To prove the lemma, we use the following expansion of the Möbius function

$$\mu(L, K) = \sum_{L=H_0 < \dots < H_n=K} (-1)^n$$

where the sum is over all chains connecting  $L$  and  $K$ .

Finally, we state a theorem on integrality of  $a^{M,A,p}$ .

**Theorem 4.15.** *Let  $H \leq G$  and assume the following condition holds:*

*Let  $\chi \in M(H)$  and for all  $T \leq U \leq H$  with  $U/T$  cyclic and  $\psi \in \mathcal{B}(T)$  fixed under  $U$ , we have*

$$m_\phi(\text{res}_T^H \chi) = \sum_{\phi \in \mathcal{B}(U)} m(U, \phi; T, \psi) m_\phi(\text{res}_U^H \chi)$$

that is the two elements  $p_T(\text{res}_T^H \chi)$  and  $\text{res}_T^H p_H(\chi)$  have the same coefficient at the basis element  $\psi$ .

Then we have

$$a_H(\chi) = \frac{1}{|H|} \sum_{\sigma \in \text{sd}(\text{ch}(H))} (-1)^n |N_H(\sigma)| m_\sigma m_{\phi_n}(\text{res}_{H_n}^H(\chi)) [H_0, \phi_0]_H \quad (4.2)$$

$$= \sum_{\sigma \in H \setminus \text{sd}(\text{ch}(H))} (-1)^n m_\sigma m_{\phi_n}(\text{res}_{H_n}^H(\chi)) [H_0, \phi_0]_H \quad (4.3)$$

Note that, the above theorem as stated above is a special case of the Theorem 9.3 in [B.98].

We prove the theorem by rewriting the sum over  $N_H(\sigma) \times \text{sd}(\text{ch}(H))$  and decomposing the set in such a way that the factor  $|H_0|$  in the alternating sum is changed with the bigger factor  $|N_H(\sigma)|$  without changing the result.

**Corollary 4.16.** *If the condition of the theorem 4.15 is satisfied for all  $H \leq G$  and  $\chi \in M(H)$ , then we have*

$$a_H^{M,A,p} = \sum_{\sigma \in H \setminus \text{sd}(\text{ch}(H))} (-1)^n m_\sigma m_{\phi_n}(\text{res}_{H_n}^H(\chi)) [H_0, \phi_0]_H$$

for all  $H \leq G$  and all  $\chi \in M(H)$ ; in particular  $a^{M,A,p}$  is integral. Also, if  $A(H) = M(H)$  and  $p_H = \text{id}_{M(H)}$  for all  $H \in \mathcal{C}(M)$ , then  $a^{M,A,p}$  is an integral canonical induction formula.

## 4.4 Examples

### 4.4.1 The character ring

Let  $M = \mathcal{R}$ ,  $A = \hat{\mathcal{R}}$ ,  $\mathcal{B}$  and  $p$  as in the previous sections. We will show that the condition of the theorem 4.15 is satisfied for all  $H \leq G$  and  $\chi \in \mathcal{R}(H)$ . Then by the corollary 4.16,  $a^{M,A,p}$  is an integral canonical induction formula. Note that,

in this case  $m_\sigma = 1$  for all chains  $\sigma \in sd(ch(H))$  and this formula is the same as the one in [B.90], which is proved to be integral in another way.

Let  $U$  be a finite group, and  $T \trianglelefteq U$  such that  $U/T$  is cyclic. Let  $\nu \in \text{Irr}(U)$  and  $\psi \in \hat{T}$  such that  $\psi : T \rightarrow \mathbb{C}^*$  is  $U$ -stable. Then we have to show that the multiplicity of  $\psi$  in  $p_T(\text{res}_T^U(\nu))$  and  $\text{res}_T^U(p_U(\nu))$  coincide. The case  $\nu(1) = 1$  is clear. Otherwise,  $\text{res}_T^U(p_U(\nu)) = 0$  and we have to show  $\psi$  does not occur in  $p_T(\text{res}_T^U(\nu))$ . This means that

$$0 = (\psi, \text{res}_T^U(\nu))_T = (\text{ind}_T^U(\psi), \nu)_U$$

by Frobenius reciprocity. Since  $\psi$  is  $U$ -stable, the cyclic subgroup  $T/\ker\psi$  is central in  $U/\ker\psi$  and since  $U/T$  is cyclic,  $U/\ker\psi$  is abelian. Therefore  $\text{ind}_T^U\psi$  splits into linear characters as it is a character inflated from a character of the abelian group  $U/\ker\psi$  and hence  $(\text{ind}_T^U(\psi), \nu)_U = 0$ .

Note that, the same proof applies to any field  $K$  of characteristic zero containing a primitive  $\exp(G)$ -th root of unity. Hence, we have an equivalent canonical induction formula for  $\mathcal{R}_K \cong \mathcal{R}$  from  $\hat{\mathcal{R}}_K \cong \hat{\mathcal{R}}$ , where  $\mathcal{R}_K$  is the ring of virtual  $K$ -characters.

### 4.4.2 The Brauer Character Ring

We will show that for  $M = \mathcal{R}_{\mathfrak{K}}$  and  $A = \hat{\mathcal{R}}_{\mathfrak{K}}$  and for  $\mathcal{B}$  and  $p$  as in the previous sections, the condition of the theorem 4.15 is satisfied and again by corollary 4.16,  $a^{\mathcal{R}_{\mathfrak{K}}, \hat{\mathcal{R}}_{\mathfrak{K}}, p}$  is an integral canonical induction formula. Note that,  $m_\sigma = 1$  for all  $\sigma \in sd(ch(H)/H)$  and that  $m_\phi([V])$  counts the multiplicity of  $\mathfrak{K}_\phi$  as a composition factor in  $V \in \mathfrak{K}H\text{-mod}$  for  $H \leq G$  and  $\phi \in \hat{H}(\mathfrak{K})$ . This is done similarly as in the case of character ring.

# Chapter 5

## Some Induction Formulae

In this chapter, we introduce some induction formulae for different character rings of a finite group. In some cases, we are able to prove (or disprove) the integrality whereas in other cases, we have just conjectured it.

### 5.1 Modular Characters

In this section, we are going to explain how we can derive a canonical induction formula for modular characters from the formula for ordinary characters.

First, note that  $\hat{\mathcal{R}}_\kappa(H)_+$  (resp.  $\hat{\mathcal{R}}_{\mathfrak{K}}(H)_+$ ) is the monomial Burnside ring  $B(C, H)$  (resp.  $B(C', H)$ ) where  $C$  (resp.  $C'$ ) is a cyclic group containing all (resp. all  $p'$ ) roots of unity and  $\kappa$  and  $\mathfrak{K}$  are sufficiently large fields.

Consider the diagram

$$\begin{array}{ccc} \mathcal{R}_\kappa(-) & \xrightarrow{a} & B(C, -) \\ \downarrow d & & \downarrow d_+ \\ \mathcal{R}_{\mathfrak{K}}(-) & \xrightarrow{\alpha} & B(C', -) \end{array}$$

where  $d$  is the decomposition map (see [S]) which is a morphism of Mackey functors. In [B.98], it is remarked that the diagram does not commute for any canonical induction formula  $\alpha$  for the modular characters and  $a$  for the ordinary characters. (see the remark 5.6 at the end of this section for the proof of this statement.) Nevertheless, we have

**Proposition 5.1.** *The above diagram commutes up to  $\ker(b)$  where  $b$  is the induction morphism, that is,*

$$\text{Im}(d_+ \circ a - \alpha \circ d) \subset \ker(b).$$

*Proof.* Use commutativity of the following diagram

$$\begin{array}{ccc} \mathcal{R}_\kappa(-) & \xleftarrow{b} & B(C, -) \\ \downarrow d & & \downarrow d_+ \\ \mathcal{R}_{\mathfrak{K}}(-) & \xleftarrow{b} & B(C', -) \end{array}$$

to calculate  $b(d_+ \circ a - \alpha \circ d)$ . □

Inspired by the above proposition we have

**Theorem 5.2.** *Let  $\lambda$  be the Brauer lift defined by*

$$\lambda(\psi)(h) = \psi(h_{p'}) \quad \text{where } \psi \in \mathbb{B}r(H) \quad \text{and } h \in H$$

*and let  $a_H$  be a complex canonical induction formula. Then,*

$$\alpha_H = d_+ \circ a_H \circ \lambda$$

*is a canonical induction formula for modular characters. The residue  $\pi_H$  of  $\alpha_H$  is given for  $\psi \in \mathbb{B}r(H)$  by*

$$\pi_H(\psi) = \sum_{\phi \in \mathcal{B}(H)} \sum_{\chi \in d^{-1}(\phi)} m_\chi(\lambda\psi)\phi$$

*where  $\mathcal{B}(H)$  is the set of one-dimensional modular characters of  $H$  and*

$$p_H(\chi) = \sum_{\phi \in \hat{H}} m_\phi(\chi)\phi$$

*is the residue of  $a_H$ .*

*Proof.* The first part follows easily as the decomposition map is a morphism of Mackey functors and the Brauer lift is a section for the decomposition map. To find the residue  $\pi_H$  of  $\alpha_H$ , recall that

$$a_H(\chi) = \frac{1}{|H|} \sum_{\sigma \in sd(ch(H))} (-1)^{|\sigma|} |H_0| m_\sigma m_{\phi_n} (\text{res}_{H_n}^H(\chi)) [H_0, \phi_0]_H.$$

Using this we get,

$$\begin{aligned} \alpha_H(\psi) &= d_+ \circ a_H \circ \lambda(\psi) \\ &= \frac{1}{|H|} \sum_{\sigma \in H \setminus sd(ch(H))} (-1)^\sigma |H_0| m_\sigma m_{\phi_n} (\text{res}_{H_n}^H(\lambda\psi)) [H_0, d\phi_0]_H \end{aligned}$$

where  $\sigma = ((H_0, \phi_0) < \dots < (H_n, \phi_n))$ .

Then, by definition, the residue  $\pi$  of  $\alpha$  is,

$$\begin{aligned} \pi_H(\psi) = \text{res}(\alpha_H)(\psi) &= \text{pr}_+ \circ \alpha_H(\psi) \\ &= \sum_{\phi \in \hat{H}} m_\phi(\lambda\psi) d\phi \\ &= \sum_{\phi \in \mathcal{B}(H)} \sum_{\chi \in d^{-1}(\phi)} m_\chi(\lambda\psi) \phi. \end{aligned}$$

□

**Theorem 5.3.** *The modular canonical induction formula  $\alpha$  is integral if the complex canonical induction formula  $a$  is.*

*Proof.* Theorem is clear since the maps  $d$  and  $\lambda$  and  $a_H$  are all integral. □

**Corollary 5.4.** *The map*

$$\mathbb{Q}\text{-Con}(G)(\mathcal{R}_\kappa, \hat{\mathcal{R}}_\kappa) \rightarrow \mathbb{Q}\text{-Con}(G)(\mathcal{R}_{\mathfrak{R}}, \hat{\mathcal{R}}_{\mathfrak{R}})$$

*of morphism of  $\mathbb{Q}$ -conjugation functors defined by  $p_H \mapsto \pi_H$  induces a map*

$$\mathbb{Q}\text{-Res}(G)(\mathcal{R}_\kappa, \hat{\mathcal{R}}_{\kappa+}) \rightarrow \mathbb{Q}\text{-Res}(G)(\mathcal{R}_{\mathfrak{R}}, \hat{\mathcal{R}}_{\mathfrak{R}+})$$

*of  $\mathbb{Q}$ -restriction functors and this map embeds  $\mathbb{Z}\text{-Res}(G)(\mathcal{R}_\kappa, \hat{\mathcal{R}}_{\kappa+})$  in  $\mathbb{Z}\text{-Res}(G)(\mathcal{R}_{\mathfrak{R}}, \hat{\mathcal{R}}_{\mathfrak{R}+})$ .*

**Remark 5.5.** *Note that if we put*

$$p_H(\chi) = \sum_{\phi \in \hat{H}} \langle \phi, \chi \rangle \phi$$

*then*

$$\pi_H(\psi) = \sum_{\phi \in \mathcal{B}(H)} \sum_{\chi \in d^{-1}(\phi)} \langle \chi, \lambda\psi \rangle \phi$$

*Hence, the modular canonical induction formula we obtain is different from the one of the section 4.4.2 as the residues are different.*

**Remark 5.6.** *To show that there is no modular canonical induction formula compatible with the decomposition map, it is sufficient to show that there exist no  $\pi \in k\text{-Con}(G)(\mathcal{R}_{\mathfrak{K}}, \hat{\mathcal{R}}_{\mathfrak{K}})$  such that the diagram commutes:*

$$\begin{array}{ccc} \mathcal{R}_{\kappa}(-) & \xrightarrow{p} & \hat{\mathcal{R}}_{\kappa} \\ \downarrow d & & \downarrow d_+ \\ \mathcal{R}_{\mathfrak{K}}(-) & \xrightarrow{\pi} & \hat{\mathcal{R}}_{\mathfrak{K}} \end{array}$$

*by Proposition 4.7. Let  $G = S_4$  and  $\phi$  be the sign character and  $\chi$  be the unique character degree 2 of  $S_4$ . Also, let  $\text{char}(\mathfrak{K}) = 3$ . Then, the character  $\psi = 1 + \phi + \chi$  is in the kernel of  $d$ , whereas the projection  $p_G(\psi)$  is not.*

## 5.2 Characters over Number Fields

Throughout this section, let  $G$  be a finite group and  $K$  be a number field and  $L/K$  be an extension of  $K$  with all  $|G|$ -th roots of unity, so that,  $L$  is a splitting field for  $G$ . Let  $\Gamma = \text{Gal}(L/K)$  be the Galois group of this extension. We denote by  $\mathcal{R}_L(G)$  the ring of  $L$ -characters of  $G$ . Note that the canonical induction formula of 4.4.1 induces a canonical induction formula for  $L$ -characters of  $G$  and these formulae are equivalent. We write  $a^L$  for this formula.

For any  $\chi \in \text{Irr}_L(G)$ , we write

$$\text{tr}_{\Gamma}\chi = m_{\chi}(\chi_1 + \dots + \chi_n)$$

where  $m_\chi$  is the Schur index of  $\chi$  and  $\chi_i$  for  $i = 1, \dots, n$  are all Galois conjugates of  $\chi$ . It is well-known that the set of characters  $\text{tr}_\Gamma \chi$  for  $\chi \in \text{Irr}_L(G)$  forms a basis for the ring  $\mathcal{R}_R(G)$  of  $K$ -characters of  $G$ .

We put, for  $H \leq G$ ,

$$\hat{H}_K = \{\text{tr}_\Gamma \chi : \chi \in \hat{H}\}$$

and define  $p_H^K$  to be the obvious projection. Then we obtain a morphism  $a^K$  of  $\mathbb{Q}$ -restriction functors and by the Witt-Berman theorem, the morphism is a canonical induction formula. Unfortunately, the formula is not integral, in general. (For example, consider  $S_3$ ).

On the other hand, the following diagram

$$\begin{array}{ccc} \mathcal{R}_L(G) & \xrightarrow{p_G^L} & \hat{\mathcal{R}}_L(G) \\ \downarrow \text{tr}_\Gamma & & \downarrow \text{tr}_\Gamma \\ \mathcal{R}_K(G) & \xrightarrow{p_G^K} & \hat{\mathcal{R}}_K(G) \end{array}$$

commutes and by proposition 4.7, we recover  $a^K$  as a special case of the complex canonical induction formula.

In particular, the above construction is equivalent to the composite

$$\text{tr}_{\Gamma+} \circ a_H$$

where

$$\text{tr}_{\Gamma+} : B(C, G) \rightarrow \mathbb{Q}B(C, G)^\Gamma$$

is defined by  $[H, \phi] \mapsto [H, \frac{1}{|\text{orb}_\Gamma \phi}| \text{tr}_\Gamma \phi]$ .

**Remark 5.7.** *Comparing these two constructions, we see that the first formula has coefficient at  $[K, \psi]$  with denominator dividing  $|G|$  and the second dividing  $|\text{tr}_\Gamma \psi|$ . So, although the formula is not integral, the denominators are divisors of  $\text{gcd}(|G|, |\text{tr}_\Gamma \psi|)$ . In particular, the formula is integral if  $\text{gcd}(|G|, |\text{tr}_\Gamma \psi|) = 1$ . In the next section, we will show that this property allows us to construct a non-canonical integral formula for the real characters of  $G$ .*

## 5.3 Real Characters

In this section, we concentrate on the real characters of a finite group  $G$ . There are two different directions to follow to get an explicit formula for real characters from real characters of dimension at most two. First one is to use the method from the previous section, which results in a non-canonical integral induction formula and the other is to use Serre's induction theorem, in which case we get a canonical induction formula that we do not know whether the formula is integral or not.

### 5.3.1 A non-canonical formula

The canonical induction formula  $a^{\mathbb{R}}$  of the previous section is not integral, in general, but we have

**Proposition 5.8.** *Let  $G$  be a finite group. Then,*

1.  $a_G^{\mathbb{R}}$  is integral if  $|G|$  is odd.
2.  $i_G \circ a_G^{\mathbb{R}}$  is integral if  $|G|$  is even where  $i_G : B(C, G)^{\Gamma} \rightarrow \mathcal{R}_{\mathbb{R}^+}(G)$  is defined below.

Let  $[H, \phi] \in B(C, G)^{\Gamma}$  with  $\phi = \lambda + \bar{\lambda}$  and let  $K = N_G(H, \phi)$ . Suppose that  $\lambda =_G \bar{\lambda}$ . Now, for any  $k \in K$ , we have  ${}^k\lambda = \lambda$  or  $\bar{\lambda}$ , hence

$$|\mathrm{tr}_K \lambda| = |K : K_{\lambda}|$$

is even and as  $H < K_{\lambda}$ ,  $|K : H|$  is also even. Thus there exists  $kH \in K/H$  of order two, that is  $k^2 \in H$ .

Now, define  $i_G : B(C, G)^{\Gamma} \rightarrow \mathcal{R}_{\mathbb{R}^+}(G)$  as

$$i_G([H, \phi]_G) = [H(k), \mathrm{ind}_H^{H(k)} \phi]_G$$

if  $\phi$  is as above and identical otherwise.

*Proof.* (of proposition) The first part follows from the remark at the end of the previous section as  $\gcd(|G|, |\text{tr}_\Gamma|) = 1$ . The second part follows from the definition of  $i_G$  as in that case  $a_G^{\mathbb{R}}$  is not integral exactly at those  $[H, \phi]_G$  (with denominator 2) where  $i_G$  is non-trivial.  $\square$

**Remark 5.9.** *The formula is not canonical as we choose  $k \in N_G(H, \phi)$  arbitrarily.*

### 5.3.2 A canonical formula for real characters

Let  $G$  be a finite group and for  $H \leq G$ , let  $\mathcal{R}_{\mathbb{R}}(H)$  be the ring of real characters of  $H$ , that is the free abelian group on the set of irreducible real characters  $\text{Irr}_{\mathbb{R}}(H)$  of  $H$ . Then  $\mathcal{R}_{\mathbb{R}}$  is a  $\mathbb{Z}$ -Mackey functor on  $G$  by the usual conjugation, restriction and induction maps. For  $H \leq G$ , let  $\hat{\mathcal{R}}_{\mathbb{R}}(H) \subset \mathcal{R}_{\mathbb{R}}(H)$  denote the  $\mathbb{Z}$ -span of the subset  $\hat{H}_{\mathbb{R}}$  of  $\text{Irr}_{\mathbb{R}}(H)$  consisting of the characters of the following types:

- $\psi \in \text{Hom}(H, C_2)$ ,
- $\psi = \lambda + \bar{\lambda}$  with  $\lambda \in \text{Hom}(H, \mathbb{C})$ ,
- $\psi$ , real with  $G/\ker\psi = D_{2m}$  for some  $m > 2$ .

Then,  $\hat{\mathcal{R}}_{\mathbb{R}}$  is a  $\mathbb{Z}$ -restriction functor on  $G$ .

Define  $p \in \mathbb{Z} - \text{Con}(G)(\mathcal{R}_{\mathbb{R}}, \hat{\mathcal{R}}_{\mathbb{R}})$  by

$$p_H(\chi) = \begin{cases} \chi, & \text{if } \chi \in \hat{\mathcal{R}}_{\mathbb{R}} \\ 0, & \text{otherwise.} \end{cases}$$

for  $\chi \in \text{Irr}_{\mathbb{R}}(H)$  and  $H \leq G$ .

**Proposition 5.10.** *Let  $G$  be a finite group and let  $a_H^{\mathbb{R}} \in \mathbb{Q} - \text{Res}(G)(\mathcal{R}_{\mathbb{R}}, \hat{\mathcal{R}}_{\mathbb{R}+})$  be the morphism defined according to 3.12. Then,*

1.  $a_H^{\mathbb{R}}$  is a canonical induction formula,

2.  $a_H^{\mathbb{R}}$  is integral if  $|G|$  is odd,
3. For  $H \leq G$  and  $\psi \in \hat{H}_{\mathbb{R}}$ , we have  $a_H^{\mathbb{R}}(\psi) = [H, \psi]_H$ ,
4. The morphism  $a^{\mathbb{R}}$  is given explicitly by

$$a_H^{\mathbb{R}}(\chi) = \frac{1}{|H|} \sum_{\sigma \in \text{sd}(\text{ch}_{\mathbb{R}}(H))} (-1)^{|\sigma|} |H_0| m_{\sigma} m_{\phi_n} (\text{res}_{H_n}^H \chi) [H_0, \phi_0]_H$$

for  $H \leq G$  and  $\chi \in \mathcal{R}_{\mathbb{R}}(H)$ .

*Proof.* 1. By proposition 4.4, we need to show that for all  $H \leq G$  and  $\chi \in \mathcal{R}_{\mathbb{R}}(H)$  we have

$$p_H(\chi) - \chi \in \sum_{K < H} \text{ind}_K^H(\mathcal{R}_{\mathbb{R}}(K)).$$

It is sufficient to show this for  $\chi \in \text{Irr}_{\mathbb{R}}(H)$ , but for  $\chi \in \hat{H}_{\mathbb{R}}$ , we have  $p_H(\chi) = \chi$  so the condition is trivial. For the other case,  $p_H(\chi) = 0$  and the condition is satisfied by the Serre's induction theorem, theorem 2.5.

2. This is trivial as the formula is equivalent to the previous formula for  $|G|$  is odd.
3. This is also trivial as  $p_K(\text{res}_K^H \psi) = \text{res}_K^H(p_H(\psi))$
4. Follows from 4.14.

□

**Remark 5.11.** *The integrality condition of Theorem 4.15 is not satisfied when we put  $M = \mathcal{R}_{\mathbb{R}}$ . However, there is no example where some coefficients are not integers among groups of order strictly less than 32, except possibly for some groups of order 16.*

**Conjecture 5.12.** *The real canonical induction formula is integral.*

## 5.4 Rational Characters of $p$ -groups

It was proved by J. Ritter [R] and G. Segal [Se], independently, that any rational representation of a finite  $p$ -group comes from the Burnside ring of the group.

Later, S. Bouc [Bo] strengthened the result and showed that any non-trivial rational module of a finite  $p$ -group is induced from a module inflated from a  $C_p$ -section of a subgroup.

In this section, we will introduce a canonical induction formula for rational characters of finite  $p$ -groups from characters of subgroups inflated from  $C_p$ -sections. But we have not succeeded in proving (or disproving) the integrality.

Let  $P$  be a finite  $p$ -group and let  $\mathcal{R}(P)$  denotes the ring of rational characters of  $P$ . With the usual conjugation, restriction and induction maps,  $\mathcal{R}$  is a  $\mathbb{Z}$ -Mackey functor on  $P$ . Let  $\hat{\mathcal{R}}(P) \subset \mathcal{R}(P)$  denote the  $\mathbb{Z}$ -span of the subset  $\mathcal{B}(P)$  of  $\text{Irr}_{\mathbb{Q}}(P)$  consisting of characters inflated from a  $C_p$ -section, that is,  $\chi \in \text{Irr}_{\mathbb{Q}}(P)$  such that  $P/\ker\chi = C_p$ . Note that  $\hat{\mathcal{R}}(P)$  is a  $\mathbb{Z}$ -restriction functor on  $P$  with the restricted conjugation and restriction maps.

We define  $\pi \in \mathbb{Z}\text{-Con}(P)(\mathcal{R}, \hat{\mathcal{R}})$  by

$$\pi_R(\chi) = \begin{cases} \chi, & \text{if } \chi \in \mathcal{B}(R); \\ 0, & \text{otherwise.} \end{cases}$$

for  $\chi \in \text{Irr}_{\mathbb{Q}}(P)$  and  $R \leq P$

**Proposition 5.13.** *Let  $a_R \in \mathbb{Q}\text{-Res}(P)(\mathcal{R}, \hat{\mathcal{R}}_+)$  be the morphism defined according to 3.12. Then*

1.  $a_R$  is a canonical induction formula.
2.  $a_R$  is integral if  $\exp(R) = p$ .
3. The morphism  $a$  is given explicitly by

$$a_R^{\mathbb{Q}}(\chi) = \frac{1}{|R|} \sum_{\sigma \in \text{sd}(\text{ch}_{\mathbb{Q}}(R))} (-1)^{|\sigma|} |R_0| m_{\sigma} m_{\phi_n} (\text{res}_{R_n}^R \chi) [R_0, \phi_0]_R$$

for  $R \leq P$  and  $\chi \in \mathcal{R}_{\mathbb{Q}}(R)$

*Proof.* The first part is clear as the condition of Proposition 4.4 is satisfied (by Theorem 2.6). To prove the second part note that when the exponent of  $R$  is

$p$ , any 1-dimensional complex character of  $R$  has Galois orbit of size at most  $p - 1$ . Hence the integrality condition 4.15 is satisfied. The last part follows from Lemma 4.14.  $\square$

The condition 4.15 is not satisfied, in general. To see this, let  $T, U$  and  $\psi$  be as in the condition. Then, the condition is equivalent to the following statement:

For any complex constituent  $\phi$  of  $\psi$ , any complex constituent  $\phi'$  of  $\text{ind}_T^U(\phi)$  has Galois orbit of size at most  $p - 1$ .

The statement does not satisfied, for example for  $C_4$ . But, we have

**Conjecture 5.14.** *The rational canonical induction formula is integral.*

**Remark 5.15.** *Theorem 2.6 implies the possibility of having an integral formula as the theorem states that any irreducible rational character is induced from an element of  $\text{ch}(P)$ . We conjecture that this can be canonical if we are allowed to have more than one character to induce from.*

**Remark 5.16.** *To see that the condition 4.15 is not satisfied by  $C_4$ , let  $U = C_4$  and  $T = C_2$  and  $\psi$  be the nontrivial irreducible character of  $C_2$ . Then  $\text{ind}_{C_2}^{C_4}\psi$  is the unique two dimensional rational character of  $C_4$ , hence is not in the span of characters inflated from  $C_2$ -sections. Thus the condition is not satisfied. However, in that case, the formula is integral as both the order of  $C_2$  and the multiplicity of  $\psi$  in the two dimensional rational character of  $C_4$  are 2.*

# Chapter 6

## Modules of Monomial Burnside Algebra

In this chapter, we describe the primitive idempotents of the character rings and find the images under the canonical induction formula. We also find the inverse composition  $a \circ b$  of the induction morphism with the canonical induction formula in the primitive idempotent basis.

Throughout, let  $\mathcal{O}$  be a complete local Noetherian commutative ring, with field of fractions  $\kappa$  of characteristic zero, and the residue field  $\mathfrak{K}$  of prime characteristic  $p$ . Let  $H$  be a finite group and assume that the field  $\kappa$  contains  $|H|$ -th root of unity. For any field  $F$ , denote by  $\mathcal{R}_F$  the  $F$ -character ring Mackey functor. Let  $\text{Irr}(H)$  (resp.  $\mathbb{B}r(H)$ ) be the set of irreducible  $\kappa$ -characters (resp. irreducible  $\mathfrak{K}$ -characters) of  $H$ . Let  $R_\kappa(H)$  (resp.  $R_{\mathfrak{K}}(H)$ ) be the set of conjugacy classes (resp.  $p'$ -conjugacy classes) of  $H$ . Note that these sets are in bijective correspondence with the set of species of the algebras  $\kappa\mathcal{R}_\kappa(H)$  and  $\kappa\mathcal{R}_{\mathfrak{K}}(H)$ , respectively. Similarly, for the monomial Burnside ring  $B(C, H)$  with the fibre group  $C$ , denote by  $\mathcal{B}(C, H)$  and  $\mathcal{B}^p(C, H)$  the set of species of  $\kappa B(C, H)$  and  $\mathfrak{K}B(C, H)$ .

## 6.1 Idempotents in Characteristic Zero

In this section, we will describe the primitive idempotents of the algebras  $\kappa\mathcal{R}_\kappa(H)$ ,  $\kappa\mathcal{R}_{\mathfrak{R}}(H)$  and introduce a formulae for a special class of primitive idempotents of  $\kappa B(C, H)$  and  $\kappa B(C', H)$  where  $C$  (resp.  $C'$ ) is a cyclic group containing all (resp. all  $p'$ ) roots of unity. Also, we will find the images of the idempotents of  $\kappa\mathcal{R}_\kappa(H)$  and  $\kappa\mathcal{R}_{\mathfrak{R}}(H)$  under the canonical induction formula and define some maps between the primitive idempotents.

### 6.1.1 Algebra of the Complex Character Ring

It is well-known that the species of  $\kappa\mathcal{R}_\kappa(H)$  are of the form  $s_h^H$ ,  $h \in R_\kappa(H)$  with  $s_h^H(\chi) = \chi(h)$  for all  $\chi \in \mathcal{R}_\kappa(H)$ . Hence we get the following explicit formula for  $e_h^H$ :

$$e_h^H = \frac{1}{|H|} \sum_{\chi \in \text{Irr}H} \chi(h^{-1})\chi$$

**Remark 6.1.** *Note that as for characters, one can find the primitive idempotents of  $\kappa\mathcal{R}_K$  where  $K$  is a subfield of the complex numbers and get similar results that we will get in the following sections. For example, if  $K = \mathbb{Q}$ , then the primitive idempotents of  $\kappa\mathcal{R}_K(H)$  are of the form*

$$e_h^{\mathbb{Q},H} = \sum_{k \in H: \langle k \rangle = \langle h \rangle} e_k^H$$

where  $e_k^H$  are the primitive idempotents of  $\kappa\mathcal{R}_L(H)$  where  $L$  is a splitting field for  $H$ .

### 6.1.2 Algebra of the Modular Character Ring

We denote by  $\epsilon_h^H$  for  $h \in R_{\mathfrak{R}}(H)$  the primitive idempotents of  $\kappa\mathcal{R}_{\mathfrak{R}}(H)$ . The following proposition describe  $\epsilon_h^H$ , explicitly.

**Proposition 6.2.** *Let  $e_h^H \in \kappa\mathcal{R}_\kappa(H)$  be a primitive idempotent defined in 6.1.1, and  $d$  be the decomposition map from 5.1. Then*

$$d(e_h^H) = \begin{cases} \epsilon_h^H & \text{if } [h]_H \in R_{\mathfrak{R}}(H) \\ 0 & \text{otherwise} \end{cases}$$

Moreover, when  $d(e_h^H) \neq 0$  we get

$$\epsilon_h^H = \frac{1}{|H|} \sum_{\psi \in \text{Br}(H)} \sum_{\chi \in \text{Irr}H} \chi(h^{-1}) d_{\psi\chi} \psi$$

where  $d_{\psi\chi}$  is the decomposition number of  $\chi$  at  $\psi$ .

*Proof.* We have, for  $k \in R_\kappa(H)$

$$\begin{aligned} s_k^H(d(e_h^H)) &= \frac{1}{|H|} \sum_{\chi \in \text{Irr}H} \chi(h^{-1}) d(\chi)(k) \\ &= \begin{cases} 1 & \text{if } k =_H h \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

by the orthogonality relation for the characters of  $H$ . Note that  $d(\chi)(k) = \chi(k)$  as  $k \in R_{\mathfrak{R}}(H)$ .

We get the last part using the following equality:

$$d(\chi) = \sum_{\psi \in \text{Br}(H)} d_{\psi\chi} \psi$$

where  $d_{\psi\chi}$  is the decomposition number of  $\chi$ . □

**Remark 6.3.** *Recall that the Brauer lift is a section of the decomposition map defined in 5.1. It is easy to see that the Brauer lift of a primitive idempotent  $\epsilon_h^H$  of  $\kappa\mathcal{R}_{\mathfrak{R}}(H)$  is*

$$\lambda(\epsilon_h^H) = \sum_{k \in R_\kappa(H): k_p =_H h} e_k^H$$

### 6.1.3 Algebra of the Monomial Burnside Ring

The species of  $\kappa B(C, H)$  are found in [Dress]. Let's just define the species.

Let  $\mathcal{B}(C, H) = \{(K, k) : K \leq_H H, kK' \in \bar{K}\}$ . Then we define

$$s_{K,k}^H : \kappa\mathcal{B}(C, H) \rightarrow \kappa$$

such that  $[V, \nu]_H \mapsto \sum_{hV \subseteq H: K \leq^h V} ({}^h\nu)(k)$ . It was proved in [Dress] that every species of  $\kappa\mathcal{B}(C, H)$  is of the form  $s_{K,k}^H$  for some  $(K, k) \in \mathcal{B}(C, H)$ . Alternatively, in [B.03],  $s_{K,k}^H$  is defined to be the composition  $s_k^K \circ \text{Res}_K \circ \text{res}_{+K}^H$ , and it is easy to see that the two definitions are equivalent.

In [LJB] and [B.03], the primitive idempotents  $e_{K,k}^H$  of  $\kappa\mathcal{B}(C, H)$  are found, explicitly. In the following we introduce a formula for the primitive idempotents  $e_{\langle h \rangle, h}^H$  of  $\kappa\mathcal{B}(C, H)$ , in terms of the basis elements  $[\langle h \rangle, \phi]_H$  where  $\phi \in \hat{H}$  and  $\epsilon_{K,k}^H$  of  $\kappa\mathcal{B}(C', H)$  in terms of  $e_{K,k}^H$ . We call the idempotent  $e_{\langle h \rangle, h}^H$  a *cyclic idempotent*. The formula for cyclic idempotents is a special case of the formulae found in [LJB] and [B.03] with a different proof.

**Proposition 6.4.** *Let  $h \in H$ . Then*

$$e_{\langle h \rangle, h}^H = \frac{1}{|N_H(\langle h \rangle)|} \sum_{\phi \in \langle \hat{h} \rangle} \phi(h^{-1})[\langle h \rangle, \phi]_H$$

*Proof.* We calculate the right hand side at  $s_{K,k}^H$ . It is sufficient to consider the case  $K = \langle k \rangle$  as otherwise  $K \not\leq \langle h \rangle$  and  $s_{K,k}^H([\langle h \rangle, h]_H) = 0$ . In that case, we get

$$\begin{aligned} s_{\langle k \rangle, k}^H(RHS) &= \frac{1}{|N_H(\langle h \rangle)|} \sum_{\phi \in \langle \hat{h} \rangle} \phi(h^{-1}) \sum_{g\langle h \rangle \subseteq H: k = ghg^{-1}} \phi(g^{-1}kg) \\ &= \begin{cases} 1 & \text{if } k \in [h]_H \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

□

**Corollary 6.5.**

$$e_h^H = \frac{1}{|N_H(\langle h \rangle)|} \sum_{\phi \in \langle \hat{h} \rangle} \phi(h^{-1}) \text{ind}_{\langle h \rangle}^H \phi$$

**Proposition 6.6.** *Let  $d_+$  be the decomposition map of section 5.1. Then*

$$d_+(e_{K,k}^H) = \begin{cases} \epsilon_{K,k}^H & \text{if } k_{p'} = k \\ 0 & \text{otherwise} \end{cases}$$

*Proof.* Follows from the commutativity of the following diagram

$$\begin{array}{ccc}
 \kappa B(C, H) & \xrightarrow{d_+} & \kappa B(C', H) \\
 & \searrow s_{K,k}^H & \downarrow s_{K,k}^H \\
 & & \kappa
 \end{array}$$

□

**Remark 6.7.** As in the character ring case, we can get an explicit formula for the cyclic idempotents  $\epsilon_{\langle h \rangle, h}^H$ .

**Remark 6.8.** Note that similar to the decomposition map, we define the Brauer lift  $\lambda_H : B(C', H) \rightarrow B(C, H)$  by  $[K, \psi]_H \mapsto [K, \lambda(\psi)]_H$  for  $K \leq H$  and  $\psi \in \hat{K}$ . Then the Brauer lift of  $\epsilon_{\langle h \rangle, h}^H$  is

$$\lambda_H(\epsilon_{\langle h \rangle, h}^H) = \sum_{k \in R_\kappa(H) : k_{p'} \in [h]_H} e_{\langle h \rangle, h}^H.$$

Finally, we describe the correspondence of the primitive idempotents of  $\kappa \mathcal{R}_\kappa(H)$  and  $\kappa B(C, H)$  under the induction morphism  $b_H$ .

**Lemma 6.9.** The following diagram commutes.

$$\begin{array}{ccc}
 \kappa B(C, H) & \xrightarrow{b_H} & \kappa \mathcal{R}_\kappa(H) \\
 & \searrow s_{\langle h \rangle, h}^H & \downarrow s_h^H \\
 & & \kappa
 \end{array}$$

*Proof.*

$$s_h^H(b_H([K, \psi]_H)) = s_h^H(\text{ind}_K^H \psi) = \text{ind}_K^H \psi(h) = \sum_{gK \subseteq H : h \in {}^g K} ({}^g \psi)(h) = s_{\langle h \rangle, h}^H([K, \psi]_H)$$

□

**Proposition 6.10.** Let  $e_{K,k}^H, (K, k) \in \mathcal{B}(C, H)$  be a primitive idempotent of  $\kappa B(C, H)$ . Then

$$b_H(e_{K,k}^H) = \begin{cases} e_k^H & \text{if } K = \langle k \rangle \\ 0 & \text{otherwise} \end{cases}$$

*Proof.* Follows from the above lemma.  $\square$

The above proposition shows that all of the simple modules of  $\kappa B(C, H)$  except those corresponding to the cyclic idempotents are annihilated by  $b_H$ .

**Remark 6.11.** *Similar result holds for the idempotents  $\epsilon_{(h),h}^H$  of  $\kappa B(C', H)$ .*

## 6.1.4 Image under Canonical Induction Formula

### Character ring

We want to find  $a_H(e_h^H) \in \kappa B(C, H)$  where  $a_H$  is the canonical induction formula of section 4.4.1.

By corollary 4.16, we have

$$a_H(e_h^H) = \sum_{((H_0, \phi_0) < \dots < (H_n, \phi_n)) \in \text{sd}(\text{ch}(H))/H} (-1)^n m_{\phi_n}(\text{res}_{H_n}^H e_h^H) [H_0, \phi_0]_H$$

where  $m_{\phi_n}(\text{res}_{H_n}^H \chi) = \langle \phi_n | \text{res}_{H_n}^H \chi \rangle$  is the multiplicity of  $\phi_n$  in  $\text{res}_{H_n}^H \chi$ .

It is easy to see that

$$\text{res}_K^H e_h^H = \sum_{k \in R_\kappa(K) : k \in [h]_H} e_k^K$$

and combining this with the formula from 6.1.1, we get

$$\begin{aligned} m_{\phi_n}(\text{res}_{H_n}^H e_h^H) &= \langle \phi_n | \sum_{h_n \in R_\kappa(H_n) \cap [h]_H} \frac{1}{|H_n|} \sum_{\chi \in \text{Irr} H_n} \chi(h_n^{-1}) \chi \rangle \\ &= \frac{1}{|H_n|} \sum_{h_n \in R_\kappa(H_n) \cap [h]_H} \phi_n(h_n^{-1}) \end{aligned}$$

Hence

$$a_H(e_h^H) = \sum_{((H_0, \phi_0) < \dots < (H_n, \phi_n)) \in \text{sd}(\text{ch}(H))/H} (-1)^n \frac{1}{|H_n|} \sum_{h_n \in R_\kappa(H_n) \cap [h]_H} \phi_n(h_n^{-1}) [H_0, \phi_0]_H$$

It is possible to refine the equality above, using an inverse process done in lemma 4.14. We use

$$\sum_{(H_0, \phi_0) < \dots < (H_n, \phi_n)} 1 = \sum_{H_0 < \dots < H_n} \sum_{\phi_n \in \widehat{H}_n/H} 1$$

to get

$$\begin{aligned} a_H(e_h^H) &= \sum_{H_0 < \dots < H_n} \sum_{\phi_n \in \widehat{H}_n/H} \frac{(-1)^n}{|H_n|} \sum_{h_n \in R_\kappa(H_n) \cap [h]_H} \phi_n(h_n^{-1}) [H_0, \text{res}_{H_0}^{H_n} \phi_n]_H \\ &= \sum_{H_0 < \dots < H_n} \frac{(-1)^n}{|H_n|} \sum_{h_n \in R_\kappa(H_n) \cap [h]_H} \sum_{\phi_n \in \widehat{H}_n/H} \phi_n(h_n^{-1}) [H_0, \text{res}_{H_0}^{H_n} \phi_n]_H \\ &= \sum_{K \leq_H H} \frac{1}{|K|} \sum_{k \in R_\kappa(K) \cap [h]_H} \sum_{\phi \in \widehat{K}/H} \phi(k^{-1}) \sum_{L < K} \mu(L, K) [L, \text{res}_L^K \phi] \end{aligned}$$

On the other hand, as  $s_{\langle k \rangle, k}^H(e_{\langle h \rangle, h}^H) = [k \in [h]_H]$  and  $b_H(e_{\langle h \rangle, h}^H) = e_h^H$  we find

$$a_H(e_h^H) = e_{\langle h \rangle, h} + \sum_{(K, k) \in \mathcal{B}(C, H): K \neq \langle k \rangle} \lambda_{K, k}^{H, h} e_{K, k}^H$$

where  $\lambda_{K, k}^{H, h} = s_{K, k}^H(a_H(e_h^H))$  and  $\sum_{(K, k) \in \mathcal{B}(C, H): K \neq \langle k \rangle} \lambda_{K, k}^{H, h} e_{K, k}^H \in \ker(b_H)$ . In the following proposition we describe the coefficients  $\lambda_{K, k}^H$ .

**Proposition 6.12.**  $\lambda_{K, k}^{H, h} = |K \setminus kK' \cap [h]_H| / |K'|$ .

*Proof.* As  $s_{K, k}^H = s_k^H \circ \text{pr}_{+K} \circ \text{res}_{+K}^H$ , we get

$$\begin{aligned} s_{K, k}^H(a_H(e_h^H)) &= (s_k^K \circ \text{pr}_{+K} \circ \text{res}_{+K}^H)(a_H(e_h^H)) \\ &= s_k^K(p_K(\text{res}_K^H(e_h^H))) \end{aligned}$$

by the commutativity of the diagram 3.11. Then,

$$\begin{aligned} s_{K, k}^H(a_H(e_h^H)) &= \frac{1}{|K|} \sum_{l \in_K K: l \in [h]_H} \sum_{\psi \in \widehat{K}} \bar{\psi}(l) \psi(k) \\ &= \frac{1}{|K|} \sum_{l \in_K K: l \in [h]_H} \sum_{\psi \in \widehat{K/K'}} \bar{\psi}(\bar{l}^{-1}) \bar{\psi}(\bar{k}) \\ &= \sum_{l \in_K kK': l \in [h]_H} \frac{1}{|K'|} \end{aligned}$$

□

**Remark 6.13.** *Since*

$$b_H(e_{(h),h}^H) = e_h^H$$

by 6.1.3, we get

$$e_h^H = \sum_{((H_0, \phi_0) < \dots < (H_n, \phi_n)) \in \text{sd}(\text{ch}(H))/H} (-1)^n \frac{1}{|H_n|} \sum_{h_n \in R_\kappa(H_n) \cap [h]_H} \phi_n(h_n^{-1}) \text{ind}_{H_0}^H \phi_0.$$

The canonical induction formula is constructed as a section for the induction morphism, that is, we have  $b \circ a = \text{id}$ . We seek a formula for  $a \circ b$ . Now, let's find this composition using the primitive idempotents.

Let  $\zeta \in \kappa B(C, H)$ . Then, we can write

$$\zeta = \sum_{(K,k) \in \mathcal{B}(C,H)} s_{K,k}^H(\zeta) e_{K,k}^H$$

By proposition 6.10, we have

$$b_H(\zeta) = \sum_{(\langle k \rangle, k) \in \mathcal{B}(C,H)} s_{\langle k \rangle, k}^H(\zeta) e_k^H$$

Hence, using the formula for  $a_H(e_k^H)$  we have found above, we get

$$\begin{aligned} a_H \circ b_H(\zeta) &= \sum_{(\langle k \rangle, k) \in \mathcal{B}(C,H)} s_{\langle k \rangle, k}^H(\zeta) a_H(e_k^H) \\ &= \sum_{(\langle k \rangle, k) \in \mathcal{B}(C,H)} s_{\langle k \rangle, k}^H(\zeta) (e_{\langle k \rangle, k}^H + \sum_{(L,l) \in \mathcal{B}(C,H): L \neq \langle l \rangle} \lambda_{L,l}^{H,k} e_{L,l}^H) \\ &= \sum_{(\langle k \rangle, k) \in \mathcal{B}(C,H)} s_{\langle k \rangle, k}^H(\zeta) e_{\langle k \rangle, k}^H + \\ &\quad \sum_{(L,l) \in \mathcal{B}(C,H): L \neq \langle l \rangle} \left( \sum_{(\langle k \rangle, k) \in \mathcal{B}(C,H)} s_{\langle k \rangle, k}^H(\zeta) \lambda_{L,l}^{H,k} \right) e_{L,l}^H \end{aligned}$$

### Brauer character ring

Similar calculations can be done for the Brauer characters, but we shall not repeat the calculations. Instead, we recall that, in section 5.1, we proved that

the diagram

$$\begin{array}{ccc} \mathcal{R}_\kappa(-) & \xrightarrow{a} & B(C, -) \\ \downarrow d & & \downarrow d_+ \\ \mathcal{R}_{\mathfrak{R}}(-) & \xrightarrow{\alpha} & B(C', -) \end{array}$$

commutes up to an error term in  $\ker(b : B(C', -) \rightarrow \mathcal{R}_{\mathfrak{R}}(-))$ . In this section, we will find this error term for the modular canonical induction formula  $\alpha_H$  of section 5.1 by finding  $\alpha_H(\epsilon_h^H)$ .

Recall that we have defined  $\alpha_H$  to be the composition  $\alpha_H = d_+ \circ a_H \circ \lambda$  and we have found, in 6.1.2,  $\lambda(\epsilon_h^H)$ . Using these and the formula from 6.13, we get

$$\begin{aligned} \alpha_H(\epsilon_h^H) &= d_+ \left( \sum_{k \in R_\kappa(H) : k_{p'} \in [h]_H} a_H(e_k^H) \right) \\ &= d_+ \circ a_H(e_h^H) + \sum_{k \in R_\kappa(H) : k_{p'} \in [h]_H, k \neq h} d_+ \circ a_H(e_k^H) \end{aligned}$$

The last summand in the above equality is the term we are looking for as  $\alpha_H(\epsilon_h^H) = \alpha_H \circ d(e_h^H)$ . By proposition 6.6, we get

$$\alpha_H(\epsilon_h^H) - d_+ \circ a_H(e_h^H) = \sum_{(L,l) \in \mathcal{B}(C', H) : L \neq \langle l \rangle} \sum_{k \in R_\kappa(H) : k_{p'} \in [h]_H, k \neq h} \lambda_{L,l}^{H,k} e_{L,l}^H.$$

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