

COMPARISON OF CRACK INITIATION CRITERIA IN FUNCTIONALLY
GRADED MATERIALS UNDER MIXED MODE LOADING

by

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ABSTRACT

COMPARISON OF CRACK INITIATION CRITERIA IN FUNCTIONALLY GRADED MATERIALS UNDER MIXED MODE LOADING

In this thesis, analytical crack initiation criteria are evaluated in detail. Finite element method is used to determine crack initiation angles for edge cracked functionally graded materials. Minimum strain energy density (S), maximum energy release rate (G), and maximum triaxial stress (M) criteria are used in numerical analysis. Several finite element models are constructed to investigate the effect of crack length, crack angle and material non-homogeneity on crack initiation angle. Results of minimum strain energy density (S), energy release rate (G), maximum triaxial stress (M) and maximum tangential stress (MTS) criteria are compared.

ÖZET

ÖZELLİKLERİ FONKSİYONEL OLARAK DERECELENDİRİLMİŞ MALZEMELERDE, KARIŞIK TİP YÜKLEME ALTINDA ÇATLAK İLERLEME KRİTERLERİNİN KARŞILAŞTIRILMASI

Bu tezde, analitik çatlak ilerleme kriterleri detaylı olarak değerlendirilmiştir. Homojen merkez çatlaklı ve özellikleri fonksiyonel olarak derecelendirilmiş kenar çatlaklı malzemelerde, çatlak ilerleme açılarının belirlenmesinde sonlu elemanlar yöntemi kullanılmıştır. Numerik analizde S, G ve M kriterleri kullanılmıştır. Çatlak boyunun, çatlak açısının ve homojen olmama durumunun, çatlak ilerleme açısı üzerindeki etkisini araştırmak için çeşitli sonlu eleman modelleri geliştirilmiştir. S, G, M ve MTS kriterlerinin çatlak ilerleme açısı sonuçları karşılaştırılmıştır.

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LIST OF SYMBOLS/ABBREVIATIONS

a	Crack length
B	Biaxiality ratio
a_{11}, a_{12}, a_{22}	Arbitrary constants
C	Arbitrary constant
D_{11}, D_{12}, D_{22}	Arbitrary constants
E	Elastic Modulus
E_1	Elastic modulus at the left edge of the plate
E_2	Elastic modulus at the right edge of the plate
E_{tip}	Elastic modulus at the crack tip
f	Angular function
G_s	Modulus of rigidity
G	Energy release rate
G_c	Fracture Toughness
H	Height of the plate
I, II	Subscripts denoting mode of loading
I_1, I_2	First and second stress invariants
J	J-integral
K	Stress intensity factor
K_{eff}	Effective stress intensity factor
K_{Ic}, K_{IIc}	Fracture toughness
M	Stress triaxiality ratio
M_e	Mixity parameter
N	Arbitrary constant
O	High order terms
Q	Shear load
r, θ	Polar co-ordinates at the crack tip
r_c	Critical radius
S	Strain energy density factor
T	T-stress

t	Thickness of the plate
U_s	Total strain energy
U	Total strain energy density
U_V	Dilatational strain energy density
U_D	Distortional strain energy density
x, y	Cartesian co-ordinates at the crack tip
W	Width of the plate
W_s	surface Energy
α	Ratio of critical radius to the crack length
β	Crack angle
δ	Infinitesimal kink length
κ	Constant
μ	Ratio of stress intensity factors
ν	Poisson's ratio
Π	Potential energy
ψ	The length scale of the material gradient
σ	Uniform stress applied on the upper boundary of the plate
σ_{xx}	Normal stress along x-direction
σ_{yy}	Normal stress along y-direction
σ_{zz}	Normal stress along z-direction
σ_{xy}	Shear stress on x-plane along y-direction
σ_{rr}	Normal stress along radial direction
$\sigma_{\theta\theta}$	Normal stress along tangential direction
$\sigma_{r\theta}$	Shear stress on r-plane along θ direction
σ_H	Hydrostatic stress
σ_{eq}	Equivalent stress
θ_c	Crack initiation angle
FEM	Finite Element Method
FGM	Functionally Graded Material
SIF	Stress Intensity Factor

1. INTRODUCTION

Functionally graded materials (FGMs) are special composites usually made from ceramics and metals. The composition in an FGM is varied gradually, thus giving a non-uniform microstructure in the material with continuously graded macro properties that are most distinctive features of FGM's. The change in microstructure induces chemical, material and microstructural gradients, and makes FGMs different in behavior from homogeneous materials and traditional composite materials.

These materials also have certain advantages over existing isotropic materials and conventional composites, especially in applications that demand, resistance to corrosion and high temperature as in furnace walls and turbine blades, wear resistance as in gears and high speed machine tools, combined with good toughness and strength characteristics.

For FGM's with compositions including a ceramic or a glass, fracture is an important design limitation. If the FGM is a brittle/brittle composite, linear elastic fracture mechanics (LEFM) can be used to characterize such failure. The knowledge of crack growth in FGMs is important in order to evaluate their integrity. It has been shown that for FGM crack problems the crack tip has a regular square-root singularity, the stress and displacement near-tip field are of the same forms as those for homogeneous materials. The problem of crack initiation under mixed mode loading is still of great interest in the field of fracture mechanics. In order to accurately predict and simulate crack propagation paths, one must define a valid criterion for initial crack extension angles. The prediction of crack initiation angles is also important in the design and implementation of techniques for arresting cracks. Because of the complex nature of the actual loading condition at the crack tip it is important to address crack initiation and propagation under mixed-mode loading and not simplify to pure mode I and/or pure mode II analysis [1].

Thesis is composed of three main parts. First part is Fracture Mechanics, the aim of this part is to give briefly information about fundamental fracture mechanics concepts. Second part includes a detailed analysis of crack initiation criteria which exist in the literature. These criteria are:

- MTS Criterion
- G Criterion
- S Criterion
- Generalized MTS Criterion
- M Criterion
- T Criterion
- N_t and Y Criterion
- Det Criterion
- I_p Criterion
- R and W Criterion

Numerical analysis is the third part of the thesis.

1.1. Introduction to Fracture Mechanics

Fracture mechanics concerns the design and analysis of structures which contain cracks or flaws. On some size-scale all materials contain flaws either microscopic, due to cracked inclusions, debonded fibres etc., or macroscopic, due to corrosion, fatigue, welding flaws etc. Thus fracture mechanics is involved in any detailed design or safety assessment of a structure. As cracks can grow during service due to e.g. fatigue, fracture mechanics assessments are required throughout the life of a structure or component, not just at start of life. Fracture mechanics answers the questions: What is the largest sized crack that a structure can contain or the largest load the structure can bear for failure to be avoided? How long before a crack which was safe becomes unsafe? What material should be used in a certain application to ensure safety?

Studies in the US in the 1970s by the US National Bureau of Standards estimated

that cost of fracture due to accidents, overdesign of structures, inspection costs, repair and replacement was on the order of 120 billion dollars a year. While fracture cannot of course be avoided, they estimated that, if best fracture control technology at the time was applied, 35 billion dollars could be saved annually. This indicates the importance of fracture mechanics to modern industrialised society [1].

In fracture mechanics, determining stress state at the crack tip is important. We define a polar or a cartesian coordinate axis with the origin at the crack tip as illustrated in the Figures 1.1-1.2.

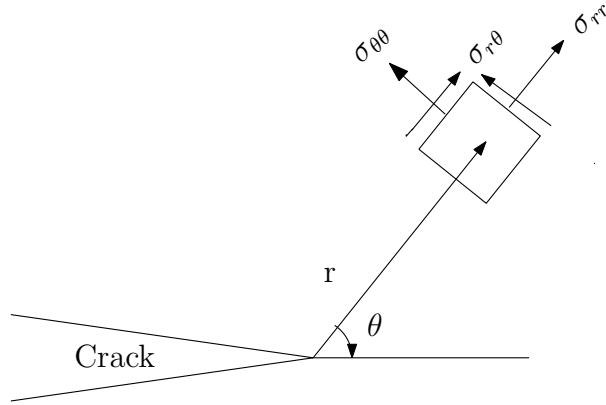


Figure 1.1. Polar coordinates centered at a crack tip

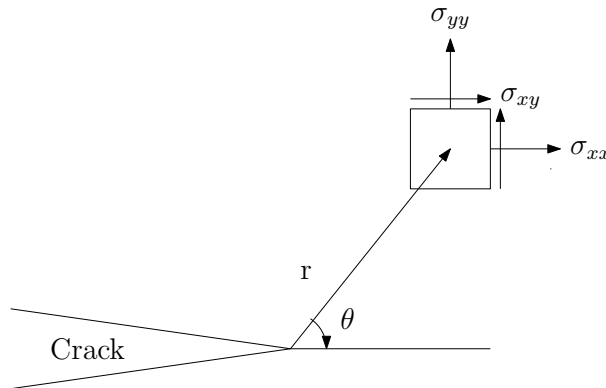


Figure 1.2. Cartesian coordinates centered at a crack tip

For a linear elastic material, state of stress may be solved by using the Airy stress function method which gives an exact solution to linear elastic stress problems [1]. The stress field is represented as an infinite series in r ,

$$\sigma_{ij} = \sum k_i r^{C_i} f_{ij}(\theta) \quad (1.1)$$

where k_i and C_i are unknown real constants and f_{ij} are unknown functions of θ . If sufficient terms are taken the exact solution to any problem can be obtained. However, for a fracture mechanics analysis, we focus on the terms which make the largest contribution to the stress in the vicinity of the crack tip, $r \rightarrow 0$. It can then be shown that the only value of C , between -1 and 0, which satisfies equilibrium and the boundary conditions is $-1/2$, i.e.

$$\sigma_{ij} = \frac{K}{\sqrt{r}} f_{ij}(\theta) + \dots \quad (1.2)$$

The dots indicate that this is the first (and most important) term in a series for the crack tip stress field.

Each mode of loading produces the $1/\sqrt{r}$ singularity at the crack tip, but the proportionality constant, k , and f_{ij} depends on mode. It is convenient at this point to replace k , by the 'stress intensity factor', K , where $K = k\sqrt{2\pi}$. The stress intensity factor is usually given a subscript to denote the mode of loading; i.e., K_I , K_{II} [2]. Stress intensity factors completely characterize the crack tip conditions in a linearly elastic material. Conventionally, the arbitrary constant K is replaced by the stress intensity factor K_I and K_{II} (Mode I (tension) and Mode II (shear)).

Asymptotic stress fields near the crack tip

in polar coordinates is:

$$\sigma_{rr} = \left[\frac{K_I}{\sqrt{2\pi r}} \cos \frac{\theta}{2} (1 + \sin^2 \frac{\theta}{2}) \right] - \left[\frac{K_{II}}{\sqrt{2\pi r}} (\sin \theta - 2 \tan \frac{\theta}{2}) \right] + T \cos^2 \theta + O(r) \quad (1.3)$$

$$\sigma_{\theta\theta} = \left[\frac{K_I}{\sqrt{2\pi r}} \cos^2 \frac{\theta}{2} \right] + \left[\frac{K_{II}}{\sqrt{2\pi r}} \left(\frac{3 \sin \theta}{2} \right) \right] + T \sin^2 \theta + O(r) \quad (1.4)$$

$$\sigma_{r\theta} = \left[\frac{K_I}{2\sqrt{2\pi r}} \cos \frac{\theta}{2} \sin \theta \right] + \left[\frac{K_{II}}{2\sqrt{2\pi r}} (3 \cos \theta - 1) \right] + T \sin \theta \cos \theta + O(r) \quad (1.5)$$

in cartesian coordinates is:

$$\sigma_{xx} = \left[\frac{K_I}{\sqrt{2\pi r}} \cos \frac{\theta}{2} (1 - \sin \frac{\theta}{2} \sin \frac{3\theta}{2}) \right] - \left[\frac{K_{II}}{\sqrt{2\pi r}} \sin \frac{\theta}{2} (2 + \cos \frac{\theta}{2} \cos \frac{3\theta}{2}) \right] + T + O(r) \quad (1.6)$$

$$\sigma_{yy} = \left[\frac{K_I}{\sqrt{2\pi r}} \cos \frac{\theta}{2} (1 + \sin \frac{\theta}{2} \sin \frac{3\theta}{2}) \right] + \left[\frac{K_{II}}{\sqrt{2\pi r}} \sin \frac{\theta}{2} \cos \frac{\theta}{2} \cos \frac{3\theta}{2} \right] + O(r) \quad (1.7)$$

$$\sigma_{xy} = \left[\frac{K_I}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \sin \frac{\theta}{2} \cos \frac{3\theta}{2} \right] + \left[\frac{K_{II}}{\sqrt{2\pi r}} \cos \frac{\theta}{2} (1 - \sin \frac{\theta}{2} \sin \frac{3\theta}{2}) \right] + O(r) \quad (1.8)$$

where T represents T-stress and O represents higher order term.

Equations 1.3-1.8 are only valid near the crack tip, where the $1/\sqrt{r}$ singularity dominates the stress field. Stresses far from the crack tip governed by the remote boundary conditions. For example, if the cracked structure is subjected to a uniform remote tensile stress (σ), σ_{yy} approaches a constant value σ . We can define a *singularity dominated zone* as the region where the Equations 1.3-1.8 describe the crack tip fields [2].

There are three possible modes of crack extension in linear elastic fracture mechanics: the opening mode, sliding mode, and tearing mode (Figure 1.3).

In the opening mode, Mode I, the two crack surfaces are pulled apart in the y direction, but the deformations are symmetric about the x - z and x - y planes. In the sliding mode, Mode II, the two crack surfaces slide over each other in the x-direction, but the deformations are symmetric about the x - y plane and skew symmetric about the x - z plane. In the tearing mode, Mode III, the crack surfaces slide over each other in the z-direction, but the deformations are skew symmetric about the x - y and x - z planes. The majority of fracture testing is carried out under Mode I conditions as this is generally the critical mode for failure. However, increasingly mixed mode fracture toughness tests (combination of Mode I and Mode II) are being carried out to cover the full range of a materials response to mechanical load [3].

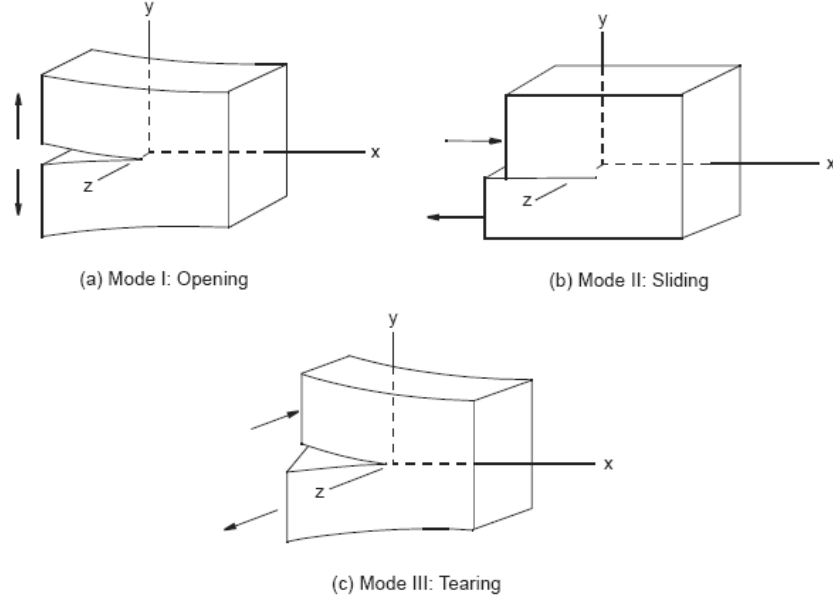


Figure 1.3. Three basic modes of crack extension

1.2. The Fracture Mechanics Approach

There are two alternative approaches to fracture analysis: the energy criterion and the stress intensity approach. These two approaches are equivalent in certain circumstances.

The energy approach states that crack extension occurs when the energy available for crack growth is sufficient to overcome the resistance of the material. The material resistance may include the surface energy, plastic work etc. Griffith was the first to propose the energy criterion for fracture, but Irwin is primarily responsible for developing the present version of the energy release rate, G , approach. At the moment of the fracture, $G = G_c$, critical energy release rate. For a crack length $2a$ and thickness B , in an infinite plate subjected to far-field tensile stress (Figure 1.4), the energy release rate is given by,

$$G = \frac{\pi \sigma^2 a}{E} \quad (1.9)$$

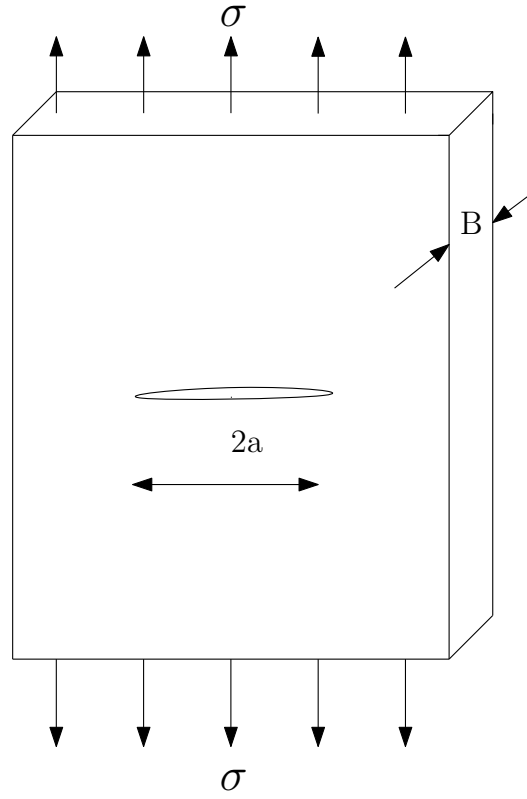


Figure 1.4. Crack in an infinite plate subjected to tensile stress

where E is Young's modulus, σ is the applied stress, and a is the half crack length. At fracture $G = G_c$, for a constant G_c value, failure stress varies with $\frac{1}{\sqrt{a}}$. One of the fundamental assumptions of fracture mechanics is that fracture toughness G_c is independent of the size and the geometry of the cracked body [2].

The stress intensity approach states that failure occurs when $K = K_{Ic}$. It can be seen that for the body shown in Figure 1.4, each stress component (Equations 1.6-1.8) is proportional with K_I . If this constant is known, the entire stress distribution at the crack tip can be computed with the Equations 1.6-1.8. This constant completely characterizes the crack tip conditions in linear elastic material. Assuming that the material fails locally at some combination of stress or strain, fracture must occur at a critical stress intensity, K_{Ic} . Thus K_{Ic} is an alternative measure of fracture toughness.

For the plate illustrated in the Figure 1.4, the stress intensity factor is given by

$$K_I = \sigma\sqrt{\pi a} \quad (1.10)$$

Failure occurs when $K = K_{Ic}$. In this case K_I is the driving force for fracture and K_{Ic} is a measure of material resistance. Comparing Equations 1.9-1.10 results in a relationship between K_I and G :

$$G = \frac{K_I^2}{E} \quad (1.11)$$

This same relationship obviously holds for K_{Ic} and G_c . Thus the energy and stress intensity approaches to fracture mechanics are equivalent for linear elastic materials [2].

For a inclined crack subjected to biaxial or uniaxial stress as shown in Figure 1.5 the stress intensity factors and the T-stress can be found as a function of the angle between the crack and the loading direction. So by changing β , different combinations of mode I and II can be obtained.

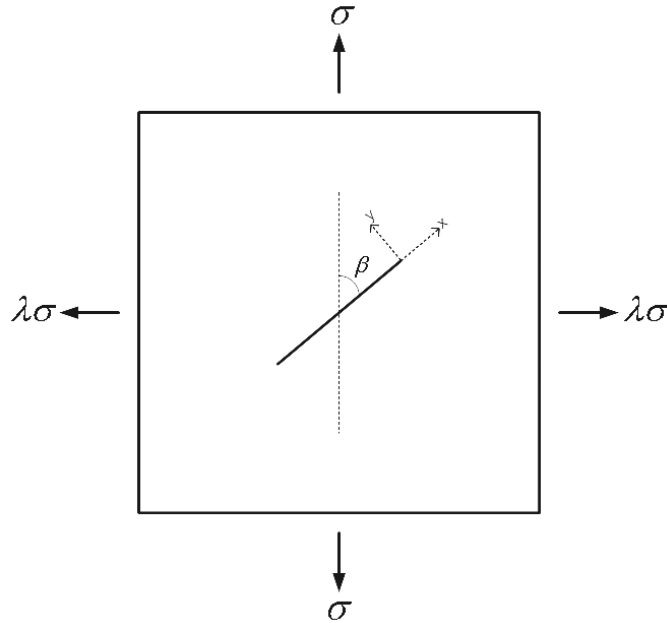
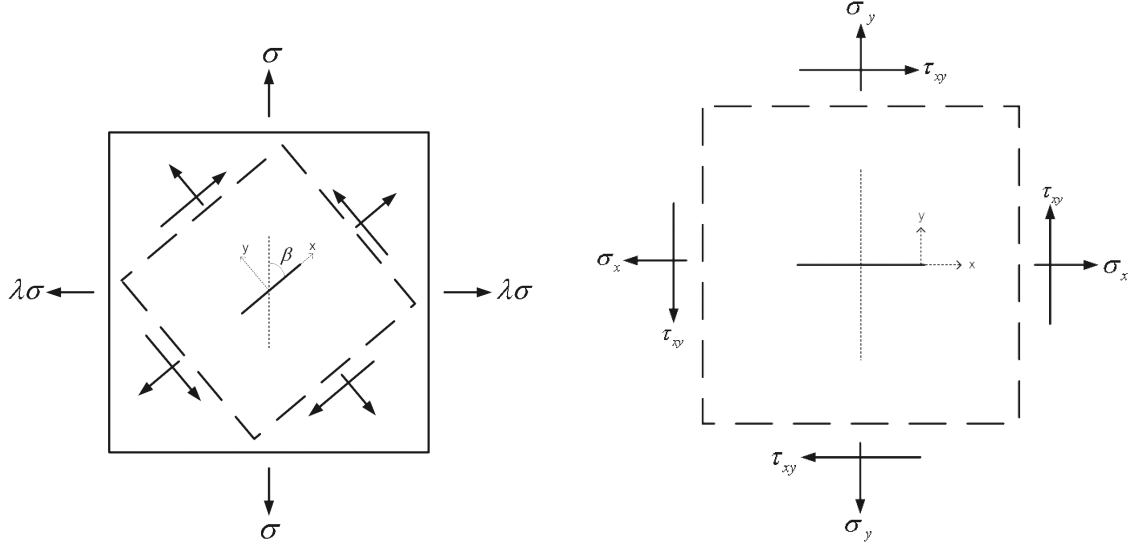


Figure 1.5. An inclined center crack in a biaxially loaded homogeneous plate with crack angle β

If the crack length is small enough, relative to the width of the plate, a closed-form solution can be obtained for K_I , K_{II} and T . This can be achieved [4] by considering another large plate inside the initial one, such that the edges of the second plate are normal or parallel to the crack as shown in Figure 1.6(a).



(a) The position of the second plate in the initial one

(b) Principal stress on the second plate

Figure 1.6. Second Plate

The far field stresses on the boundary of the secondary plate can be found using Mohr circle.

$$\sigma_x = \sigma(\cos^2 \beta + \lambda \sin^2 \beta) \quad (1.12)$$

$$\sigma_y = \sigma(\lambda \cos^2 \beta + \sin^2 \beta) \quad (1.13)$$

$$\tau_{xy} = \sigma(1 - \lambda)(\cos \beta \sin \beta) \quad (1.14)$$

where σ_x , σ_y , τ_{xy} are shown in Figure 1.6(b).

The stress intensity factors can be obtained easily for the second plate as

$$K_I = \sigma_y \sqrt{\pi a} = \sigma \sqrt{\pi a} (\lambda \cos^2 \beta + \sin^2 \beta), \quad (1.15)$$

$$K_{II} = \tau_{xy} \sqrt{\pi a} = \sigma \sqrt{\pi a} (1 - \lambda) (\cos \beta \sin \beta) \quad (1.16)$$

For the second plate, the $T - stress$ is given by using the following formula:

$$T = \sigma_x - \sigma_y = \sigma(1 - \lambda) \cos 2\beta \quad (1.17)$$

To obtain a complete range of mixed-mode, theoretically, biaxially or uniaxially loaded plate with an inclined crack can be used. But using biaxially loaded plate in experimental studies has some difficulties as failure due to buckling and absence of the suitable testing machine. Therefore in general, in mixed mode experiments by using an inclined crack, uniaxially loaded plate is used [5]. For an infinite plate with an inclined crack subjected to uniaxial tensile loading where $\lambda = 0$, Equations 1.15 - 1.17 become

$$K_I = \sigma \sqrt{\pi a} \sin^2 \beta, \quad (1.18)$$

$$K_{II} = \sigma \sqrt{\pi a} \cos \beta \sin \beta, \quad (1.19)$$

$$T = \sigma \cos 2\beta, \quad (1.20)$$

$$B = \frac{\cos 2\beta}{\sin \beta}. \quad (1.21)$$

$$\frac{K_I}{K_{II}} = \tan \beta \quad (1.22)$$

2. CRACK INITIATION CRITERIA

Crack propagation path can not be predicted without having the knowledge of the angle of crack initiation. The study of crack initiation angle is also as much important in dealing with arresting the crack. The problem needs to be studied in mixed mode since the state of loading at the crack tip could be very complex and may result in mixed mode fracture. Several criteria have been developed for predicting crack initiation angles but there is no consensus on the most satisfactory theory to predict crack growth direction (crack initiation angle). The geometry used in analytical problems is illustrated in Figure 2.1.

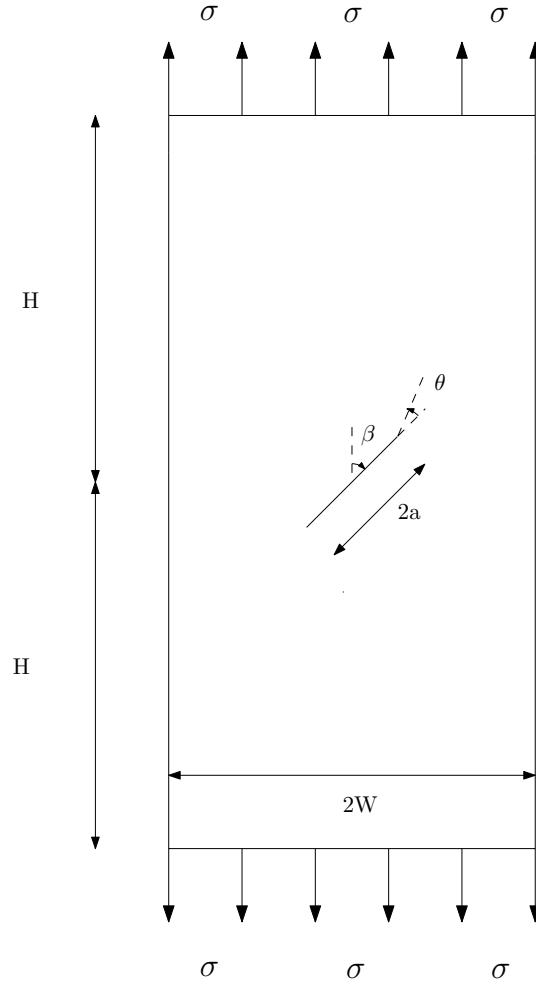


Figure 2.1. Description of the center crack problem

The work of Erdogan and Sih [6] published in 1963 stated an initiation criterion based on Maximum Tangential Stress (*MTS*) criterion. Williams and Ewing [7] made

first detailed experimental study in brittle fracture and modified MTS criterion in 1972. In 1973, Sih [8] analyzed the effect of minimum strain energy density criterion on the crack initiation, and S criterion was developed. In 1974, Hussain [9] proposed G criterion based on Griffith energy principle which is called Maximum Energy Release Rate Criterion. In 1982, Theocaris and Andrianopoulos [10], [11] developed a strain energy density criterion, called T criterion. They pointed out that the strain energy density may be divided into two components, dilatational strain energy density and distortional strain energy density. It is postulated that crack will occur in the direction with the maximum dilatational strain energy density value. In 1987, a criterion to study crack initiation direction was developed by Papadopoulos [12] called Det criterion. It postulates that the crack propagates in the direction of maximum value of second stress invariants that is evaluated along a circle around the crack tip. It can be clearly seen that the maximum value of second stress invariant and dilatational components have the same meaning. The difference is the former criterion is evaluated along a circle and the later one is calculated along a Misses elastic-plastic boundary. Also, in 1987 Yehia [13] stated that they modified T criterion and proposed Nt criterion. Yehia [14] presented another initiation criterion, Y criterion, for calculating crack initiation by using the distortional strain energy density in 1991. It is stated that crack will initiate in the direction of minimum distortional strain energy density defined along a circle. Later a criterion proposed by Ukadgaonker and Asaware [15] states that the crack will initiate in the direction defined by a maximum value of I_P , based on the first and second stress invariant, along the elastic-plastic boundary. It is assumed that the minimum radius of core region along the direction of crack initiation must be equal or greater than the critical radius. The criterion is also an algebraic variation of the T -criterion. In 1995, Kong et. al. [16] proposed maximum trial stress, M , criterion which assumes a constant core region radius. In 2000 Wasiluk [17] proposed W -criterion and in 2004 Khan [18] proposed R -criterion which both of them are presented a direct correlation between size and shape of the core region and crack initiation angles.

2.1. MTS Criterion

The use of maximum tangential stress theory to calculate crack initiation angles was originally proposed by Erdogan and Sih [6] to predict the direction of crack initiation in an isotropic brittle material under plane stress or plane strain conditions. This criterion states that the crack will start to grow from the tip in the direction along which the tangential stress $\sigma_{\theta\theta}$ (Figure 2.2) is maximum at a constant radius under mixed mode loading. The region within this constant radius is known as the core region, and it is assumed that LEFM does not hold in this core region.

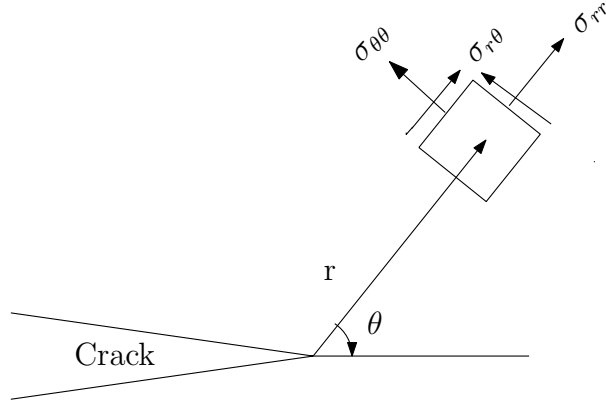


Figure 2.2. 2-D element in polar coordinate system

In K dominant region $\sigma_{\theta\theta}$ is defined as:

$$\sigma_{\theta\theta} = \frac{1}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \left(K_I \cos^2 \frac{\theta}{2} - \frac{3}{2} K_{II} \sin \theta \right) \quad (2.1)$$

where K_I and K_{II} are Mode I and Mode II stress intensity factors.

To find maximum $\sigma_{\theta\theta}$:

$$\frac{\partial \sigma_{\theta\theta}}{\partial \theta} = 0 \quad \text{and} \quad \frac{\partial^2 \sigma_{\theta\theta}}{\partial \theta^2} < 0 \quad (2.2)$$

Taking the derivative of $\sigma_{\theta\theta}$ with respect to θ , we have

$$\frac{\partial \sigma_{\theta\theta}}{\partial \theta} = \frac{1}{\sqrt{2\pi r}} \left(-\frac{3}{4} \cos \frac{\theta}{2} \right) (K_I \sin \theta + K_{II}(3 \cos \theta - 1)) \quad (2.3)$$

Then taking the second derivative we have,

$$\frac{\partial^2 \sigma_{\theta\theta}}{\partial \theta^2} = \frac{1}{\sqrt{2\pi r}} \left[\left(-\frac{3}{8} (3 \cos \theta - 1) \right) \left(K_I \cos \frac{\theta}{2} - K_{II} \sin \frac{\theta}{2} \right) + \frac{9}{4} K_{II} (1 + \cos \theta) \sin \frac{\theta}{2} \right] \quad (2.4)$$

Thus, θ_c corresponding to maximum $\sigma_{\theta\theta}$ can be obtained from the following equation:

$$\left(\cos \frac{\theta_c}{2} \right) (K_I \sin \theta_c + K_{II}(3 \cos \theta_c - 1)) = 0 \quad (2.5)$$

$$\Rightarrow \left(-\frac{3}{8} (3 \cos \theta_c - 1) \right) \left(K_I \cos \frac{\theta_c}{2} - K_{II} \sin \frac{\theta_c}{2} \right) < 0 \quad (2.6)$$

where θ_c is the crack initiation angle.

Equation 2.5 has two solutions:

$$\cos \frac{\theta_c}{2} = 0 \implies \theta_c = \pm \pi \quad (2.7)$$

which is the trivial solution, and

$$(K_I \sin \theta_c + K_{II}(3 \cos \theta_c - 1)) = 0 \quad (2.8)$$

For pure Mode I, $K_{II} = 0$, and Equation 2.8 yields

$$K_I \sin \theta_c = 0 \implies \theta_c = 0^\circ. \quad (2.9)$$

For pure Mode II, $K_I = 0$, then

$$K_{II}(3 \cos \theta - 1) = 0 \implies \theta_c = \pm 70.53^\circ \quad (2.10)$$

For mixed mode condition Equation 2.8 can be rewritten as:

$$2 \tan^2 \frac{\theta_c}{2} - \mu \tan \frac{\theta_c}{2} - 1 = 0 \quad \text{where} \quad \mu = \frac{K_I}{K_{II}} \quad (2.11)$$

Figure 2.3 illustrates the solution of Equation 2.11.

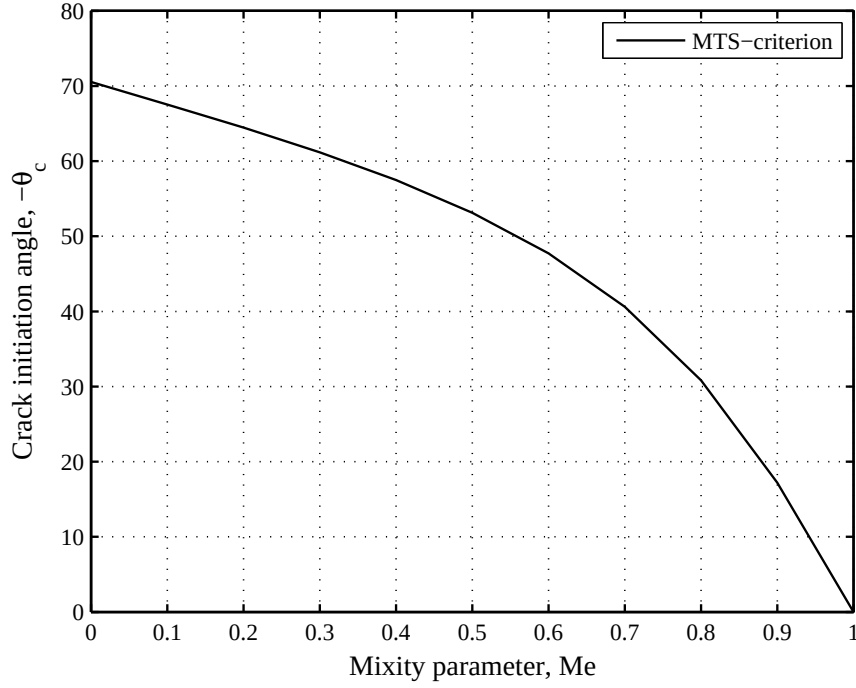


Figure 2.3. Crack Initiation Angle under mixed mode loading in MTS criterion

where θ_c is the crack initiation angle and M_e is the mixity parameter,

$$M_e = \frac{2}{\pi} \tan^{-1} \left(\frac{K_I}{K_{II}} \right).$$

M_e is defined as the relative amount of Mode I to Mode II and for mixed mode, parameter takes values between 0 and 1.

To test the validity of theoretical results, a series of experiments are performed by Sih and Erdogan [6]. A large plate with a central straight crack was subjected to equal and opposite concentrated shear loads applied to crack surface at a distance b from axis of symmetry (Fig:2.4). The results of the experiments are shown in the Table 2.1.

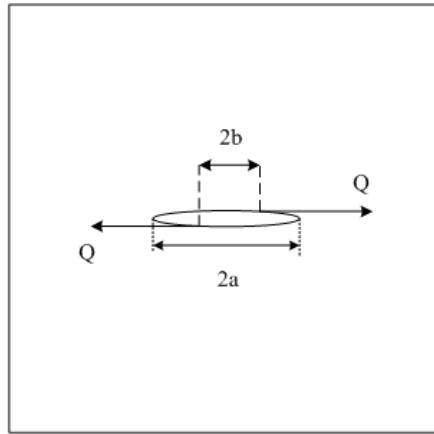


Figure 2.4. cracked plate under shear load

Table 2.1. Crack initiation angles from experiments of Sih and Erdogan[6]

$2a(\text{in.})$	$2b(\text{in.})$	$h(\text{in.})$	$Q(\text{lb})$	θ_c
1.90	0.50	0.119	167	
1.95	0.50	0.116	157	71.0
2.18	0.40	0.119	167	70.5
2.80	0.50	0.122	182	70.0
3.20	0.41	0.117	185	68.0
3.50	0.40	0.112	193	69.0
3.60	0.50	0.121	217	69.5
4.00	0.54	0.118	191	70.0
4.05	0.45	0.116	216	71.5

where Q is the shear load per unit thickness, h is the plate thickness, $2a$ is the crack length and θ_c is the crack initiation angle.

2.2. G-Criterion

Griffith derived a criterion for crack growth using an energy approach. It is based on the concept that energy must be conserved in all processes. Griffith's energy theory which was originally formulated for Mode I cracking suggested that an existing crack will only grow when the decrease in potential energy associated with crack growth is at least equal to the energy required to create new surfaces.

$$-\frac{d\Pi}{dA} = \frac{dW_s}{dA} \quad (2.12)$$

where Π is potential energy supplied by the internal strain energy and external forces, W_s is the work required to create new surfaces and dA is the crack area.

Later Irwin, proposed an energy approach for fracture that is essentially equivalent to the Griffith model, except that Irwin's approach is in a form that is more convenient for solving engineering problems. Irwin defined energy release rate G , which is a measure of energy available for an increment of crack extension.

$$G = -\frac{d\Pi}{dA} \quad (2.13)$$

G is rate of change in potential energy with crack area. G depends on the geometry of the structure and the current loading. Griffith states that crack extension starts when G reaches a critical value. The energy balance can be expressed as:

$$G = G_c$$

where G_c is called fracture toughness of the material which is a material property and

it is determined from experiments.

$$G_c = -\frac{dW_s}{dA} = 2W_f \quad (2.14)$$

where W_f is fracture energy which could include plastic, viscoelastic or viscoplastic effects, depending on the material. It was found that while Griffith's theory worked well for very brittle materials such as glass it could not be used for more ductile materials such as metals or polymers [2].

If Griffith's energy theory is accepted as the valid criteria explaining crack growth, then the crack will grow in the direction along which the elastic energy release per unit crack extension will be maximum and the crack will start to grow when this energy reaches a critical value (or $G = G(\delta, \theta)$). Evaluation of $G(\delta, \theta)$ poses insuperable mathematical difficulties. Hussain et al [9], Palaniswamy [19] and Wu [20] studied crack initiation angle prediction by the Griffith's energy theory. This problem was solved in 1974, [9]. Fundamentally, Hussain et al. have solved for the stress intensity factor of a major crack with an infinitesimal kink at an angle θ , $K_I(\theta)$ and $K_{II}(\theta)$ in terms of the stress intensity factors of the original crack K_I and K_{II} (Figure 2.5) [3].

$$\begin{Bmatrix} K_I(\theta) \\ K_{II}(\theta) \end{Bmatrix} = \left(\frac{4}{3 + \cos^2 \theta} \right) \left(\frac{1 - \frac{\theta}{\pi}}{1 + \frac{\theta}{\pi}} \right)^{\frac{\theta}{2\pi}} \begin{Bmatrix} K_I \cos \theta + \frac{3}{2} K_{II} \sin \theta \\ K_{II} \cos \theta - \frac{1}{2} K_I \sin \theta \end{Bmatrix} \quad (2.15)$$

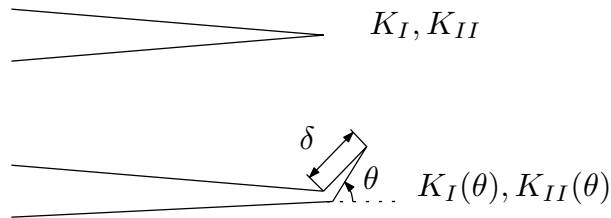


Figure 2.5. Crack with an infinitesimal kink at angle θ

Those expressions of $K_I(\theta)$ and $K_{II}(\theta)$ were then substituted into Irwin's generalized expression for the energy release rate

$$G(\theta) = \frac{1}{E} (K_I^2(\theta) + K_{II}^2(\theta)) \quad (2.16)$$

yielding,

$$G(\theta) = \frac{4[(1 + \cos^2 \theta)K_I^2 - 8K_I K_{II} \sin \theta \cos \theta + (9 - 5 \cos^2 \theta)K_{II}^2]}{E(3 + \cos^2 \theta)^2} \left[\frac{1 + \frac{\theta}{\pi}}{1 - \frac{\theta}{\pi}} \right]^{-\theta/\pi} \quad (2.17)$$

The angle of crack initiation θ_c is found by maximizing $G(\theta)$;

$$\frac{\partial G}{\partial \theta} = 0 \quad \text{and} \quad \frac{\partial^2 G}{\partial \theta^2} < 0 \quad \implies \quad \theta = \theta_c \quad (2.18)$$

Figure 2.6 shows the $G(\theta)$ distribution around the crack tip. It can be seen that for $\mu = 1$ ($\beta = 45^\circ$), the $G(\theta)$ has a maximum value between -60° and -40° .

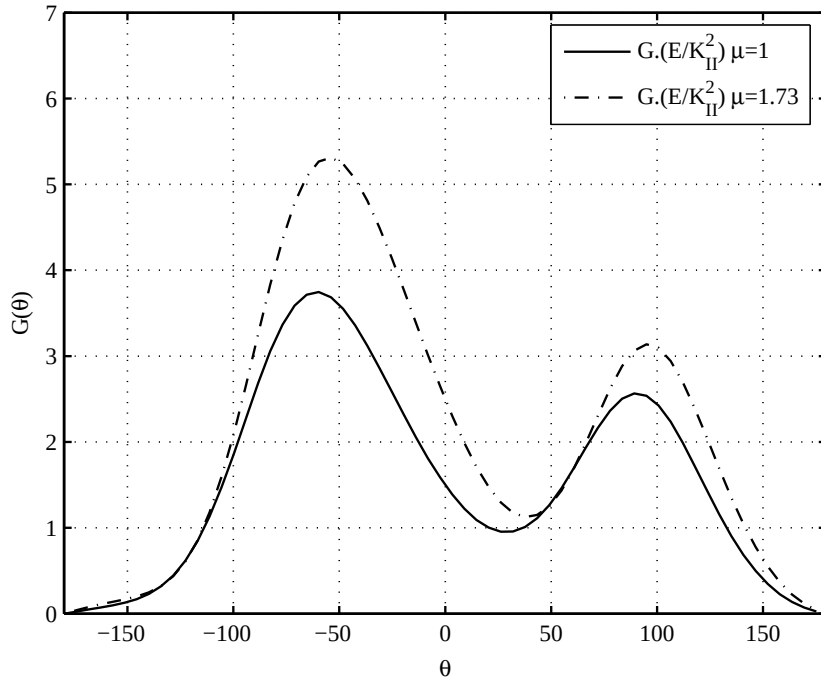


Figure 2.6. Distribution of energy release rate around the crack tip

Analytical crack initiation angle results [9] and experimental results done by Palaniswamy [19] are presented in the Figure 2.7.

Taking the first and second derivative of Equation 2.17 is very difficult. To represent K_I - K_{II} interaction curve derived from G criterion, Palaniswamy [19] used

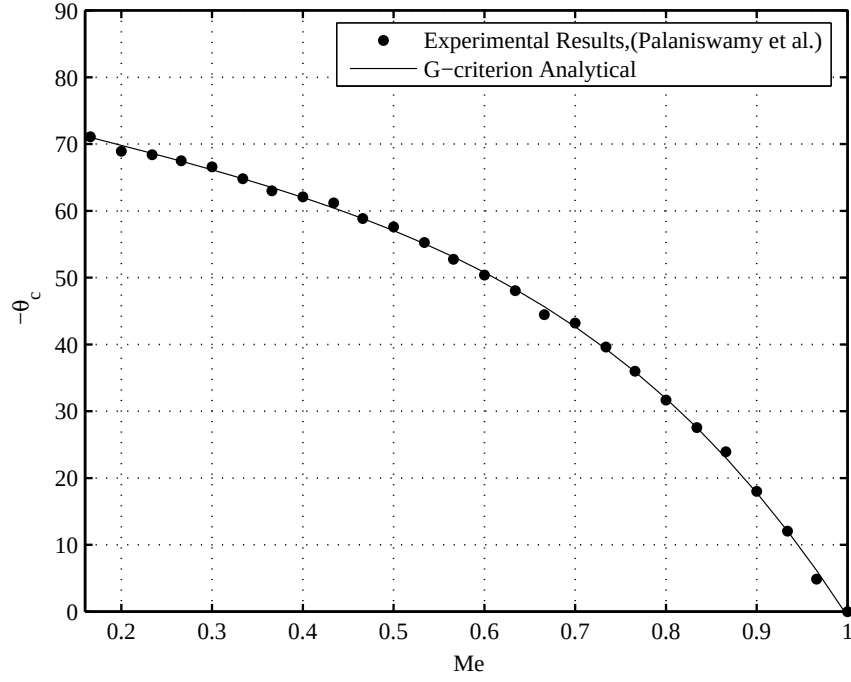


Figure 2.7. Crack Initiation Angle under mixed mode loading in G criterion

the expression:

$$2\left(1 - \frac{K_I}{K_{IC}}\right) - 3\left(\frac{K_{IIC}}{K_{IC}}\right) = 0 \quad (2.19)$$

Wu [20] states that their numerical results are in a good agreement with Palaniswamy's Equation 2.19.

2.3. S-Criterion

The minimum strain energy density theory, formulated by Sih [8], postulates that a crack initiation will start in a radial direction along which the strain energy density is a minimum (i.e. crack propagates along path of minimum resistance).

S-criterion is the only criterion that shows dependence of crack initiation angle on the material properties such as ν (Poisson's ratio) and the state of stress. Strain energy stored in an element can be expressed as:

$$U_s = \int \int \int \left(\int \sigma_{ij} d\varepsilon_{ij} \right) dV \quad (2.20)$$

For a linearly elastic homogeneous material, the strain energy density stored in the element under loading:

$$U = \frac{1}{2} \sigma_{ij} \varepsilon_{ij} \quad (2.21)$$

By arranging Equation 2.21, we get:

$$U = \frac{1 + \nu}{2E} \left[\frac{\kappa + 1}{4} (\sigma_{xx} + \sigma_{yy})^2 - 2(\sigma_{xx}\sigma_{yy} - \tau_{xy}^2) \right] \quad (2.22)$$

where $\kappa = 3 - 4\nu$ for plane strain
and $\kappa = \frac{3 - \nu}{1 + \nu}$ for plane stress

By inserting Equations 1.6 - 1.8 for 2D elasticity problem into Equation 2.22, we obtain

the following quadratic form denoted by S :

$$S = (a_{11}K_I^2 + 2a_{12}K_I K_{II} + a_{22}K_{II}^2) \quad (2.23)$$

where

$$\begin{aligned} a_{11} &= \frac{1}{16\pi G_s} (1 + \cos \theta)(\kappa - \cos \theta) \\ a_{12} &= \frac{1}{16\pi G_s} (\sin \theta)(2 \cos \theta - (\kappa - 1)) \\ a_{22} &= \frac{1}{16\pi G_s} ((\kappa + 1)(1 - \cos \theta) + (1 + \cos \theta)(3 \cos \theta - 1)) \end{aligned}$$

Figure 2.8 shows the $S(\theta)$ distribution around the crack tip for plane stress, $\nu = 0.3$. By taking the first derivative of Equation 2.23 and equating it to zero we get:

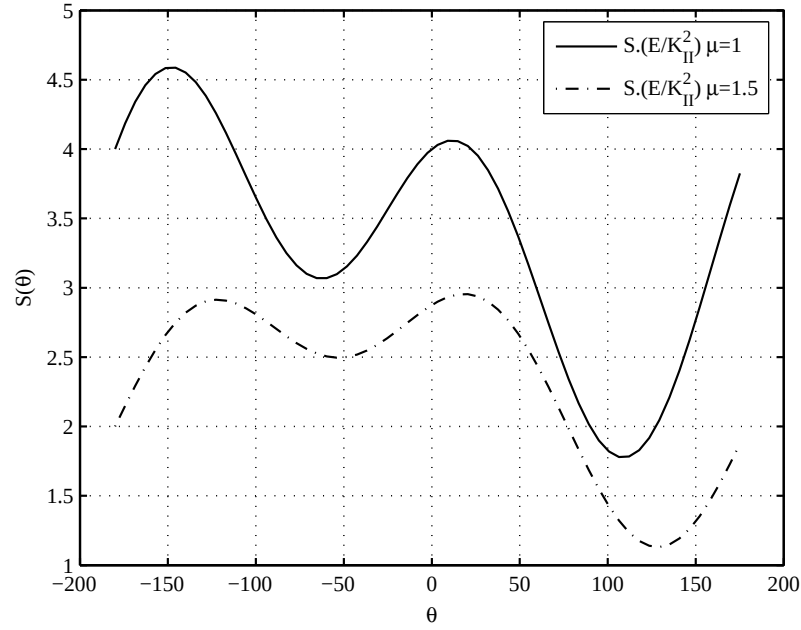


Figure 2.8. Distribution of strain energy density around the crack tip

$$\begin{aligned} \frac{K_I^2}{K_{II}^2} [\sin \theta_c (2 \cos \theta_c + 1 - \kappa)] + 2 \frac{K_I}{K_{II}} [2 \cos 2\theta_c + (1 - \kappa) \cos \theta_c] \\ + \sin \theta_c [(\kappa - 1) - 6 \cos \theta_c] = 0 \end{aligned} \quad (2.24)$$

And taking the second derivative:

$$\begin{aligned} \frac{K_I^2}{K_{II}^2} [2 \cos 2\theta_c - (\kappa - 1) \cos \theta_c] + 2 \frac{K_I}{K_{II}} [-4 \sin 2\theta_c + (\kappa - 1) \sin \theta_c] \\ + \cos \theta_c [(\kappa - 1) - 6 \cos 2\theta_c] > 0 \end{aligned} \quad (2.25)$$

For pure Mode I loading ($K_{II}=0$):

$$K_{II}^2 \sin \theta_c (2 \cos \theta_c + 1 - \kappa) = 0 \quad \text{and} \quad (2.26)$$

$$K_{II}^2 (2 \cos 2\theta_c - (\kappa - 1) \cos \theta_c) > 0. \quad (2.27)$$

Equation 2.26 has two solutions

$$\theta_c = \arcsin 0 \quad \text{and} \quad \theta_c = \arccos \frac{\kappa - 1}{2} \quad (2.28)$$

Equation 2.27 is satisfied by $\theta_c=0^\circ$. Therefore for Mode I loading θ_c does not depend on material property.

For pure Mode II loading:

$$\begin{aligned} K_{II} [-6 \cos \theta_c + (\kappa - 1)] \sin \theta_c &= 0 \\ K_{II} [-6 \cos 2\theta_c + (\kappa - 1)] \cos \theta_c &> 0 \end{aligned} \quad (2.29)$$

Equation 2.29 has two solutions

$$\theta_c = \arcsin 0 \quad \text{and} \quad \theta_c = \arccos \frac{\kappa - 1}{6} \quad (2.30)$$

Equation 2.25 is satisfied by $\theta_c = \arccos \frac{\kappa - 1}{6}$.

for $\nu = 0.3$ $\kappa = 2.08$ and $\theta_c = 79.66^\circ$.

Figure 2.9 shows solution of the Equation 2.24 for $\nu=0.2, 0.3$ and 0.4 . Figure

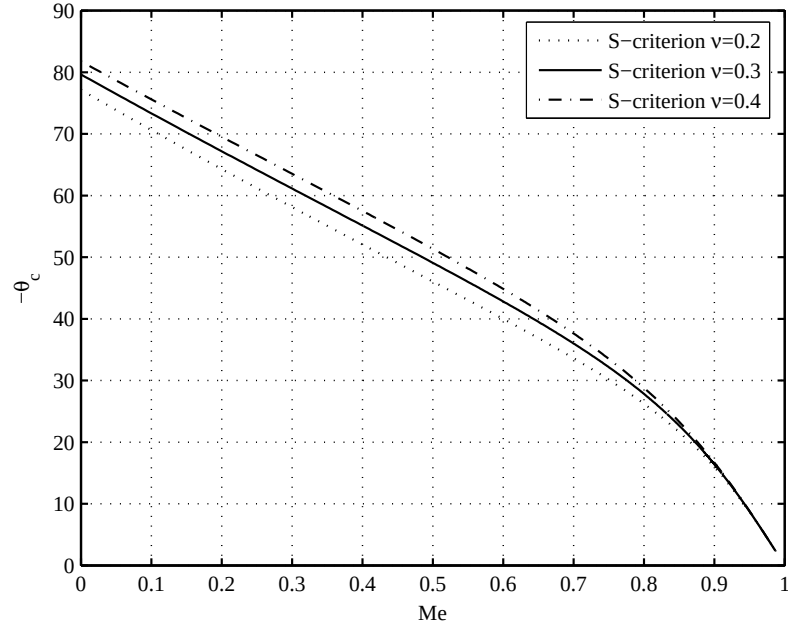


Figure 2.9. Crack initiation angle under mixed mode loading in S criterion

2.10 represents the comparison of crack initiation angles for MTS, G, S criterion.

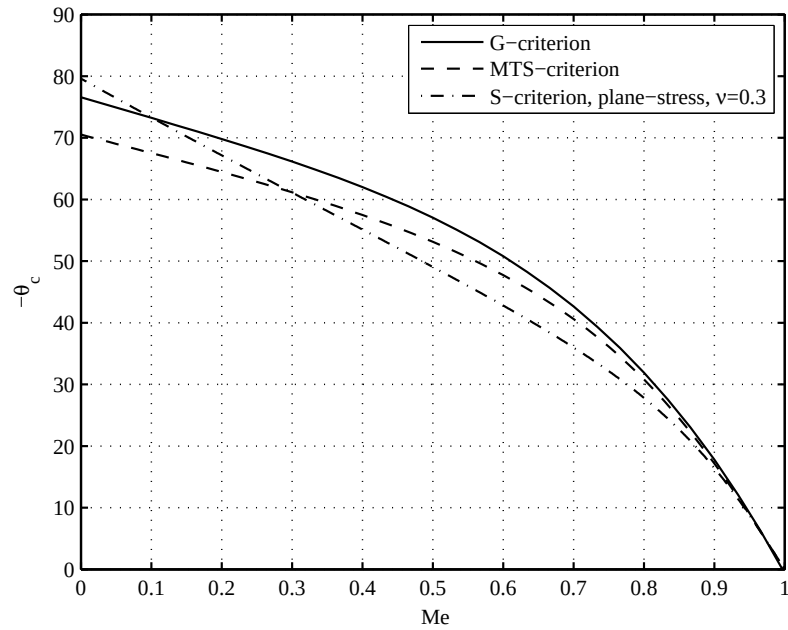


Figure 2.10. Comparison of crack initiation criteria for brittle fracture under mixed mode loading

2.4. Generalized MTS Criterion

Williams and Ewing [7] performed experiments using PMMA specimens and measured crack initiation angle (θ_c) against crack angle (β). Williams and Ewing showed that there was a discrepancy between mixed-mode fracture predicted by the MTS criterion and their experiments, with largest errors occurring at small loading angles. They observed that for $\beta < 40^\circ$ the points are definitely above the theoretical line (work of Erdogan and Sih [6]) and tend to the value of -90° , not to -70.5° for $\beta = 0$. T-stress which is a constant stress parallel to the crack is the reason for difference between experimental results and MTS criterion results.

The experimental work by Williams and Ewing demonstrated that non-singular terms in the expansion of stresses ahead of the crack tip have significant influence on brittle fracture of linear elastic materials subjected to mixed mode loading [4]. And in the further analysis of stress field around the crack tip, Williams and Ewing modified the MTS criterion which is proposed by Sih and Erdogan [6] and added T-stress term to stress equations as shown below.

$$\sigma_{\theta\theta} = \frac{1}{\sqrt{2\pi r}} \cos \frac{\theta}{2} \left(K_I \cos^2 \frac{\theta}{2} - \frac{3}{2} K_{II} \sin \theta \right) + T \sin^2 \theta \quad (2.31)$$

and maximizing $\sigma_{\theta\theta}$ as in MTS criterion with this extra term,

$$\frac{\partial \sigma_{\theta\theta}}{\partial \theta} = 0 \quad \text{and} \quad \frac{\partial^2 \sigma_{\theta\theta}}{\partial \theta^2} < 0 \quad \implies \quad \theta = \theta_c \quad (2.32)$$

By taking the first derivative of Equation 2.31 we get:

$$\frac{\partial \sigma_{\theta\theta}}{\partial \theta} = \frac{1}{\sqrt{2\pi r_c}} \left(-\frac{3}{4} \cos \frac{\theta_c}{2} \right) (K_I \sin \theta_c + K_{II} (3 \cos \theta_c - 1)) + 2T \sin \theta_c \cos \theta_c = 0 \quad (2.33)$$

And taking the second derivative we have,

$$\begin{aligned} \frac{\partial^2 \sigma_{\theta\theta}}{\partial \theta^2} = \frac{1}{\sqrt{2\pi r_c}} \left[\left(-\frac{3}{8}(3 \cos \theta_c - 1) \right) \left(K_I \cos \frac{\theta_c}{2} - K_{II} \sin \frac{\theta_c}{2} \right) \right. \\ \left. + \frac{9}{4} K_{II} (1 + \cos \theta_c) \sin \frac{\theta_c}{2} \right] + 2T \cos 2\theta_c < 0 \end{aligned} \quad (2.34)$$

For pure Mode I loading ($K_{II} = 0$)

$$\left(-\frac{3}{4} \cos \frac{\theta_c}{2} \sin \theta_c \right) \frac{1}{\sqrt{2\pi r_c}} K_I + 2T \sin \theta_c \cos \theta_c = 0 \quad (2.35)$$

$$\sin \theta_c \left[\left(-\frac{3}{4} \cos \frac{\theta_c}{2} \right) \frac{1}{\sqrt{2\pi r_c}} K_I + 2T \cos \theta_c \right] = 0 \quad (2.36)$$

Solving for θ_c one gets,

$$\begin{aligned} \sin \theta_c = 0 &\Rightarrow \theta_c = 0 \quad \text{or} \\ \cos \frac{\theta_c}{2} K_I - \frac{8}{3} \sqrt{2\pi r_c} T \cos \theta_c &= 0 \end{aligned}$$

$$\Rightarrow \theta_c = 2 \arccos \left[\left(\frac{3K_I}{32T\sqrt{2\pi r_c}} \right) \pm \sqrt{\left(\frac{3K_I}{32T\sqrt{2\pi r_c}} \right)^2 + \frac{1}{2}} \right] \quad (2.37)$$

For pure Mode I, the T-term vanishes in the expansion for the tangential stress in front of the crack tip along the crack line. Consequently, if it is assumed that brittle fracture occurs along the crack line, then the T-stress has no influence. In mixed mode conditions, the crack growth is not in the same plane as the initial crack. For this case, T-stress contributes to tangential stress. This is along the direction of initiation of crack growth and hence influences the apparent mixed mode fracture toughness.

To normalize the effect of the T-stress, relative to the stress intensity factor in mode I, biaxiality B is used [4],

$$B = (T\sqrt{\pi a})/K_I. \quad (2.38)$$

Equation 2.38 can be extended to mixed mode loading by:

$$B = (T\sqrt{\pi a})/K_{eff} \quad (2.39)$$

where effective stress intensity factor K_{eff} is defined as:

$$K_{eff} = \sqrt{(K_I^2 + K_{II}^2)} \quad (2.40)$$

Equation 2.33 can be written as

$$[K_I \sin \theta_c + K_{II}(3 \cos \theta_c - 1)] - 16(B\alpha K_{eff})/3 \sin \theta_c \cos \theta_c = 0 \quad (2.41)$$

where

$$\gamma = 16(B\alpha K_{eff})/3 \quad (2.42)$$

and $\alpha = \sqrt{\frac{2r_c}{a}}$. Equation 2.31 can be written as

$$\sqrt{2\pi r}(\sigma_{\theta\theta})_c = \cos \frac{\theta_c}{2} \left(K_I \cos^2 \frac{\theta_c}{2} - \frac{3}{2} K_{II} \sin \theta_c \right) + \sqrt{2\pi r} T \sin^2 \theta_c \quad (2.43)$$

where $(\sigma_{\theta\theta})_c$ is the critical value of the tangential stress at a critical radius r_c . For pure Mode I when K_{II} , T and $\theta_c=0$, K_I at fracture can be replaced by the mode I fracture toughness K_{Ic} and Equation 2.43 reduces to:

$$\sqrt{2\pi r}(\sigma_{\theta\theta})_c = K_{Ic} \quad (2.44)$$

K_{Ic} is the fracture toughness of the material.

Then Equation 2.43 becomes:

$$\cos \frac{\theta_c}{2} \left(K_I \cos^2 \frac{\theta_c}{2} - \frac{3}{2} K_{II} \sin \theta_c \right) = K_{Ic} - T \sqrt{2\pi r} \sin^2 \theta_c \quad (2.45)$$

or in terms of B

$$\cos \frac{\theta_c}{2} \left(K_I \cos^2 \frac{\theta_c}{2} - \frac{3}{2} K_{II} \sin \theta_c \right) = -B\alpha K_{eff} \sin^2 \theta_c + K_{Ic} \quad (2.46)$$

Equations 2.43, 2.45, 2.46 describe a generalized MTS criterion where mixed mode crack initiation angle is predicted for any geometry for which K_I , K_{II} , T are known. The crack initiation results for the range $-0.6 < B\alpha < 0.375$ are presented in the Figure 2.11. As shown in the Figure 2.11, the crack initiation angle increases for positive T-stress values and decreases for negative T-stress values and effect of T-stress is large in Mode II dominant region.

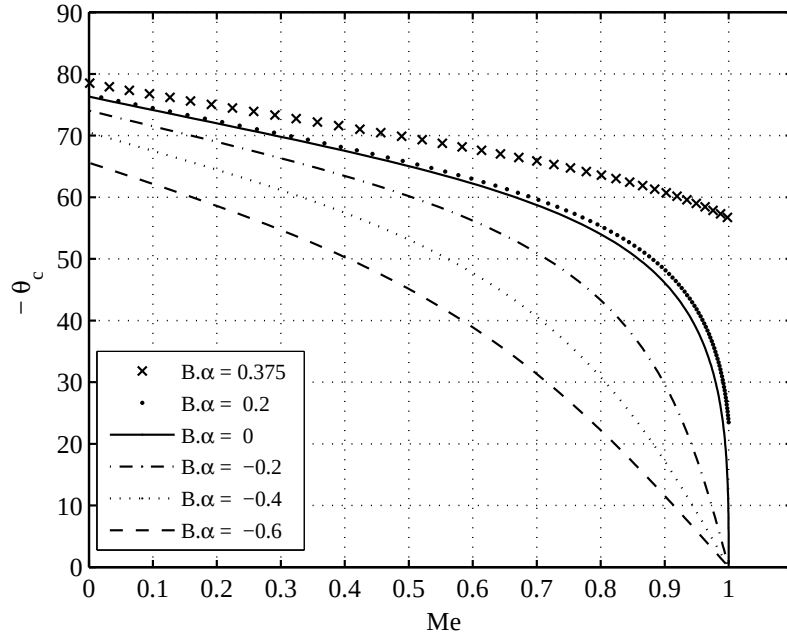


Figure 2.11. The effect of T stress on the direction of the crack initiation

The fracture loci is illustrated in Figure 2.12. It is observed that fracture toughness increases by negative values of T-stress.

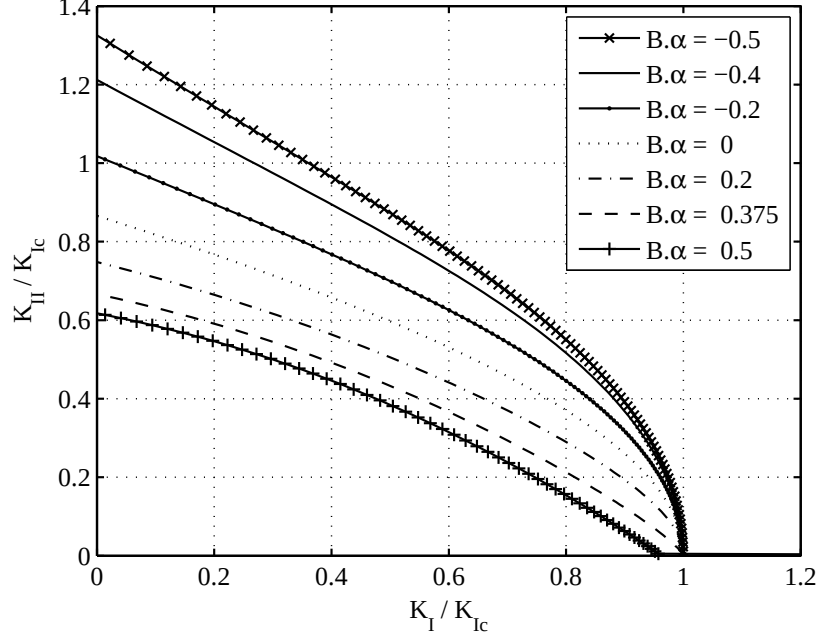


Figure 2.12. Mixed mode fracture loci on the generalized MTS criterion

2.5. M-Criterion

Another stress based criterion is the maximum stress triaxiality M criterion. Stress triaxiality is defined as the ratio of the hydrostatic stress (σ_H) to the equivalent stress (σ_{eq}). Where

$$M = \frac{\sigma_H}{\sigma_{eq}} \quad (2.47)$$

Hydrostatic stress is:

$$\sigma_H = \frac{\sigma_x + \sigma_y + \sigma_z}{3} \quad (2.48)$$

and can be shown as:

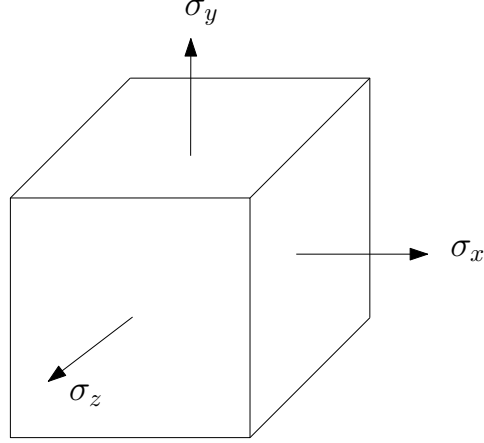


Figure 2.13. Hydrostatic Stress Components

for plane strain, $\epsilon_z = 0$ and

$$\sigma_z = \nu(\sigma_x + \sigma_y)$$

$$\sigma_H = \frac{2(1+\nu)}{3\sqrt{2\pi r}} \left[K_I \cos \frac{\theta}{2} - K_{II} \sin \frac{\theta}{2} \right] \quad (2.49)$$

$$\sigma_{eq} = \sqrt{\left[\frac{(\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6\tau_{xy}^2}{2} \right]} \quad (2.50)$$

and for plane stress, $\sigma_z = 0$:

$$\sigma_H = \frac{1}{\sqrt{2\pi r}} \left[K_I \cos \frac{\theta}{2} - K_{II} \sin \frac{\theta}{2} \right] \quad (2.51)$$

$$\sigma_{eq} = \sqrt{\left[\sigma_x^2 + \sigma_y^2 - \sigma_x \sigma_y + 3\tau_{xy}^2 \right]} \quad (2.52)$$

and substituting stress field Equations 1.6- 1.8 into Equation 2.52 we get:

$$\begin{aligned} \sigma_{eq} = \frac{1}{\sqrt{2\pi r}} & \left[K_I^2 \left(\cos^2 \frac{\theta}{2} + 3 \cos^2 \frac{\theta}{2} \sin^2 \frac{\theta}{2} \sin^2 \frac{3\theta}{2} + 3 \cos^2 \frac{\theta}{2} \sin^2 \frac{\theta}{2} \cos^2 \frac{3\theta}{2} \right) \right. \\ & + K_{II}^2 \left(-6 \sin \frac{\theta}{2} \cos^2 \frac{\theta}{2} \sin \frac{3\theta}{2} + 4 \sin^2 \frac{\theta}{2} + 3 \cos^2 \frac{\theta}{2} + 6 \sin^2 \frac{\theta}{2} \cos \frac{\theta}{2} \cos \frac{3\theta}{2} \right) \\ & \left. + K_I K_{II} \left(6 \cos^2 \frac{\theta}{2} \sin \frac{\theta}{2} \cos \frac{3\theta}{2} + 6 \cos \frac{\theta}{2} \sin^2 \frac{\theta}{2} \sin \frac{3\theta}{2} - 2 \cos \frac{\theta}{2} \sin \frac{\theta}{2} \right) \right]^{\frac{1}{2}} \quad (2.53) \end{aligned}$$

Then by using simple trigonometric equations, Equation 2.53 simplifies to:

$$\begin{aligned} \sigma_{eq} = \frac{1}{\sqrt{2\pi r}} & \left[K_I^2 \cos^2 \frac{\theta}{2} \left(1 + 3 \sin^2 \frac{\theta}{2} \right) + K_{II}^2 \left(3 + \sin^2 \frac{\theta}{2} (1 - 9 \sin^2 \frac{\theta}{2}) \right) \right. \\ & \left. + K_I K_{II} \sin \theta (3 \cos \theta - 1) \right]^{\frac{1}{2}} \quad (2.54) \end{aligned}$$

M-criterion is presented by Kong [16] and states that the direction of initial crack extension coincides with the direction of maximum stress triaxiality along a constant radius around the crack tip.

$$\frac{\partial M}{\partial \theta} = 0 \quad \text{and} \quad \frac{\partial^2 M}{\partial \theta^2} < 0 \quad \implies \quad \theta = \theta_c \quad (2.55)$$

we get:

$$N^4 - 3\mu N^3 - (1 - 2\mu^2)N^2 + \frac{1}{2}(1 - \mu^2)\mu N - \frac{1}{2}(1 + \mu^2) = 0 \quad (2.56)$$

$$\begin{aligned} & [2(\mu^2 + 5)] \sin \frac{\theta}{2} + [27(\mu^2 + 1)] \sin \frac{3\theta}{2} + [5(5\mu^2 - 3)] \sin \frac{5\theta}{2} \\ & - [2(\mu^2 + 5)] \cos \frac{\theta}{2} [9(\mu^2 + 1)\mu] \cos \frac{3\theta}{2} - [5(\mu^2 - 7)\mu] \cos \frac{5\theta}{2} < 0 \quad (2.57) \end{aligned}$$

where $N = \tan(\theta/2)$ and $\mu = K_I/K_{II}$

For pure Mode I loading, Equation 2.56 becomes:

$$\begin{aligned} -\frac{1}{2}N &= 0 \\ \text{then } \theta_c &= 0^\circ \end{aligned} \quad (2.58)$$

For pure Mode II loading, Equation 2.56 becomes:

$$N^4 - N^2 - \frac{1}{2} = 0 \quad (2.59)$$

N has 4 roots, two of which are imaginary and the other two are 1.1688 and -1.1688. From here, we get $\theta_c = 99^\circ$ which satisfies Equation 2.57.

Kong [16] proposed that triaxial stress has a large influence on fracture behavior of materials and stated that increased triaxial tension stress raises the flow curve and fracture occurs at a relatively small strains, in contrast, decreased triaxial tension stress results in a lower flow curve and fracture takes place at a relatively large strain.

Table 2.2. Crack initiation angles and maximum triaxial stress

β	5°	30°	45°	60°	90°
θ	$-94, 1^\circ$	$-72, 9^\circ$	$-59, 6^\circ$	$-45, 4^\circ$	0°
$\nu = 0, 5$	0, 89	1, 26	1, 8	3, 15	∞
$\nu = 0, 35$	0, 78	1, 1	1, 43	2, 06	3
$\nu = 0, 3$	0, 73	1, 01	1, 27	1, 8	2, 17

In the Table 2.2, the relation between crack initiation angle (θ) and inclination angle (β) is shown. The maximum triaxial stress around the crack tip increases with increasing inclination angle and decreases with decreasing Poisson's ratio. Kong [16]

suggested that M-criterion has a clear physical meaning which is this triaxial stress has only a maximum point around the crack tip ($-180^\circ \leq \theta \leq 180^\circ$). However, according to S and G -criteria, there are more extremum values around the crack tip.

Figure 2.14 shows the distribution of $M(\theta)$ around the crack tip for $\beta = 45^\circ$ and $\beta = 60^\circ$. It is seen that for $\beta = 45^\circ$, the $M(\theta)$ has a maximum between -100° and 50° .

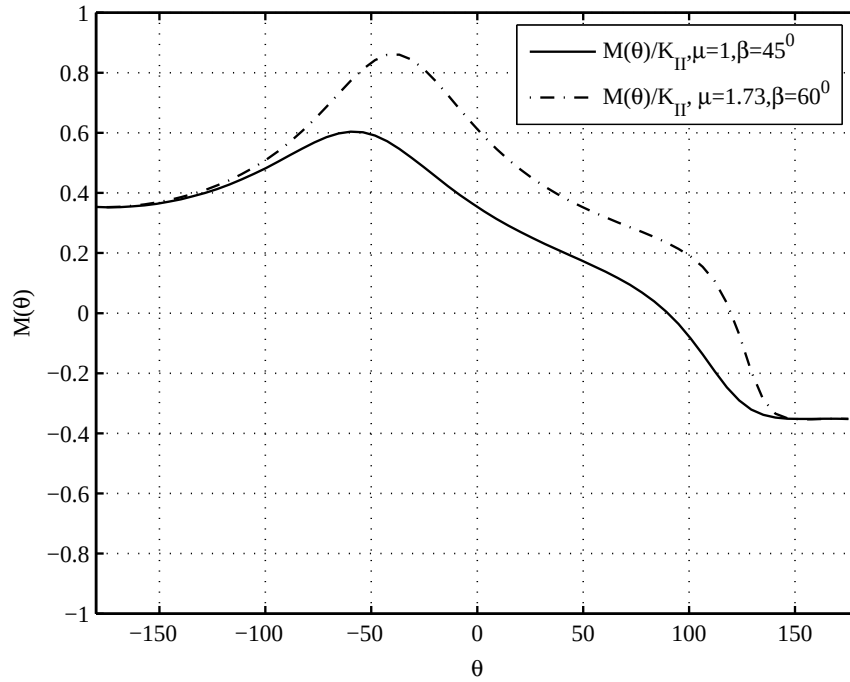


Figure 2.14. Distribution of stress triaxiality ratio around the crack tip

To check the theoretical predictions from the criterion, especially to check the predictions at small inclination angles for mixed mode fracture of steels, Kong [16] conducted experiments with a steel FeE 550 and the results are shown in Table 2.3

Table 2.3. Crack initiation angles from experiments

β		7°	16.5°	33.5°	62.5°	90°
θ	analytical	-92.8°	-84.5°	-69.8°	-42.6°	0°
	experiment	91.7°	79.9°	65.4°	46.4°	0°

Crack initiation angle results obtained by analytical M-criterion is presented in Figure 2.15.

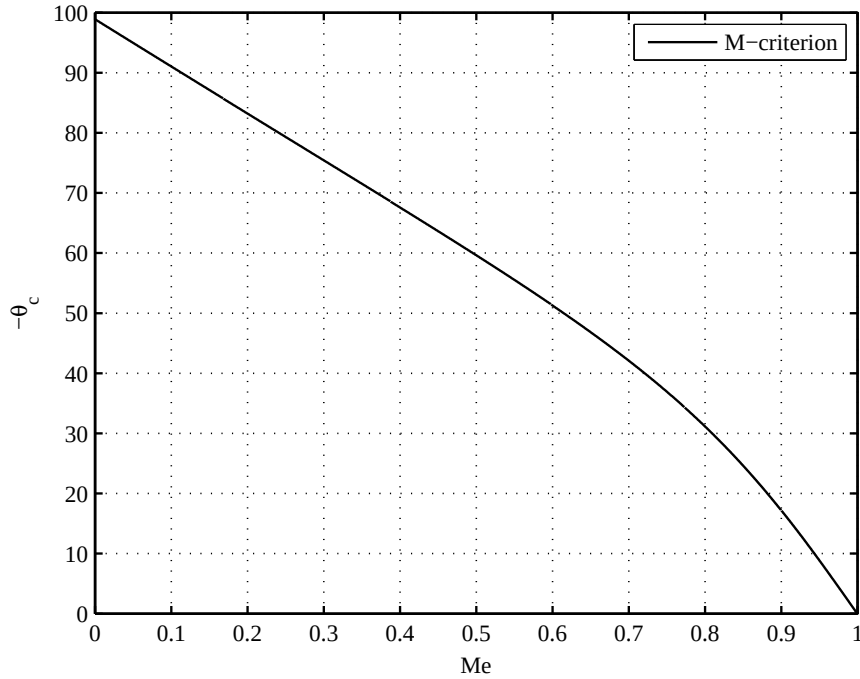


Figure 2.15. Crack Initiation angles under mixed mode loading using M-criterion

According to Kong [16], results from different experiments of tensile smooth and pre-notched round bar have shown that fracture initiation often occurs where triaxial stress is maximum.

2.6. T-Criterion

To predict crack initiation angle, most criteria use a singular stress field with linear elastic plastic fracture mechanics (LEFM) and assume a core region at the crack tip due to inability of LEFM to accurately define the stress state in the immediate vicinity of the crack. The simplest form of the core region is a circle with a constant radius.

Really there must be a certain area around the crack tip called the "core region", inside which the use of linear elasticity expressions for stresses is not possible, since

the material is not linearly elastic inside. Theocaris and Andrianopoulos [10], [11] proposed the T-criterion, (maximum dilatational strain energy criterion) suitable for ductile fracture where the Von-Misses elastic plastic boundary is used as the core region. T-criterion states that:

“When the dilatational strain energy U_v at a point in the vicinity of the crack tip reaches a critical value $U_{v,cr}$; the crack starts to propagate.”

$$\frac{\partial U_V}{\partial \theta} = 0 \quad \text{and} \quad \frac{\partial^2 U_V}{\partial \theta^2} < 0 \quad \implies \quad \theta = \theta_c \quad (2.60)$$

In the evaluation of U_V around the crack tip, the elastic plastic boundary (which is obtained from mises yield condition) is used.

Strain energy density, U , for 2-D plane elasticity problem is:

$$U = \frac{1 + \nu}{2E} \left[\frac{\kappa + 1}{4} (\sigma_{xx} + \sigma_{yy})^2 - 2(\sigma_{xx}\sigma_{yy} - \sigma_{xy}^2) \right] \quad (2.61)$$

$$\begin{aligned} \text{where} \quad \kappa &= 3 - 4\nu && \text{for plane strain,} \\ \kappa &= \frac{3 - \nu}{1 + \nu} && \text{for plane stress} \end{aligned}$$

Dilatational Strain Energy Density (energy per unit volume) is:

$$U_V = \frac{(1 - 2\nu)}{6E} (\sigma_x + \sigma_y)^2 \quad \text{for plane stress} \quad (2.62)$$

To calculate U_D , we can subtract U_V from the general strain energy density (U) equation.

$$\begin{aligned} U_D &= U - U_V \\ U_D &= \frac{(1 + \nu)}{3E} \left(\sigma_x^2 + \sigma_y^2 - \sigma_x \sigma_y + 3\tau_{xy}^2 \right) \end{aligned} \quad (2.63)$$

Because the Von-Misses yield condition gives the elastic plastic boundary as a curve of constant distortional strain energy U_D , the T-criterion implies that crack initiation direction is in the direction of maximum value of U_v along constant U_D .

To express U_V in terms of θ , we will replace the stress field equations into U_v and U_D equations.

$$\begin{aligned}\sigma_x &= \frac{1}{\sqrt{2\pi r}} f_x(\theta) \\ \sigma_y &= \frac{1}{\sqrt{2\pi r}} f_y(\theta) \\ \tau_{xy} &= \frac{1}{\sqrt{2\pi r}} f_{xy}(\theta)\end{aligned}\tag{2.64}$$

$$\begin{aligned}U_V &= \frac{1-2\nu}{12E\pi r} (f_x + f_y)^2 \\ U_D &= \frac{(1+\nu)}{6E\pi r} (f_x^2 + f_y^2 - f_x f_y + 3f_{xy}^2)\end{aligned}\tag{2.65}$$

T-criterion assumes that along the Von-Misses elastic-plastic boundary, U_D is constant.

$$U_{D,0} = \frac{(1+\nu)}{6E\pi r} (f_x^2 + f_y^2 - f_x f_y + 3f_{xy}^2)\tag{2.66}$$

$$\text{then r is } r = \frac{(1+\nu)}{6E\pi U_{D,0}} (f_x^2 + f_y^2 - f_x f_y + 3f_{xy}^2)\tag{2.67}$$

And by replacing r into U_V ,

$$U_V = \frac{(1-2\nu)6E\pi U_{D,0}}{(1+\nu)12E\pi} \frac{(f_x + f_y)^2}{(f_x^2 + f_y^2 - f_x f_y + 3f_{xy}^2)}\tag{2.68}$$

where f_x , f_y , f_{xy} are:

$$\begin{aligned}
 f_x &= \left[K_I \cos \frac{\theta}{2} (1 - \sin \frac{\theta}{2} \sin \frac{3\theta}{2}) \right] - \left[K_{II} \sin \frac{\theta}{2} (2 + \cos \frac{\theta}{2} \cos \frac{3\theta}{2}) \right] \\
 f_y &= \left[K_I \cos \frac{\theta}{2} (1 + \sin \frac{\theta}{2} \sin \frac{3\theta}{2}) \right] + \left[K_{II} \sin \frac{\theta}{2} \cos \frac{\theta}{2} \cos \frac{3\theta}{2} \right] \\
 f_{xy} &= \left[K_I \cos \frac{\theta}{2} \sin \frac{\theta}{2} \cos \frac{3\theta}{2} \right] + \left[K_{II} \cos \frac{\theta}{2} (1 - \sin \frac{\theta}{2} \sin \frac{3\theta}{2}) \right]
 \end{aligned} \tag{2.69}$$

Replacing Equations 2.69 into Equation 2.68 and applying T-criterion we get:

$$\begin{aligned}
 N^5 - 4\mu N^4 + (5\mu^2 - 1)N^3 + \frac{(3 - 5\mu^2)\mu}{2}N^2 \\
 + \frac{(\mu^4 - 2\mu^2 - 1)}{2}N + \frac{(1 + \mu^2)\mu}{2} = 0,
 \end{aligned} \tag{2.70}$$

and

$$\begin{aligned}
 &[1 - 20\mu^2 - 5\mu^4] \cos \theta + [8(3 + 2\mu^2 - \mu^4)] \cos 2\theta \\
 &+ [3(3 - 12\mu^2 + \mu^4)] \cos 3\theta + [2(13 + 5\mu^2)\mu] \sin \theta \\
 &+ [32(1 + \mu^2)\mu] \sin 2\theta - [6(5 - 3\mu^2)\mu] \sin 3\theta < 0
 \end{aligned} \tag{2.71}$$

where $N = \tan(\theta_c/2)$, $\mu = K_I/K_{II}$.

For Mode I loading ($K_{II}=0$), if we divide Equation 2.70 by μ^4 , then we only have

$$\frac{(\mu^4 - 2\mu^2 - 1)}{2\mu^4}N = 0 \tag{2.72}$$

$$\frac{1}{2} \tan\left(\frac{\theta_c}{2}\right) = 0 \implies \theta_c = 0 \tag{2.73}$$

where Equation 2.71 is satisfied.

For Mode II loading ($K_I=0$), $\mu=0$ Equation 2.70 only have

$$N^5 - N^3 - \frac{1}{2}N = 0 \quad (2.74)$$

Equation 2.74 has 5 roots, two of which are complex, one of them is 0, and other two are 1.1688 and -1.1688.

$$\text{for } N = \tan \frac{\theta_c}{2} = -1.1688 \Rightarrow \theta_c = -98.9 \quad (2.75)$$

which is the only one root that satisfies Equation 2.71

Theocaris and Andrianapolos [10] also performed experiments on PCBA specimens to compare theoretical and experimental results (Table 2.4). Also they compared the experimental and theoretical results with S and MTS criteria. And they state that the angle θ of propagation is in good agreement at small angles β where the predictions of the other two criteria differ significantly from these of T criterion[10].

Crack initiation angle results obtained from T, MTS and S criteria are plotted in Figure 2.16.

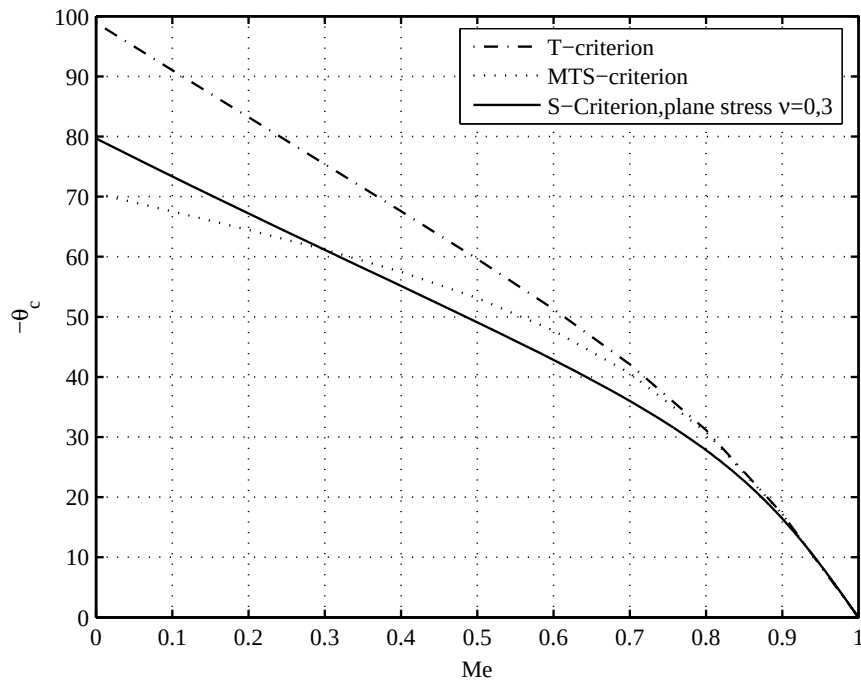


Figure 2.16. Crack Initiation Angle according to T, MTS and S criteria

Table 2.4. Crack initiation angles and fracture loads from experiments[10]

Crack angle(β)	Mean value of initiation angle	Mean value of fracture load
5	-91.3	8325
10	-86.8	7828
20	-77.1	7083
30	-69.5	6450
40	-59.0	6342
50	-53.6	6247
60	-42.5	5884
70	-31.1	5815
80	-18.0	5780
90	0.0	5775

2.7. Nt-Criterion and Y-criterion

There has been a discussion between Yehia, Theocaris and Andrianopolous on the application of the T-criterion in fracture mechanics, and its ability to predict fracture load for a propagating crack under load increments. Yehia and co-workers modified the T-criterion to incorporate the critical value of the plastic core region radius to determine the fracture load [13].

Andrianopolous and Theocaris rejected this conclusion and Yehias modification of the T-criterion. Furthermore, Yehia [14] discussed the possibility of defining a criterion completely based on the elastic-plastic core region, and presented the Y-criterion. However, the formulation of the Y-criterion itself has a consistency problem. It is not clear what critical parameter was used for determining crack initiation angles. Yehia first defined the Nt-criterion based on the Von-Misses elastic plastic core region radius, but later used the distortional energy form in Y-criterion that was obtained by rearranging the parameters. The distortional strain energy cannot be minimized on the Von-Misses elastic-plastic boundary because it is constant along that boundary. Hence, Yehia must have used a constant radius for the core region and that the Y-criterion is based on a constant radius for the core region.

2.7.1. Nt-Criterion

Yehia [13] explained that they altered the T-criterion and produced the Nt criterion.

As in the T-criterion, Yehia divides strain energy density into two and used N notation for strain energy density instead of U .

$$N = N_D + N_V \quad (2.76)$$

For plane stress condition:

$$N_V = \frac{(1-2\nu)}{6E}(\sigma_x + \sigma_y)^2 \quad (2.77)$$

$$N_D = \frac{(1+\nu)}{3E} \left(\sigma_x^2 + \sigma_y^2 - \sigma_x \sigma_y + 3\tau_{xy}^2 \right) \quad (2.78)$$

Employing elastic stress singularity field:

$$N_V = \frac{1}{36\psi\pi r}(f_x + f_y)^2 \quad \text{where} \quad \psi = \frac{E}{3(1-2\nu)} \quad (2.79)$$

$$N_D = \frac{1}{12G\pi r}(f_x^2 + f_y^2 - f_x f_y + 3f_{xy}^2) \quad (2.80)$$

Assuming that Von-Misses yield condition is applicable in this case, the N_D reaches its critical value $N_{D,0}$ along the elastic plastic boundary.

$$2\sigma_y^2 = (\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 \quad (2.81)$$

$$\text{then} \quad \sigma_y^2 = \sigma_x^2 + \sigma_y^2 - \sigma_x \sigma_y + 3\tau_{xy}^2 \quad (2.82)$$

where σ_y is material's yield strength.

$$N_D = N_{D,0} = \frac{(1+\nu)}{3E} \left(\sigma_x^2 + \sigma_y^2 - \sigma_x \sigma_y + 3\tau_{xy}^2 \right) \quad (2.83)$$

$$E = 2G(1+\nu) \quad \text{then} \quad (2.84)$$

$$N_{D,0} = \frac{\sigma_y^2}{6G} \quad (2.85)$$

The distance to the elastic plastic boundary at any angle θ at the crack tip can be determined from the Equation 2.80.

$$r_b = \frac{1}{12G\pi r N_{D,0}}(f_x^2 + f_y^2 - f_x f_y + 3f_{xy}^2) \quad (2.86)$$

According to the Nt-criterion, the direction of the crack initiation angle determined by maximizing the N_V on the elastic plastic boundary.

$$N_V = \frac{GN_{D,0}}{3\psi} \frac{(f_x + f_y)^2}{(f_x^2 + f_y^2 - f_x f_y + 3f_{xy}^2)} \quad (2.87)$$

Nt-criterion postulates that:

- the anticipated crack propagation direction θ_0 is defined as:

$$\frac{\partial N_V}{\partial \theta_0} = 0 \quad \text{and} \quad \frac{\partial^2 N_V}{\partial \theta_0^2} < 0 \quad (2.88)$$

In Figure 2.17 analytical crack initiation angle result obtained by Nt-criterion is plotted. It is obvious that Nt-criterion is identical to the T criterion up to this point.

- The crack propagates if the distance from crack tip to the elastic plastic boundary r_b in the direction θ_0 reaches a material critical value r_{bcr}

$$r_b(\theta_0) \geq r_{bcr} \quad (2.89)$$

where the material critical constant, r_{bcr} , is obtained from a single Mode I plane strain fracture (Equation 2.86 is replaced into Equation 2.89 f_y and f_{xy} terms are vanished due to Mode I loading)

$$r_{bcr} = (1 - 2\nu)^2 \frac{K_{IC}^2}{2\pi\sigma_y^2} \quad (2.90)$$

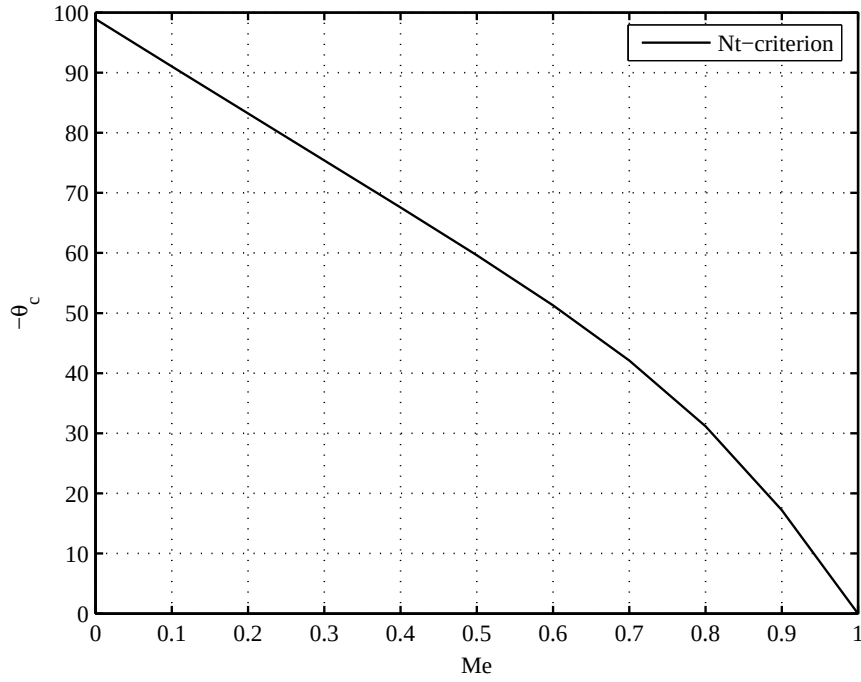


Figure 2.17. Crack Initiation angle under mixed mode loading in Nt-criterion

$$K_{IC} = \sigma \sqrt{\pi a} \quad (2.91)$$

where σ is fracture load.

The altered part of the Nt-criterion from T-criterion starts here that Yehia proposed that fracture load can be determined from Equations 2.86, 2.90, 2.91.

2.7.2. Y Criterion

Yehia [14] proposed another fracture criterion, the Y-criterion, which is completely based upon the crack tip core region. The crack tip core region in this criterion is defined by elastic-plastic boundary near the crack tip as determined through a suitable yield criterion and employing the elastic singular stress field near a crack tip in an infinite plane plate loaded at infinity. The criterion uses a physically and mathematically sound yield criterion to determine the elastic-plastic boundary at the tip. It defines the direction of minimum distance to the crack tip as the direction of the

anticipated propagation of crack. In this criterion the elastic-plastic boundary at the crack tip is defined by the Von-Misses yield criterion and separates total strain energy density to its two components; dilatational (volumetric) component Y_v and the distortional component Y_d

$$\begin{aligned} Y_v &= \frac{1}{18K} (\sigma_x + \sigma_y + \sigma_z)^2 \\ Y_d &= \frac{1}{6G} (\sigma_x^2 + \sigma_y^2 + \sigma_z^2 - \sigma_x \sigma_y - \sigma_y \sigma_z - \sigma_z \sigma_x + 3\tau_{xy}^2) \end{aligned} \quad (2.92)$$

$$K = \text{Bulk modulus} = \frac{E}{3(1 - 2\nu)} \quad (2.93)$$

$$G = \text{Shear modulus} = \frac{E}{2(1 + \nu)} \quad (2.94)$$

$$\sigma_z = 0 \quad \text{for plane stress case} \quad (2.95)$$

$$= \nu(\sigma_x + \sigma_y) \quad \text{for plane strain case} \quad (2.96)$$

Employing the crack tip elastic singular stress field in the expressions above, one gets;

$$\begin{aligned} Y_v &= \frac{1}{36K\pi r_0} (f_x + f_y + f_z)^2, \\ Y_d &= \frac{1}{12G\pi r_0} (f_x^2 + f_y^2 + f_z^2 - f_x f_y - f_y f_z - f_z f_x + 3f_{xy}^2) \end{aligned} \quad (2.97)$$

where

$$\begin{aligned}
 f_x &= K_I \left(\cos \frac{\theta}{2} - \sin \frac{\theta}{2} \sin \frac{3\theta}{2} \right) - K_{II} \left(2 \sin \frac{\theta}{2} + \frac{1}{2} \sin \theta \cos \frac{3\theta}{2} \right) \\
 f_y &= K_I \left(\cos \frac{\theta}{2} + \frac{1}{2} \sin \theta \sin \frac{3\theta}{2} \right) + K_{II} \left(\frac{1}{2} \sin \theta \cos \frac{3\theta}{2} \right) \\
 f_{xy} &= K_I \left(\frac{1}{2} \sin \theta \cos \frac{3\theta}{2} \right) + K_{II} \left(\cos \frac{\theta}{2} - \frac{1}{2} \sin \theta \sin \frac{3\theta}{2} \right) \\
 f_z &= 0 \quad \text{for plane stress} \\
 &= \nu(f_x + f_y) \quad \text{for plane strain}
 \end{aligned}$$

The material critical distortional strain energy density, according to the Von-Mises yield condition, is given by:

$$Y_{d,0} = \frac{1}{6G} \sigma_y^2 \quad (2.98)$$

and σ_y is the material uniaxial yield strength. The Y-criterion postulates that:

- The crack will propagate in a direction of the minimum distortional strain energy density determined through a proper yield condition employing LEFM.
- The crack propagation will commence when this minimum reaches a material critical value.
- A core distance r_0 is used to evaluate the parameter controlling propagation direction and fracture load.

The core distance r_o at which Y_d is evaluated, is determined from pure Mode I fracture. Equating 2.98 to 2.97 in pure Mode I fracture, one can define the core distance as:

$$\begin{aligned}
r_{oc} &= \frac{K_I^2}{2\pi\sigma^2} && \text{plane stress} \\
r_{oc} &= \frac{K_I^2}{2\pi\sigma^2}(1 - 2\nu^2) && \text{plane strain}
\end{aligned}$$

These postulations above are translated into the following two conditions:

$$\begin{aligned}
\frac{\partial y_d}{\partial \theta} = 0 \quad \text{and} \quad \frac{\partial^2 y_d}{\partial \theta^2} > 0 \\
Y_{d,\theta_0} \geq Y_{d,o}
\end{aligned} \tag{2.99}$$

Comparison of crack initiation angle results obtained by Nt and Y-criterion are illustrated in Figure 2.18.

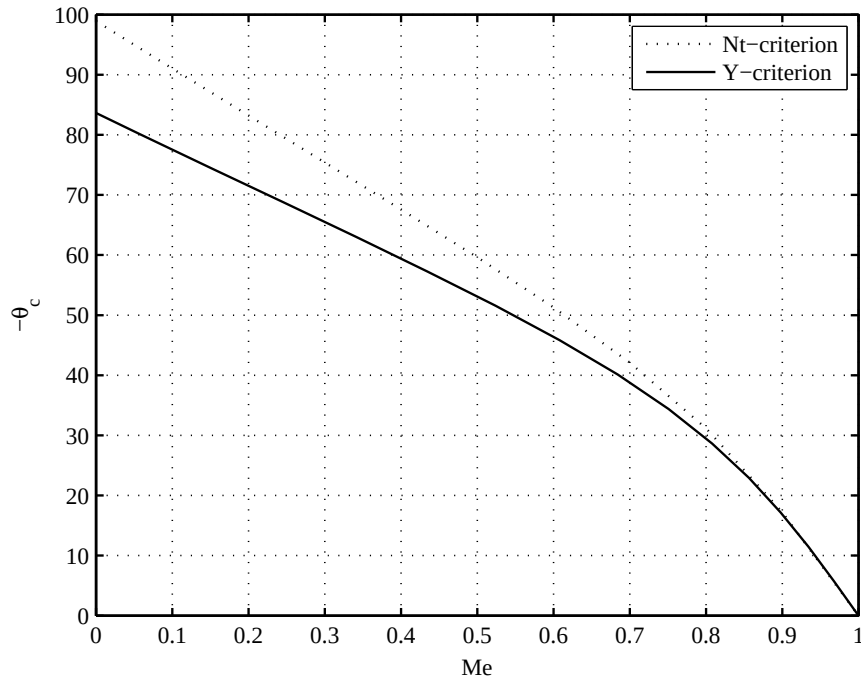


Figure 2.18. Crack Initiation angle under mixed mode loading in Nt and Y-criterion

2.8. Det Criterion

Det-criterion is proposed by Papadopoulos [12] which is a simple criterion of fracture based on the second stress invariant.

According to Det-criterion, the angle θ of the initial crack extension in mixed mode stress field governed by the K_I , K_{II} stress intensity factors is determined from condition that the determinant of the stress tensor must take maximum value.

This means:

$$\frac{\partial Det(\sigma_{ij})}{\partial \theta} = 0 \quad \text{and} \quad \frac{\partial^2 Det(\sigma_{ij})}{\partial \theta^2} < 0 \Rightarrow \theta = \theta_c \quad (2.100)$$

Then lets explain the second stress invariant. As we know total strain energy density divides into two components, T_V and T_D , dilatational and distortional strain energy density respectively.

For plane stress

$$T_V = \frac{(1 - 2\nu)}{6E} (\sigma_x + \sigma_y)^2 \quad (2.101)$$

$$T_D = \frac{(1 + \nu)}{3E} [(\sigma_x + \sigma_y)^2 - 3(\sigma_x \sigma_y - \tau_{xy}^2)] \quad (2.102)$$

For plane strain

$$T_V = \frac{(1 - 2\nu)(1 + \nu)^2}{6E} (\sigma_x + \sigma_y)^2 \quad (2.103)$$

$$T_D = \frac{(1 + \nu)}{3E} [(\nu^2 - \nu + 1)(\sigma_x + \sigma_y)^2 - 3(\sigma_x \sigma_y - \tau_{xy}^2)] \quad (2.104)$$

T_D equals to the maximum constant value $T_{D,0}$ in the elastic plastic boundary.

$$T_D = T_{D,0} = \frac{(1+\nu)}{3E} \sigma_y^2 \quad (2.105)$$

If we substitute T_V into T_D

for plane stress

$$T_D = \frac{2(1+\nu)}{1-2\nu} T_V - \frac{(1+\nu)}{E} (\sigma_x \sigma_y - \tau_{xy}^2) \quad (2.106)$$

$$Det(\sigma_{ij}) = (\sigma_x \sigma_y - \tau_{xy}^2) \quad (2.107)$$

From Equation 2.106, it can be seen that T_D depends on T_V and $Det(\sigma_{ij})$

According to T-criterion we know that at the elastic plastic boundary T_D is constant and equals to $T_{D,0}$ and T-criterion states that the crack initiation occurs at the maximum value of T_V on the elastic plastic boundary. If we look at Equation 2.106 in order to maximize T_V , $Det(\sigma_{ij})$ must have the maximum value.

The Det-criterion postulates that the crack propagates along the direction defined by a maximum value of the second stress invariant, when this distribution is evaluated along a circle around the crack tip.

If we put stress field equations into equation 2.107:

$$Det(\sigma_{ij}) = \frac{D}{2\pi r} \quad \text{where} \quad (2.108)$$

$$D = D_1 K_I^2 + D_{12} K_I K_{II} + D_2 K_{II}^2 \quad (2.109)$$

where

$$D_1 = \cos^4 \frac{\theta}{2}, \quad (2.110)$$

$$D_{12} = -4 \sin \frac{\theta}{2} \cos^3 \frac{\theta}{2}, \quad (2.111)$$

$$D_2 = \cos^2 \frac{\theta}{2} \left(3 \sin^2 \frac{\theta}{2} - 1 \right) \quad (2.112)$$

This equation gives a variation of the det of the stress tensor around the crack tip. By using Equations 2.100 and 2.108-2.112

$$\left[-2 \cos^3 \frac{\theta_c}{2} \sin \frac{\theta_c}{2} \right] \mu^2 + \left[\frac{3}{2} \sin^2 \theta_c - 2 * \cos^4 \frac{\theta_c}{2} \right] \mu + \left[\frac{3}{4} \sin 2\theta_c + \frac{1}{2} \sin \theta_c \right] = 0 \quad (2.113)$$

$$\left[-\cos^4 \frac{\theta_c}{2} + \frac{3}{4} \sin^2 \theta_c \right] \mu^2 + \left[\sin \frac{\theta_c}{2} \left(4 \cos^2 \frac{\theta_c}{2} + \frac{3}{2} \right) \right] \mu + \left[\frac{3}{2} \cos 2\theta_c + \frac{1}{2} \cos \theta_c \right] < 0 \quad (2.114)$$

For Mode I loading:

$$-2 \cos^3 \frac{\theta_c}{2} \sin \frac{\theta_c}{2} = 0 \quad (2.115)$$

$$\theta_c = 0^\circ \quad \text{or} \quad \theta_c = \pi$$

$$-\cos^4 \frac{\theta_c}{2} + \frac{3}{4} \sin^2 \theta_c < 0 \quad (2.116)$$

if $\theta_c = \pi$ Equation 2.116 is satisfied For Mode II loading:

$$\frac{3}{4} \sin 2\theta_c + \frac{1}{2} \sin \theta_c = 0 \quad (2.117)$$

$$\theta_c = 0^\circ \quad \text{or} \quad \theta_c = \pm 109.5^\circ$$

$$\frac{3}{2} \cos 2\theta_c + \frac{1}{2} \cos \theta_c < 0 \quad (2.118)$$

if $\theta_c = 109.5^\circ$ Equation 2.118 is satisfied

Solution of the Equation 2.116 is represented in graphical form in Figure 2.19.

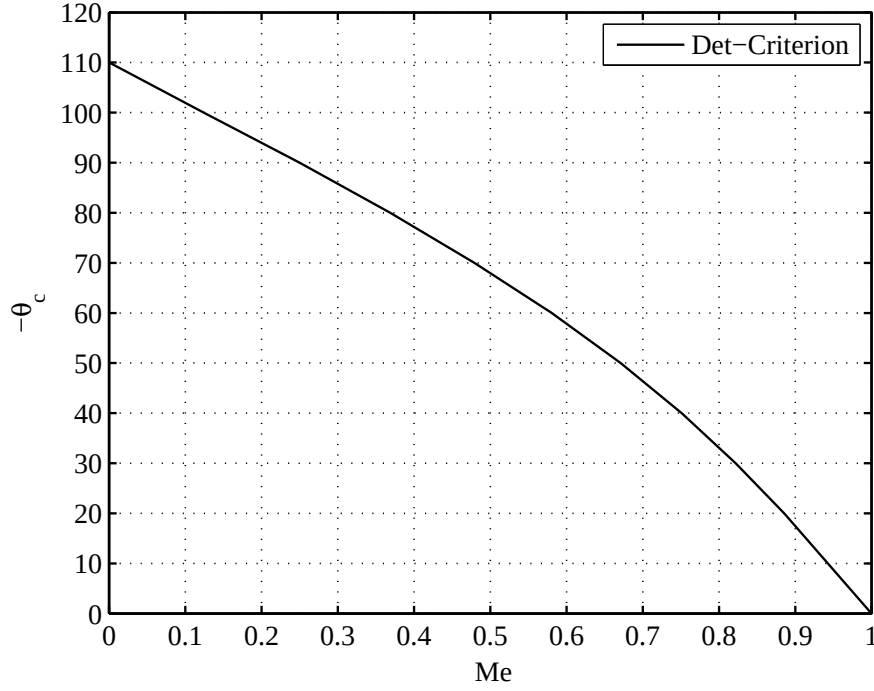


Figure 2.19. Crack Initiation angle under mixed mode loading in Det-criterion

2.9. I_P Criterion

Ukadgoeneker and Awasare [15] proposed I_P -criterion which is a modification of the T-criterion and determines the critical radius and angular location of the crack initiation in case of ductile materials. This criterion is a simple criterion of crack initiation, which is based on the first and second stress invariants.

Description of the problem is shown below. As we know from T-criterion, strain energy is divided into two components, dilatational and distortional strain energy:

For plane stress

$$U_V = \frac{(1-2\nu)}{6E}(\sigma_x + \sigma_y)^2 \quad (2.119)$$

$$U_D = \frac{(1+\nu)}{3E}[(\sigma_x + \sigma_y)^2 - 3(\sigma_x\sigma_y - \tau_{xy}^2)] \quad (2.120)$$

For plane strain

$$U_V = \frac{(1-2\nu)(1+\nu)^2}{6E}(\sigma_x + \sigma_y)^2 \quad (2.121)$$

$$U_D = \frac{(1+\nu)}{3E}[(\nu^2 - \nu + 1)(\sigma_x + \sigma_y)^2 - 3(\sigma_x\sigma_y - \tau_{xy}^2)] \quad (2.122)$$

For plane stress by extracting $\sigma_x\sigma_y$ from Equation 2.120,

$$-\frac{3E}{(1+\nu)}U_D + \sigma_x^2 + \sigma_y^2 + 2\sigma_x\sigma_y = 3\sigma_x\sigma_y - 3\tau_{xy}^2 \quad (2.123)$$

$$\sigma_x\sigma_y = \sigma_x^2 + \sigma_y^2 + 3\tau_{xy}^2 - \frac{3E}{(1+\nu)}U_D \quad (2.124)$$

substituting Equation 2.124 into Equation 2.119, we obtain

$$U_V = \frac{(1-2\nu)}{2E}(\sigma_x^2 + \sigma_y^2 + 2\tau_{xy}^2) - \frac{3E}{(1+\nu)}U_D \quad (2.125)$$

$$I_1 = \sigma_x + \sigma_y \quad (2.126)$$

$$I_2 = \sigma_x\sigma_y - 2\tau_{xy}^2 \quad (2.127)$$

Then inserting stress invariants (Equations 2.126 and 2.127) into Equation 2.125

$$U_V = \frac{(1-2\nu)}{2E}(I_1^2 - 2I_2) - \frac{3E}{(1+\nu)}U_D \quad (2.128)$$

$$\text{then } I_P = I_1^2 - 2I_2$$

T-criterion states that crack occurs when U_v is maximum. As can be seen from Equation 2.128 to maximize U_v at constant U_D , I_P must be maximum. Therefore this criterion states that crack initiation starts when I_P is maximum.

$$\frac{\partial I_P}{\partial \theta} = 0 \quad \text{and} \quad \frac{\partial^2 I_P}{\partial \theta^2} < 0 \Rightarrow \theta = \theta_c \quad (2.129)$$

$$\begin{aligned} I_P &= I_1^2 - 2I_2 \\ I_P &= (\sigma_x + \sigma_y)^2 - 2(\sigma_x\sigma_y - 2\tau_{xy}^2) \end{aligned} \quad (2.130)$$

By inserting stress field Equations into equation 2.130 and arranging we have:

$$\begin{aligned} I_P &= \frac{1}{2\pi r}(c_{11}K_1^2 + c_{22}K_{II}^2 + c_{12}K_I K_{II}) \\ c_{11} &= \frac{1}{2}(3 + 2\cos\theta - \cos^2\theta) \\ c_{22} &= \frac{1}{2}(3 - 2\cos\theta + 3\cos^2\theta) \\ c_{12} &= 2\sin\theta(\cos\theta - 1) \end{aligned} \quad (2.131)$$

By taking the first derivative of Equation 2.131:

$$\begin{aligned} \mu^2 (-\sin \theta + \cos \theta \sin \theta) + \mu(2 \cos^2 \theta - 2 \cos \theta) \\ + (\sin \theta - 3 \sin \theta \cos \theta) = 0 \end{aligned} \quad (2.132)$$

And by taking the second derivative of Equation 2.131:

$$\mu^2(-\cos \theta + \cos 2\theta) + \mu(-4 \sin 2\theta + 2 \sin \theta) + (\cos \theta - 3 \cos 2\theta) < 0 \quad (2.133)$$

For Mode I loading:

$$\begin{aligned} (-\sin \theta + \cos \theta \sin \theta) = 0 \quad \text{and} \quad (-\cos \theta + \cos 2\theta) < 0 \\ \theta_c = 0^\circ \end{aligned} \quad (2.134)$$

For Mode II loading:

$$\begin{aligned} (\sin \theta - 3 \sin \theta \cos \theta) = 0 \quad \text{and} \quad (\cos \theta - 3 \cos 2\theta) < 0 \\ \theta_c = \pi \quad \text{or} \quad \theta_c = \arccos\left(\frac{1}{3}\right) \end{aligned} \quad (2.135)$$

Equation 2.133 is satisfied by $\theta_c = \pi$

Results of crack initiation angle based on ductile criteria as I_p , Det and T-criteria are presented in Figure 2.20.

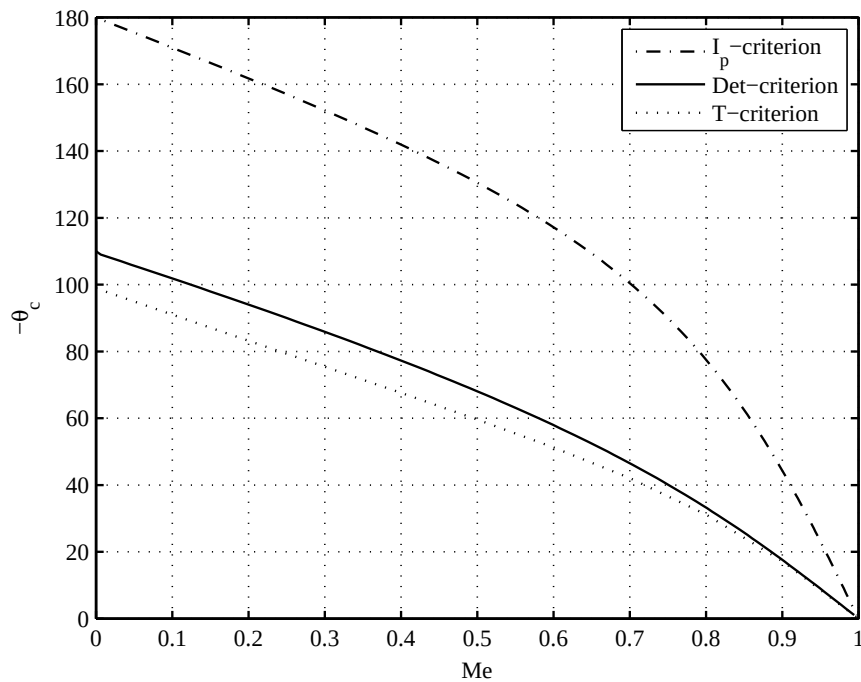


Figure 2.20. Crack Initiation angle under mixed mode loading in I_p , Det and T-criteria

2.10. Critical Radius: R and W Criteria

The parameter r_c , the core region radius, is represented by most researchers as the fracture process zone size which is the critical distance away from the crack tip where the critical parameter in order to predict the crack initiation angle is defined. They define r_c and the value of the critical parameter as the material property (stress, strain, energy) that can be assumed to be independent of the geometry of specimen and loading configuration. Using this approach, the direction of fracture initiation and the onset of crack growth is investigated at the critical radius r_c from the crack tip and not at the crack tip where r is equal to zero.

The simplest form of the core region is a circle with a constant radius which is used by MTS and M-criteria. On the other hand the T, Nt, Y, Det, I-p criteria which are all for ductile materials and the modified MTS criterion which is proposed by Khan and Khraisheh [18] by including a variable core region radius into the MTS criterion to make it applicable for ductile materials, use a variable core region radius.

The description of the core region radius is more important for the fracture of ductile materials than that of brittle materials. The reason is the relatively extensive plastic deformation ahead of the crack in ductile materials.

From a mathematical point of view, if a criterion assumes a constant radius for the core region and uses only the singular elastic stress field to define stresses at the crack tip, then the value of that radius r_c does not affect the crack initiation angle. However if the non-singular term, T-stress, is added to the elastic solution in conjunction with the singular elastic field, then the value of r_c affects the crack initiation angle.

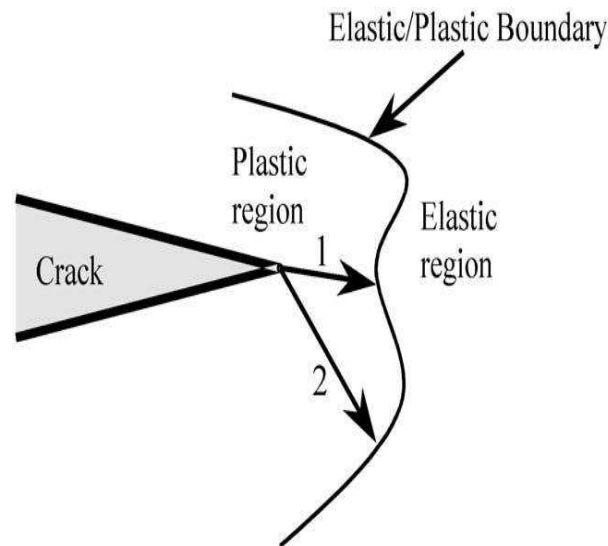


Figure 2.21. Plastic work done per unit surface area created for two different paths

The core region presents the highly strained area. The crack tip has to propagate through this highly strained area to reach elastically loaded material outside the core radius. The crack will follow the “easiest path” to the outside material. It make sense to assume that the easiest path is the shortest distance to the ‘outside’ material.

The shortest distance from the crack tip to the elastic plastic boundary represents the minimum work needed to create the crack surfaces. As can be seen from Figure 2.21, plastic work done by path 1 is smaller than path 2.

It is established that the nature of the core region at crack tip plays an important

role in determining the crack initial direction of the crack propagation. Different definitions of the core region were used in deriving crack initiation criteria. These criteria are named “R-criterion” by Khan [18] and “W-criterion” by Wasiluk [17] presents a direct correlation between size and shape of the core region and the crack initiation angles for various loading conditions.

2.10.1. R-Criterion

This criterion uses the Von-Misses yield function to define the variable radius of the core region. The Von-Misses yield condition:

$$2\sigma_y^2 = (\sigma_x - \sigma_y)^2 + (\sigma_y - \sigma_z)^2 + (\sigma_z - \sigma_x)^2 + 6\tau_{xy}^2 \quad (2.136)$$

substituting crack tip stress field equations into Equation 2.136 and solving for r.

$$\frac{1}{2\pi r_{p(\theta)}} \left[K_I^2 h_1(\theta) + K_{II}^2 h_2(\theta) + K_I K_{II} h_{12}(\theta) \right] = 2\sigma_y^2 \quad (2.137)$$

where h_1 , h_2 , h_{12} are angular functions.

For Mode I loading, terms of K_{II} drop and by arranging equation 2.137 we get:

$$r_{p(\theta)} = \left(\frac{K_I}{\sigma_y} \right)^2 \frac{1}{4\pi} \left[\frac{3}{2} \sin^2 \theta + \cos \theta + 1 \right] \quad (2.138)$$

Non dimensionality, the $r_{p(\theta)}$

$$R_{p(\theta)} = \frac{r_{p(\theta)}}{\left(\frac{K_I}{\sigma_y} \right)^2} = \frac{1}{4} \left[\frac{3}{2} \sin^2 \theta + \cos \theta + 1 \right] \quad (2.139)$$

The R-criterion states that the direction of the crack initiation angle coincides with

the direction of the minimum distance from the crack tip to the core region.

$$\frac{\partial R_p}{\partial \theta} = 0 \quad \text{and} \quad \frac{\partial^2 R_p}{\partial \theta^2} > 0 \quad (2.140)$$

For Mode I loading,

$$\begin{aligned} \frac{\partial R_{p1}}{\partial \theta} &= \frac{1}{4} \left[\frac{3}{2} 2 \sin \theta \cos \theta - \sin \theta \right] = 0 \\ \sin \theta (3 \cos \theta - 1) &= 0 \end{aligned} \quad (2.141)$$

$\theta_c = 0^\circ$ or $\theta_c = 70.5^\circ$

$$\begin{aligned} \frac{\partial^2 R_{p1}}{\partial \theta^2} &= \frac{1}{4} \left[\cos \theta (3 \cos \theta - 1) + \sin \theta (-3 \sin \theta) \right] > 0 \\ (3 \cos 2\theta - \cos \theta) &> 0 \end{aligned} \quad (2.142)$$

for $\theta_c = 0^\circ$ equation 2.142 is satisfied. Therefore for Mode I loading $\theta_c = 0^\circ$

For Mode II loading,

$$\begin{aligned} \frac{\partial R_{p2}}{\partial \theta} &= \frac{1}{4} \left[\frac{-9}{4} \sin^2 \theta + \sin^2 \frac{\theta}{2} + 3 \right] = 0 \\ \frac{-1}{2} \sin \theta (9 \cos \theta - 1) &= 0 \end{aligned} \quad (2.143)$$

From Equation 2.143, $\theta_c = 0^\circ$ or $\theta_c = 83.6^\circ$

$$\begin{aligned} \frac{\partial^2 R_{p2}}{\partial \theta^2} &= \frac{-9}{2} \left[\cos 2\theta + \frac{1}{2} \cos \theta \right] > 0 \\ \frac{1}{2} (-9 \cos 2\theta + \cos \theta) &> 0 \end{aligned} \quad (2.144)$$

Only for $\theta_c = 83.6^\circ$ Equation 2.144 is satisfied.

For mixed mode loading:

$$R_{p(\theta)} = \frac{r_{p(\theta)}}{a \left(\frac{\sigma}{\sigma_y} \right)^2} \quad (2.145)$$

$$K_I = \sigma \sqrt{\pi a} f_{KI}(\beta)$$

$$K_{II} = \sigma \sqrt{\pi a} f_{KII}(\beta)$$

$$\sigma_x = \frac{\sigma \sqrt{a}}{\sqrt{2r}} f_x(\theta, f_{Ki}) \quad (2.146)$$

$$\sigma_y = \frac{\sigma \sqrt{a}}{\sqrt{2r}} f_y(\theta, f_{Ki}) \quad (2.147)$$

$$\tau_{xy} = \frac{\sigma \sqrt{a}}{\sqrt{2r}} f_{xy}(\theta, f_{Ki}) \quad (2.148)$$

Again by putting these terms into Von-Misses yield condition:

$$R_{p(\theta), f_{Ki}} = \frac{r_{p(\theta), f_{Ki}}}{a \left(\frac{\sigma}{\sigma_y} \right)^2} \quad (2.149)$$

$$R_{p(\theta), f_{Ki}} = \frac{1}{2} \left[f_x^2 + f_y^2 - f_x f_y + 3f_{xy}^2 \right]$$

replacing f_x, f_y, f_{xy} into Equation 2.149

$$R_{p(\theta), f_{Ki}} = f_{KI}^2 g_1(\theta) + f_{KII}^2 g_2(\theta) + f_{KI} f_{KII} g_{12}(\theta) \quad (2.150)$$

calculating by the help of Matlab, we get:

$$g_1(\theta) = \frac{7 + 4 \cos \theta - 3 \cos 2\theta}{16} \quad (2.151)$$

$$g_2(\theta) = \frac{19 - 4 \cos \theta + 9 \cos 2\theta}{16} \quad (2.152)$$

$$g_{12}(\theta) = \frac{-2 \sin \theta + 3 \sin 2\theta}{4} \quad (2.153)$$

Applying R-criterion:

$$\frac{\partial R_p}{\partial \theta} = 0 \quad \text{and} \quad \frac{\partial^2 R_p}{\partial \theta^2} > 0$$

$$\begin{aligned} (2\mu) \tan^4 \left(\frac{\theta}{2} \right) - (5 - 2\mu^2) \tan^3 \left(\frac{\theta}{2} \right) - (9\mu) \tan^2 \left(\frac{\theta}{2} \right) \\ + (\mu^2 - 4) \tan \left(\frac{\theta}{2} \right) + \mu = 0 \end{aligned} \quad (2.154)$$

and

$$(1 - \mu^2) \cos(\theta) + 3(\mu^2 - 3) \cos(2\theta) + 2\mu \sin(\theta) - 12\mu \sin(2\theta) > 0 \quad (2.155)$$

From Equations 2.154, we can plot Me (mixity parameter) versus θ_c .

2.10.2. W Criterion

The size and the shape of the plastic zone can be determined theoretically by applying appropriate yield criterion in the crack tip stress field equations. The Huber-Mises yield criterion states that yielding has occurred when

$$2\sigma_{ys}^2 = (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \quad (2.156)$$

where σ_1 , σ_2 , σ_3 are principal stresses, and σ_{ys} is the yield stress in tension of the material.

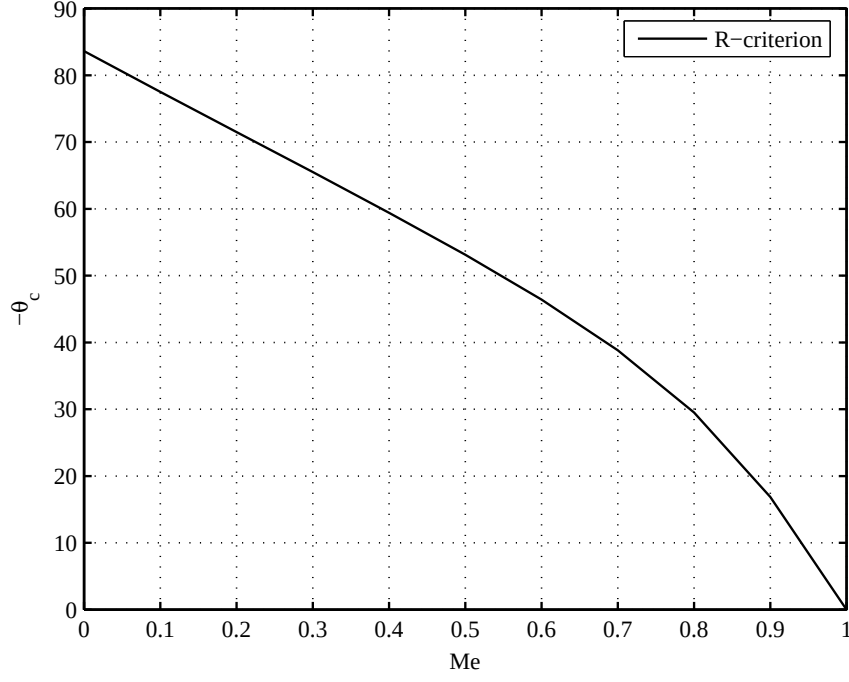


Figure 2.22. Crack Initiation angle under mixed mode loading in R criterion

For mixed mode I and II loading conditions, an expression for the plastic zone boundary, for the case of plane stress has the form of:

$$r_p = \frac{1}{2\pi\sigma_{ys}^2} \left[K_I^2 \cos^2 \frac{\theta}{2} (1 + 3 \sin^2 \frac{\theta}{2}) + K_I K_{II} \sin \theta (3 \cos \theta - 1) + K_{II}^2 \left(3 + \sin^2 \frac{\theta}{2} (1 - 9 \sin^2 \frac{\theta}{2}) \right) \right] \quad (2.157)$$

Stress intensity factors K_I and K_{II} have the form

$$K_I = \sigma \sqrt{\pi a} \sin^2 \beta \quad (2.158)$$

$$K_{II} = \sigma \sqrt{\pi a} \sin \beta \cos \beta \quad (2.159)$$

where β is inclined angle of crack to the y axis.

Inserting the nomenclature

$$\begin{aligned}\frac{\sigma}{\sigma_{ys}} &= \bar{\sigma} \\ \frac{r_p}{a} &= W\end{aligned}\tag{2.160}$$

and transforming Equation 2.157, then the W-factor for plane stress has the form:

$$\begin{aligned}W = \frac{1}{2}\bar{\sigma} \left[\sin^4 \beta \cos^2 \frac{\theta}{2} (1 + 3 \sin^2 \frac{\theta}{2}) + \sin^3 \beta \cos \beta \sin \theta (3 \cos \theta - 1) \right. \\ \left. + \sin^2 \beta \cos^2 \beta (3 + \sin^2 \frac{\theta}{2} (1 - 9 \sin^2 \frac{\theta}{2})) \right]\end{aligned}\tag{2.161}$$

Under mixed mode conditions to calculate the crack initiation angle, this criterion assumes that crack growth initiates in a direction along which the W-factor possesses a minimum value.

$$\frac{\partial W}{\partial \theta} = 0 \quad \text{and} \quad \frac{\partial^2 W}{\partial \theta^2} > 0 \quad \implies \quad \theta = \theta_c\tag{2.162}$$

where $-\pi < \theta < \pi$

Applying Equation 2.162 into Equation 2.161,

$$\begin{aligned}\frac{\partial W}{\partial \theta} = \left[\tan^4 \beta \left(-\frac{1}{2} \sin \theta + \frac{3}{4} \sin 2\theta \right) + \tan^3 \beta \left(3 \cos 2\theta - \cos \theta \right) \right. \\ \left. + \tan^2 \beta \left(\frac{1}{2} \sin \theta - 9 \sin^2 \frac{\theta}{2} \sin \theta \right) \right] = 0\end{aligned}\tag{2.163}$$

$$\begin{aligned}\frac{\partial^2 W}{\partial \theta^2} = \left[\tan^4 \beta \left(-\frac{1}{2} \cos \theta + \frac{3}{2} \cos 2\theta \right) + \tan^3 \beta \left(-6 \sin 2\theta + \sin \theta \right) \right. \\ \left. + \tan^2 \beta \left(\frac{1}{2} \cos \theta - 9 \left(\frac{1}{2} \sin^2 \theta + \sin^2 \frac{\theta}{2} \cos \theta \right) \right) \right] > 0\end{aligned}\tag{2.164}$$

Solution of the Equation 2.163 is illustrated in Figure 2.23.

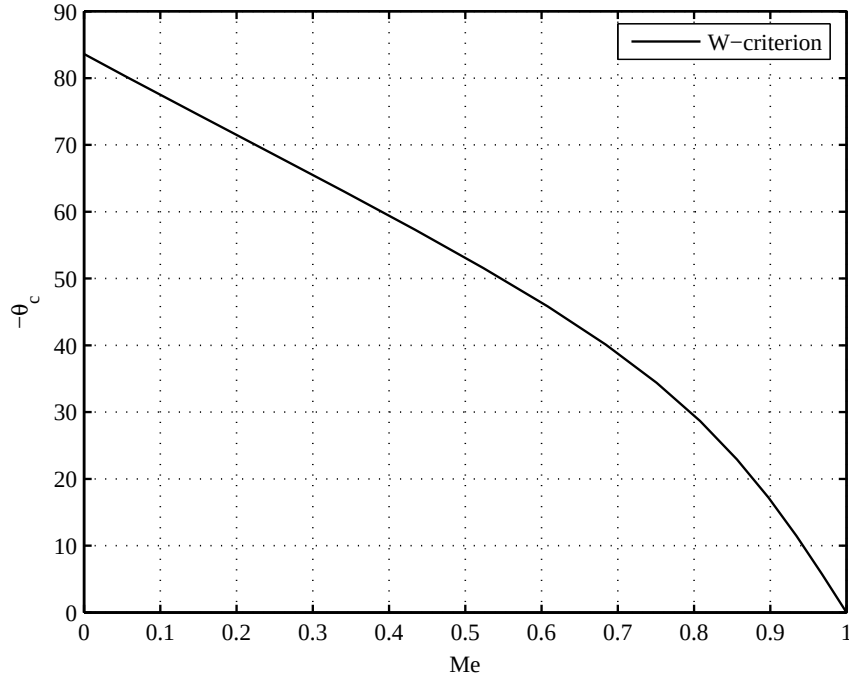


Figure 2.23. Crack Initiation angle under mixed mode loading in W criterion

For Mode I loading:

$$-\frac{1}{2} \sin \theta + \frac{3}{4} \sin 2\theta = 0 \quad (2.165)$$

Equation 2.165 has two solutions:

$$-\frac{1}{2} + \frac{3}{2} \cos \theta = 0 \Rightarrow \theta_c = 70,53^\circ \quad (2.166)$$

$$\sin \theta_c = 0 \Rightarrow \theta_c = 0^\circ \quad (2.167)$$

Equation 2.164 is only satisfied by Equation 2.167. The proposed model is connected to the material property that is characterized by the plastic zone shape. This is the direction of the minimum energy required in the plastic zone.

The W -criterion is compared with MTS criterion. Predicted crack growth direction at the initial crack tip for different values of initial orientation angles are listed in Table 2.5.

Table 2.5. Comparison between W and MTS criteria [17]

β	MTS criterion	W criterion
[deg]	[deg]	[deg]
30	-68.17	-63.49
40	-63.01	-56.66
50	-56.74	-49.47
60	-48.53	-41.49
70	-37.13	-31.80
80	-20.92	-18.49

3. FINITE ELEMENT MODELING AND ANALYSIS

The purpose of this section is to predict crack initiation angles using different crack initiation criteria and to compare theoretical and numerical results in homogeneous and functionally graded materials under mixed mode loading. S, G and M criteria are used to determine crack initiation angles for edge and center cracks, numerically. The reason for choosing S, G is these two criteria are the mostly used criteria for brittle materials in literature. Numerical results of MTS [5], S, G and M criteria are compared. In this study, Marc Mentat/Marc 2005 and Patran 2005 are used to calculate stress intensity factors, J-integrals and crack initiation angles. The geometry, material properties and boundary conditions are given below.

3.1. Geometry and Material Properties

The geometry of the edge crack plate is given in Figure 3.1 and center cracked plate is given Figure 3.2. Edge crack plate is used for FGM and homogeneous materials analysis. Center crack analysis is made for a homogeneous material to check results with analytical ones. The width of the 2-D plate (model) is 100 mm and the height is 200 mm ($H/w=0,5$). Quadrilateral elements with four nodes are used for FE model in 2-D under plane stress conditions, . Triangular elements involving degenerated nodes, are also allowed. Circular mesh is used around the crack tip . Element are arranged in order by five degrees. Out of the circular mesh, there is a region between refined and coarse mesh, the mesh is more coarser in this region. Outer part, which is the most coarse region, consists of rectangular mesh . An example of the mesh is presented in Figure 3.3. The crack tip singularities are modeled using crack tip elements, which enabled crack tip fields and fracture parameters to be obtained. To improve solution accuracy, the mesh is refined around the crack.

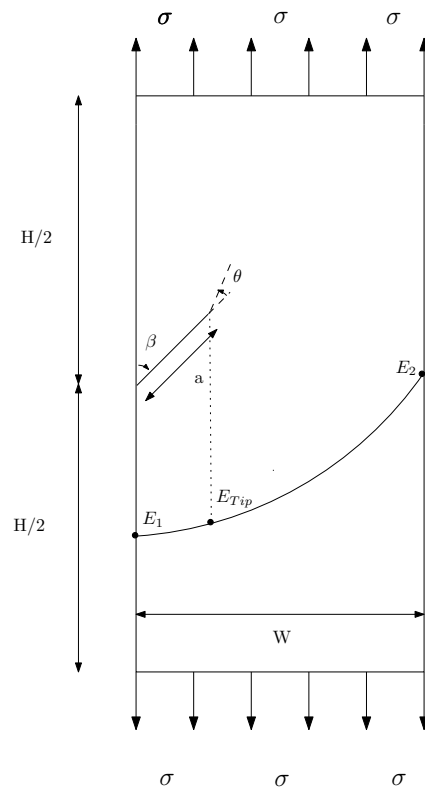


Figure 3.1. The geometry of the problem for edge cracked plate

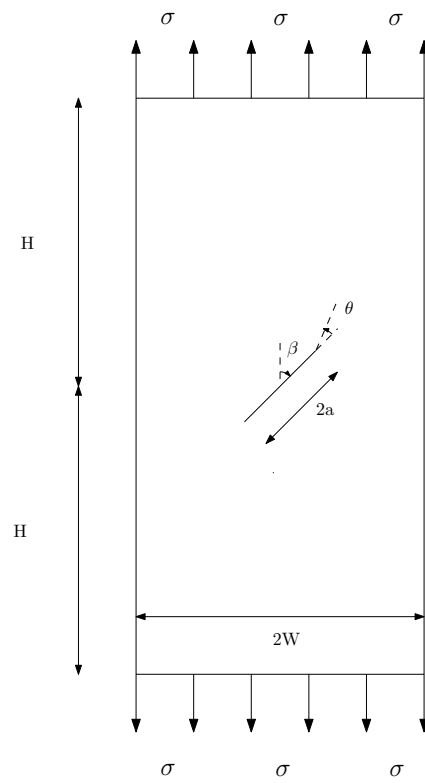


Figure 3.2. The geometry of the problem for center cracked plate

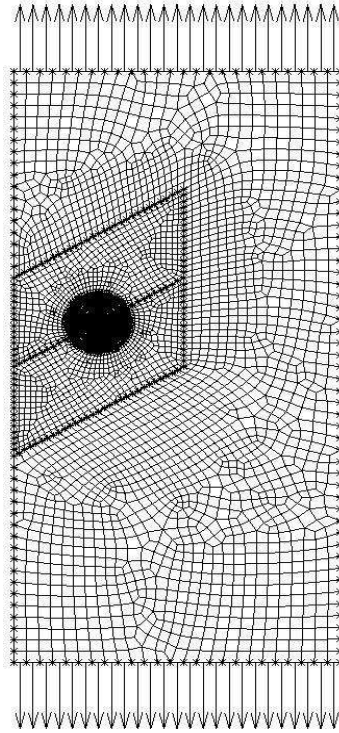


Figure 3.3. The mesh of the plate

To determine the number of elements and the number of nodes for the 2-D FE model, 7 different models with an edged flat crack are studied using homogeneous material. J-integrals and Stress Intensity Factors (SIF) are calculated for every model, and results are compared. It can be seen that from Table 3.1, J integral results are close to each other and a circular mesh around the crack tip gives better results than those of a rectangular mesh. Analytical value of F_I is 1.661. With an admissible error of approximately 1 percent, using 3500-4500 elements seems to be sufficient. Increasing number of elements depends on crack geometry such as crack length, crack angle and crack position.

For homogeneous materials, Young's Modulus is taken to be 20 GPa and Poisson's ratio is 0.3. For functionally graded materials Young's modulus is changing exponentially along the x-axis as shown below while Poisson's ratio is assumed to be constant.

Table 3.1. J-integral and SIF values

Model	Number of Elements	Mesh Type	J-integral	F_I	Error
1	2369	circular	31.645	1.640	1.20
2	3777	circular	31.691	1.641	1.14
3	4000	rectangular	31.200	1.629	1.87
4	8024	circular	32.121	1.651	0.54
5	8000	rectangular	31.478	1.635	1.50
6	11766	circular	32.087	1.650	0.60
7	12000	rectangular	31.737	1.642	1.08

$$E(x) = E_1 e^{\psi x} \quad (3.1)$$

$$\psi = \frac{1}{w} \ln\left(\frac{E_2}{E_1}\right) \quad (3.2)$$

E_1 is the elastic modulus at $x=0$, (20 GPa), and E_2 is elastic modulus at $x = w$.

Material discretization of graded materials is complicated because FE programs do not allow variation in assigned material properties across elements directly. To overcome this difficulty, first the temperature field is defined as a function of x axis in Patran. These temperature values are assigned to nodes. Finally, results are imported to Marc Mentat and material properties are changed as a function of temperature. This provided the variation of Young's Modulus along x -axis. The coefficient of thermal expansion is set to zero to avoid thermal stresses.

The analyses are made for different crack lengths, $a/w=0.2$, 0.3 and 0.4 and an edge load, $p=-50 \text{ N/mm}$ is applied.

3.2. Analysis

In this section numerical analysis of S, G and M-criteria are studied. Also deficiency of M-criterion is realized and M-criterion is modified. All the crack initiation angle results are tabulated.

3.2.1. Numerical Analysis of S-criterion

First, *S* Criterion is used to determine the crack initiation angles. *S* Criterion states that crack initiates in direction of minimum strain energy density. Advancing the crack tip along different angles, strain energy is obtained from FE program's outputs. In Figure 3.4 a kinked crack mesh is illustrated.

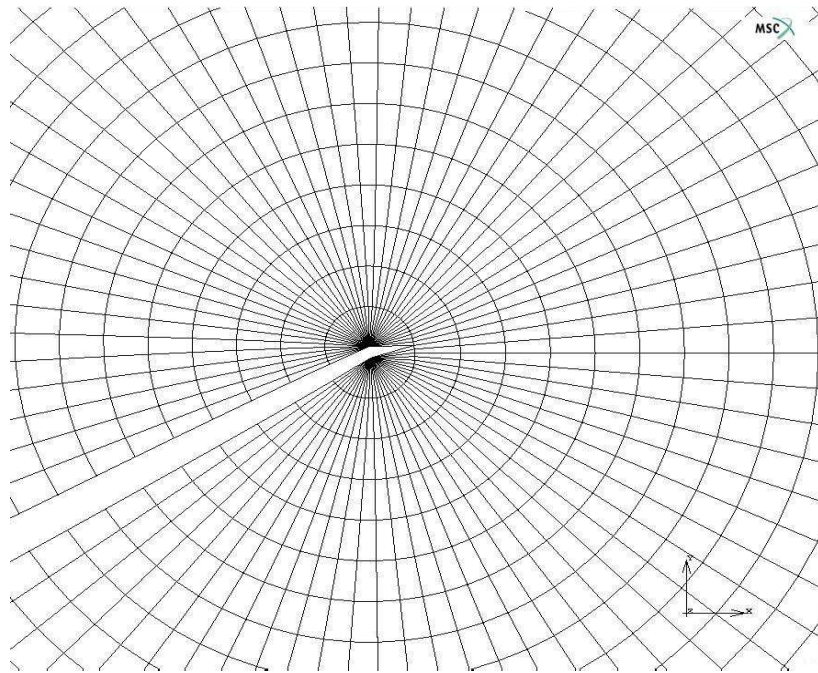


Figure 3.4. Detail of the finite element mesh near the kinked crack

The data of strain energy obtained from FE are plotted for a homogeneous edge cracked plate with $a/w=0.3$, and $\beta = 60^\circ$, in Figure 3.5. It can be seen that strain energy value has a minimum between -20° and -40° . To determine the initiation angle precisely, a second degree polynomial is fitted to data, the derivative taken with

respect to θ is equated to zero to get the initiation angle according to S criterion. For this plate, $\theta_c = -30,6^\circ$ is obtained. In Figure 3.6, comparison of crack initiation angles for center cracked homogeneous material is illustrated. Crack initiation angles determined using S criterion are tabulated in Tables 3.2-3.6.

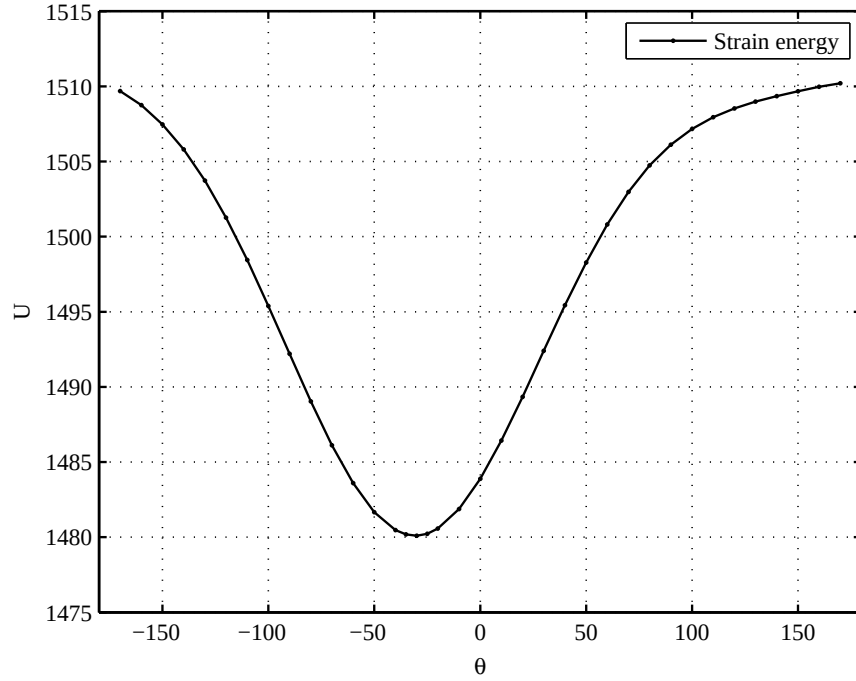


Figure 3.5. Strain energy around the crack tip

Table 3.2. Crack initiation angles for edge cracked homogeneous plate using
S-criterion

Crack Angle, β	$-\theta_c$, a/w=0.2	$-\theta_c$, a/w=0.3	$-\theta_c$, a/w=0.4
[deg]	[deg]	[deg]	[deg]
30	55.54	56.10	54.98
45	44.07	44.06	43.03
60	30.62	30.60	30.00
75	16.28	15.77	15.51
90	0.00	0.00	0.00

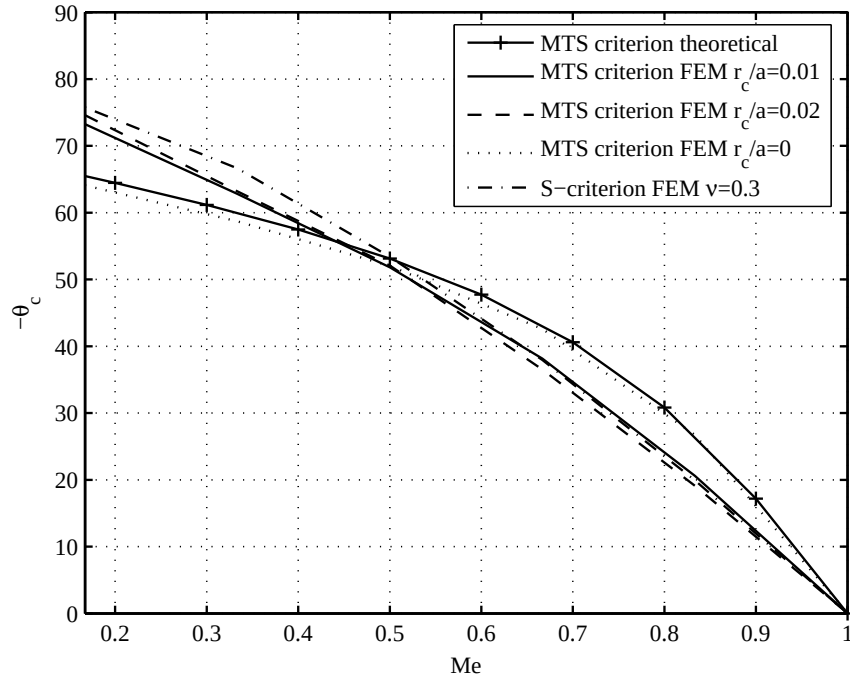


Figure 3.6. Comparison of crack initiation angle results based on S and MTS Criteria for center cracked homogeneous plate

Table 3.3. Crack initiation angles for edge cracked FGM plate ($E_2/E_1 = 0.1$) using S-criterion

Crack Angle, β	$-\theta_c$, $a/w=0.2$	$-\theta_c$, $a/w=0.3$	$-\theta_c$, $a/w=0.4$
[deg]	[deg]	[deg]	[deg]
30	58.07	59.19	57.94
45	47.00	46.37	45.53
60	33.31	32.18	31.79
75	17.25	16.56	16.41
90	0.00	0.00	0.00

Table 3.4. Crack initiation angles for edge cracked FGM plate ($E_2/E_1 = 0.2$) using
S-criterion

Crack Angle, β	$-\theta_c$, a/w=0.2	$-\theta_c$, a/w=0.3	$-\theta_c$, a/w=0.4
[deg]	[deg]	[deg]	[deg]
30	57.06	58.10	57.00
45	45.72	45.61	44.85
60	32.28	31.74	31.31
75	16.93	16.32	16.17
90	0.00	0.00	0.00

Table 3.5. Crack initiation angles for edge cracked FGM plate ($E_2/E_1 = 10$) using
S-criterion

Crack Angle, β	$-\theta_c$, a/w=0.2	$-\theta_c$, a/w=0.3	$-\theta_c$, a/w=0.4
[deg]	[deg]	[deg]	[deg]
30	54.11	53.88	51.48
45	42.55	41.61	39.50
60	29.74	28.68	27.29
75	15.24	14.71	14.07
90	0.00	0.00	0.00

Table 3.6. Crack initiation angles for edge cracked FGM plate ($E_2/E_1 = 100$) using S-criterion

Crack Angle, β	$-\theta_c$, a/w=0.2	$-\theta_c$, a/w=0.3	$-\theta_c$, a/w=0.4
[deg]	[deg]	[deg]	[deg]
30	52.57	50.64	46.40
45	40.35	38.11	34.12
60	28.14	25.73	23.11
75	14.52	13.04	11.90
90	0.00	0.00	0.00

Table 3.7. Crack initiation angle for center cracked homogeneous plate

Crack Angle, β	Initiation angle, $-\theta_c$, a/w=0.3
[deg]	[deg]
0	89.99
15	79.79
30	66.65
45	53.39
60	37.96
75	19.92
90	0.00

To check whether a finer mesh around the crack tip gives different results or not, a new model is developed for edge cracked plate. Results are tabulated in Table 3.8.

Table 3.8. Crack initiation angle obtained by different Δa values for edge cracked plate

Crack Angle, $\beta = 60^\circ$	$-\theta$ for $\Delta a = 0.25$	$-\theta$ for $\Delta a = 0.50$
homogeneous	30.74	30.60
FGM10	28.43	28.68
FGM100	25.30	25.73

The effect of material non-homogeneity and crack angle for different crack lengths is represented in Figures 3.7-3.9. From Figures 3.7-3.9, it can be seen that for a constant β , as E_2/E_1 increases for the same geometry, crack initiation angle decreases. Also it can be concluded that the effect of material non-homogeneity is increases for the small values of β .

To see the effect of crack length on crack initiation angles, Figures 3.10-3.14 are plotted. For homogeneous case the effect of crack length is too small as can be seen from Figure 3.10, but for non-homogeneous case especially for $E_2/E_1 = 100$, the effect of crack length is more significant.

Figures 3.15-3.17 represent the effect of crack length and the material non-homogeneity for specific crack angle, β . The scale of the x-coordinate is a logarithmic scale. For the same geometry, as the material non-homogeneity increases, crack initiation angle decreases, but the change depends on crack length. From Figures 3.15-3.17, it can be seen that as the crack length increases, a/w , the change in crack initiation angle increases.

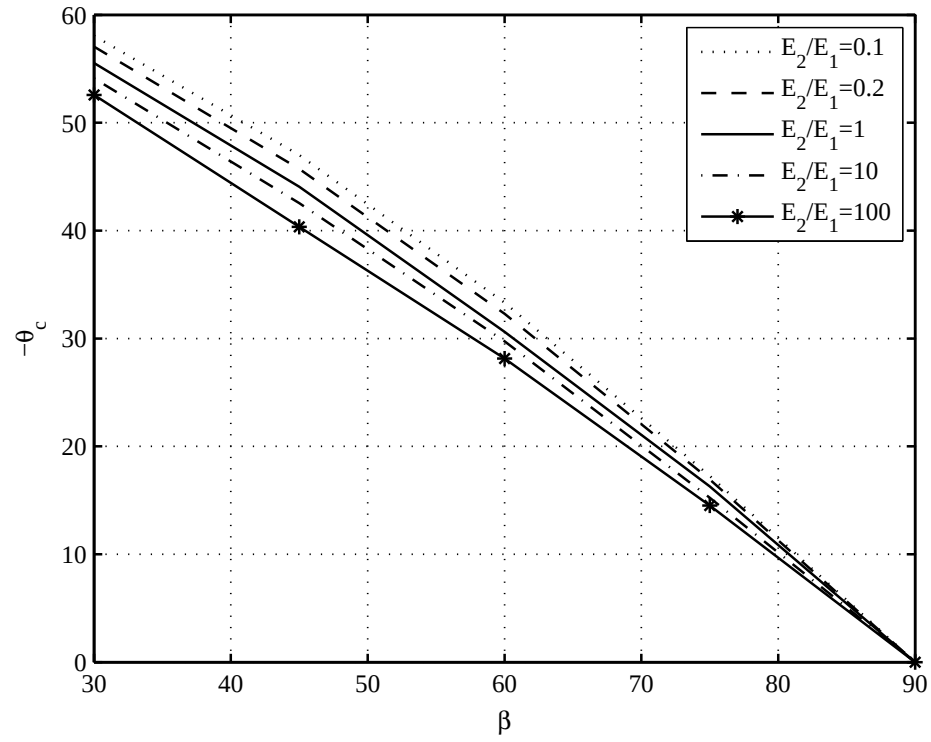


Figure 3.7. Crack initiation angles using S-criterion for different E_2/E_1 , for $a/w=0.2$

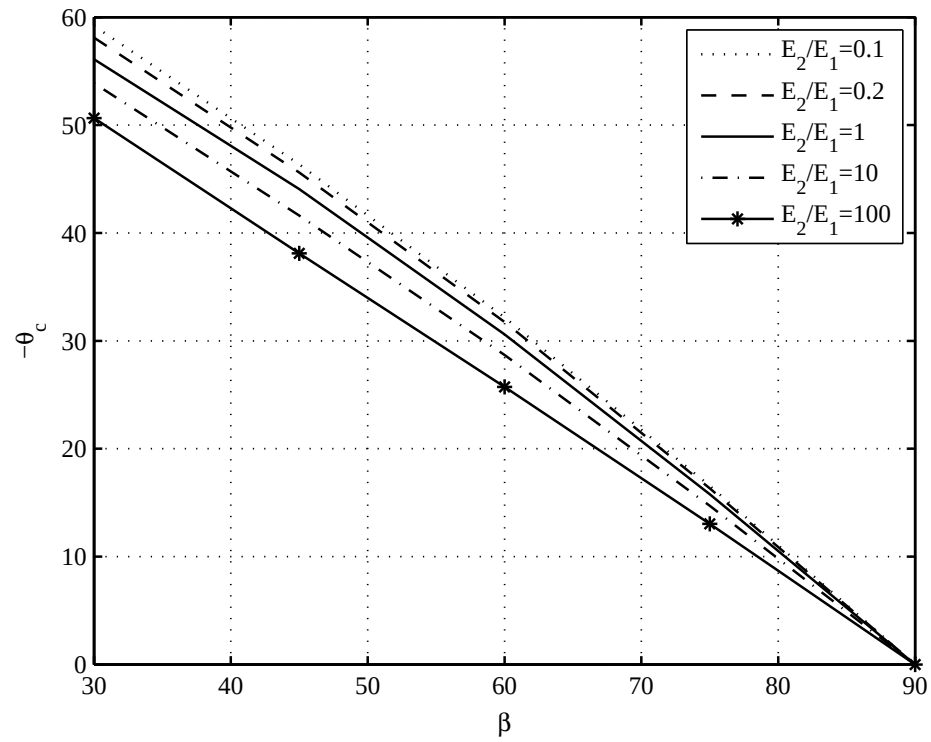


Figure 3.8. Crack initiation angles using S-criterion for different E_2/E_1 , for $a/w=0.3$

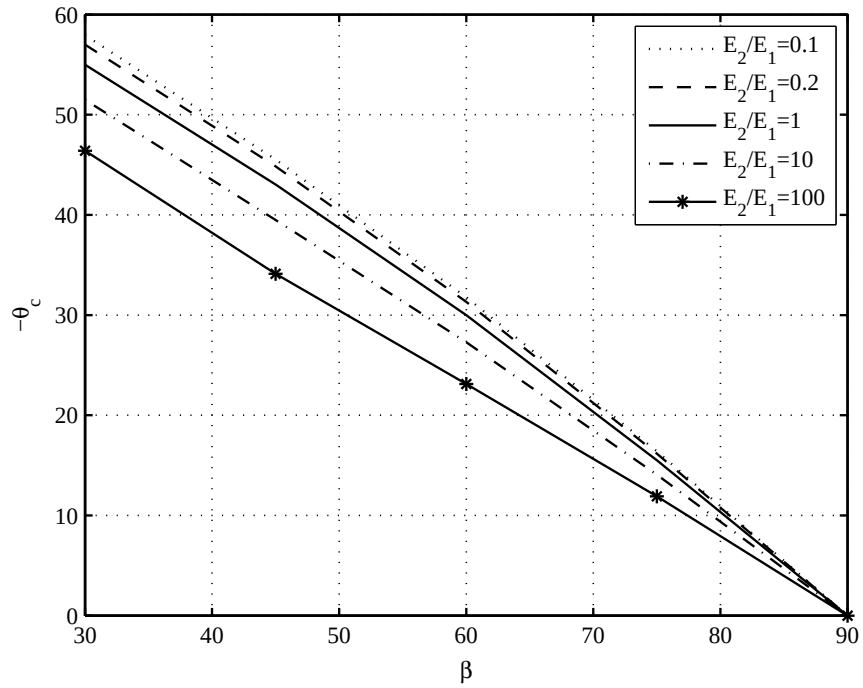


Figure 3.9. Crack initiation angles using S-criterion for different E_2/E_1 , for $a/w=0.4$

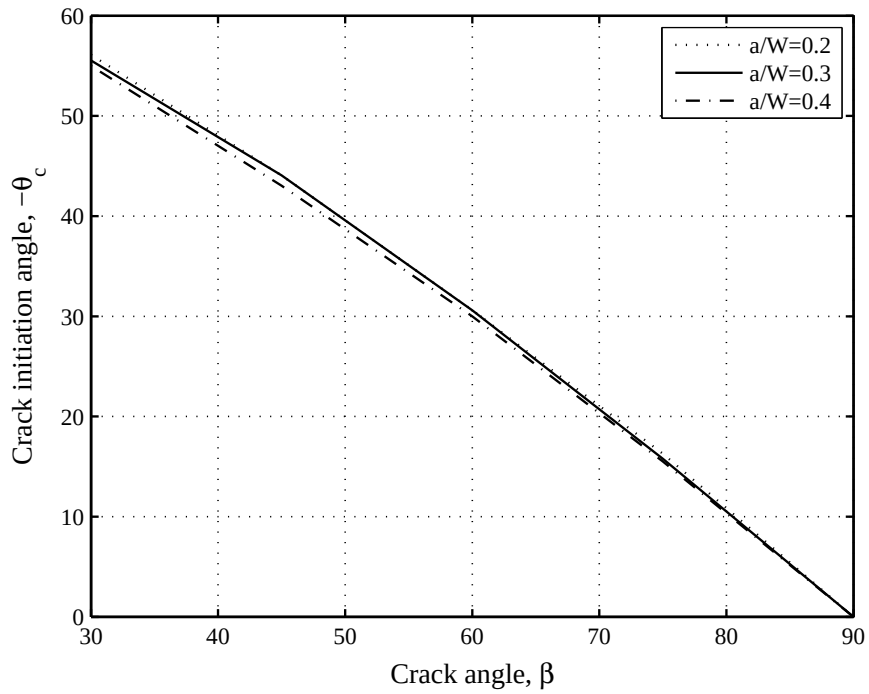


Figure 3.10. Crack initiation angles using S-criterion, for different a/w ratios, for homogeneous plate

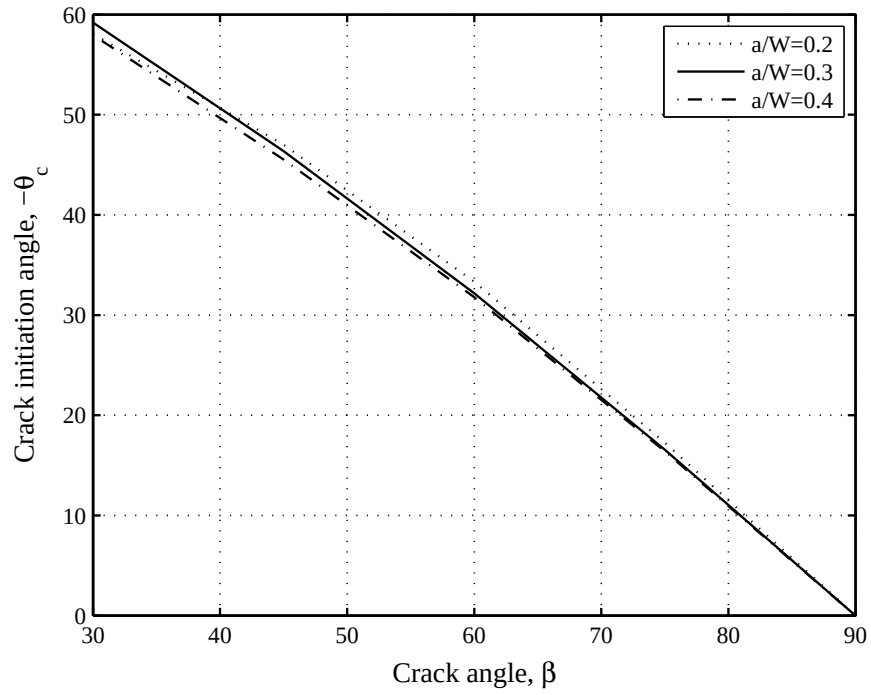


Figure 3.11. Crack initiation angles using S-criterion, for different a/w ratios, for FGM ($E_2/E_1 = 0.1$) plate

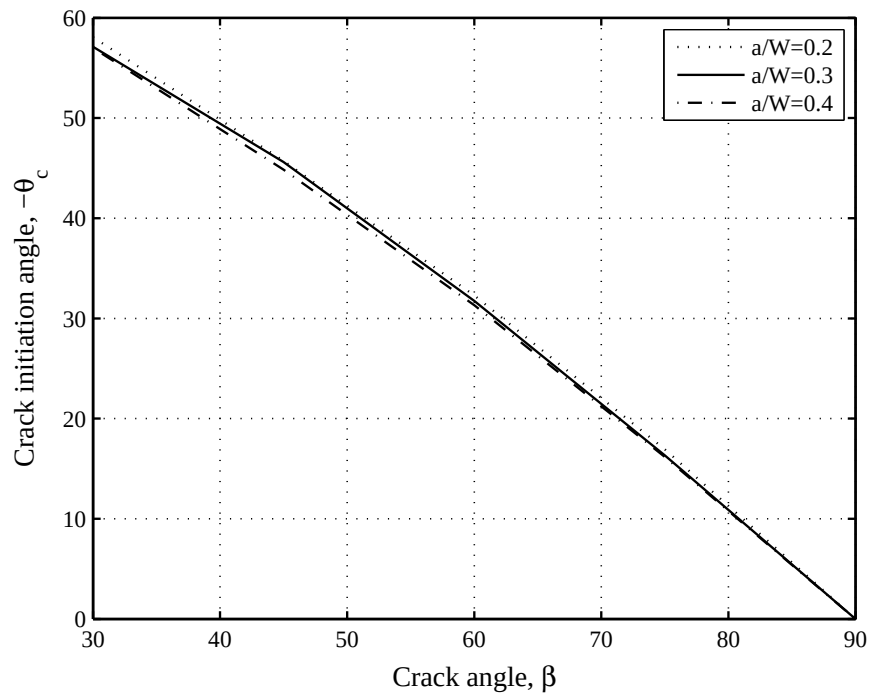


Figure 3.12. Crack initiation angles using S-criterion, for different a/w ratios, for FGM ($E_2/E_1 = 0.2$) plate

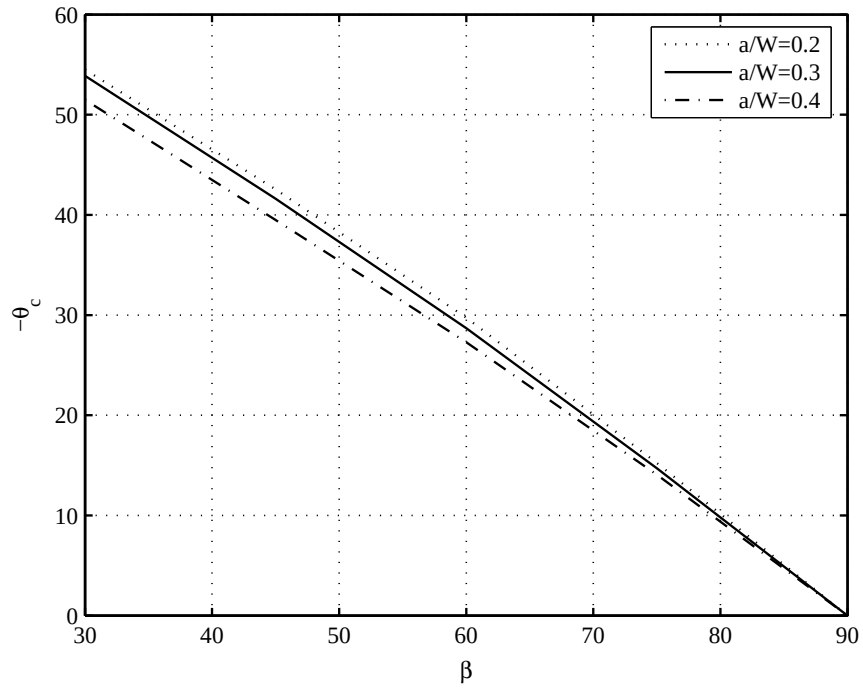


Figure 3.13. Crack initiation angles using S-criterion, for different a/w ratios, for FGM ($E_2/E_1 = 10$) plate

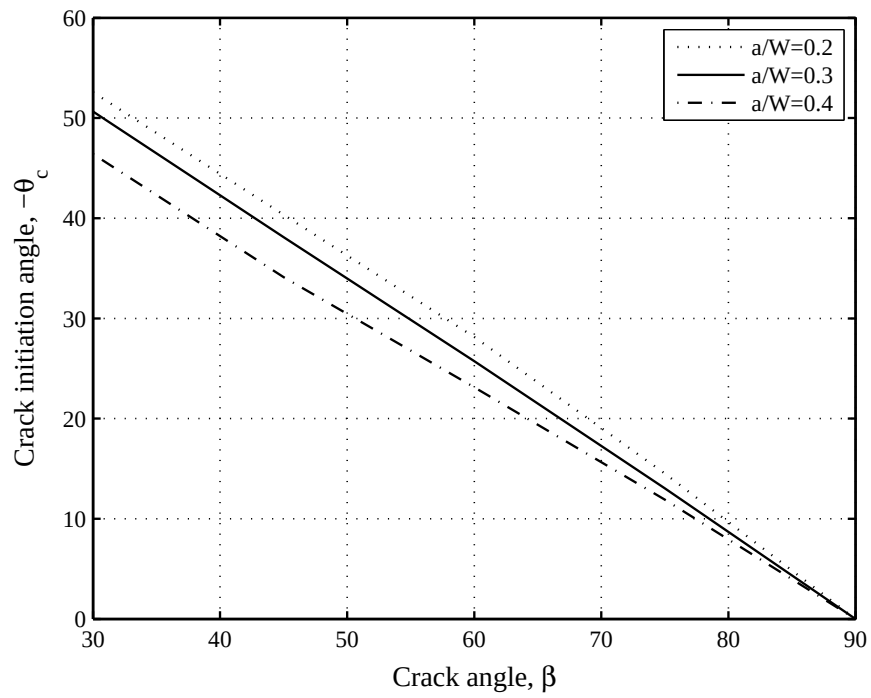


Figure 3.14. Crack initiation angles using S-criterion, for different a/w ratios, for FGM ($E_2/E_1 = 100$) plate

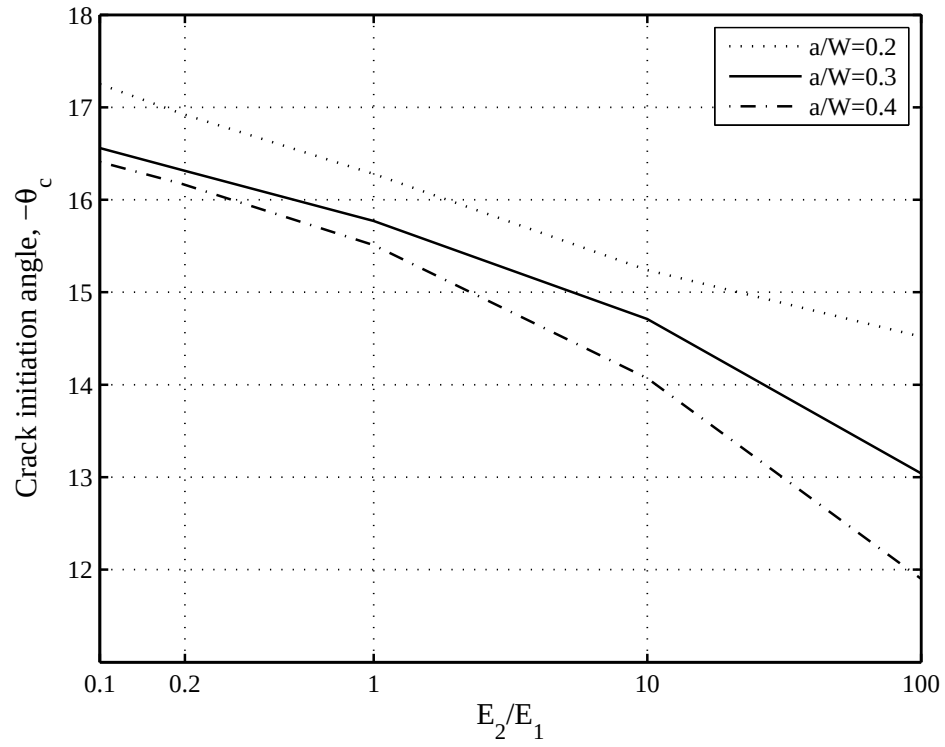


Figure 3.15. Crack initiation angles using S-criterion for different E_2/E_1 , for $\beta = 75^\circ$

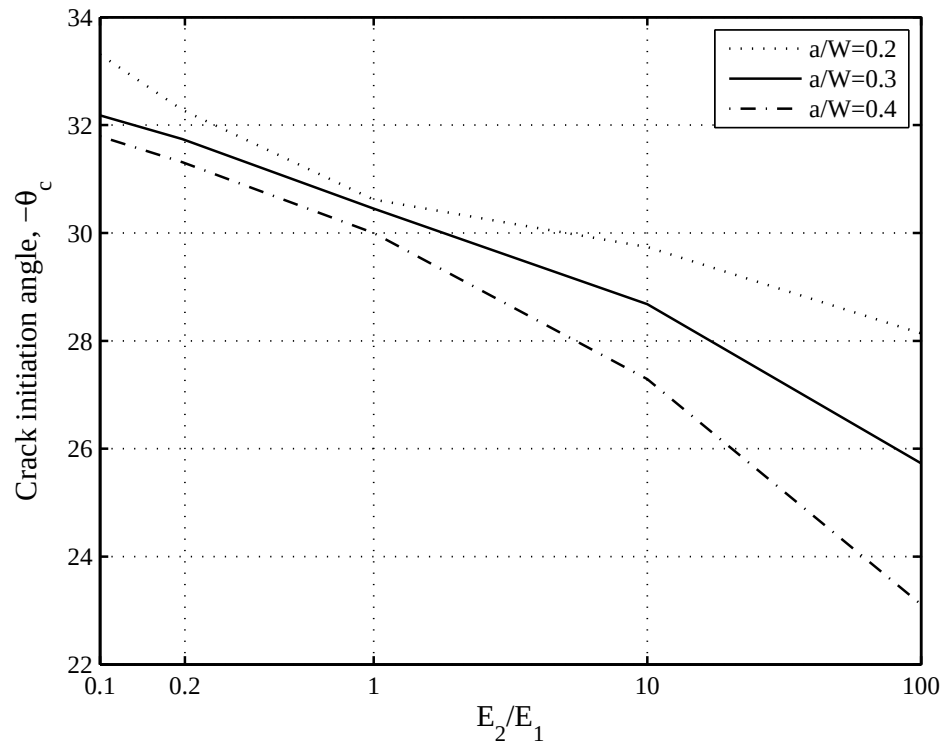


Figure 3.16. Crack initiation angles using S-criterion for different E_2/E_1 , for $\beta = 60^\circ$

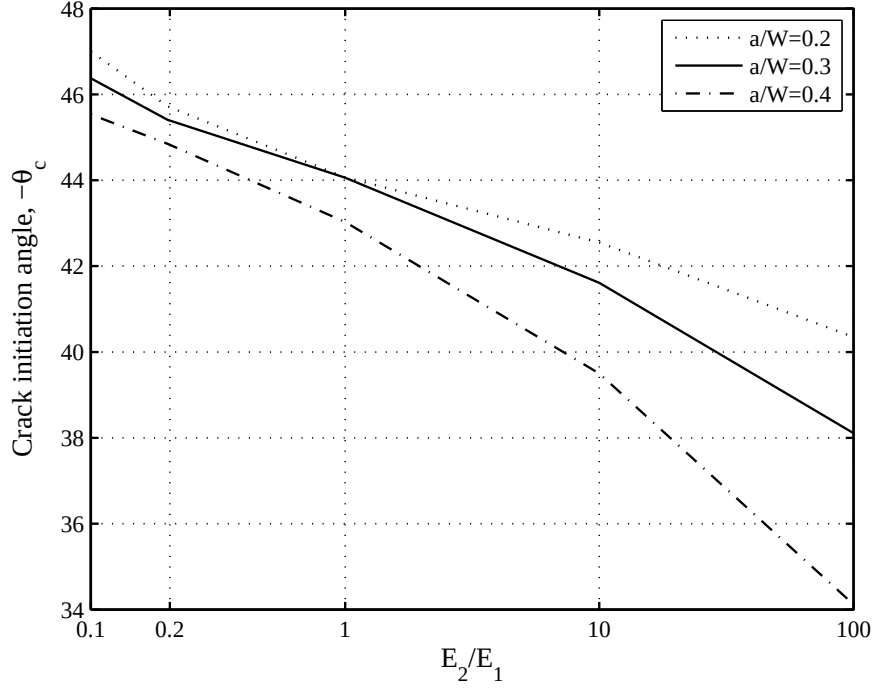


Figure 3.17. Crack initiation angles using S-criterion for different E_2/E_1 , for $\beta = 45^\circ$

3.2.2. Numerical Analysis of G-criterion

Secondly, G -Criterion is used to determine crack initiation angles. G -Criterion predicts that crack initiates when maximum energy release rate is reached. In this criterion, to determine the crack initiation angle, a small kink is extended from the crack tip, and energy release rate is calculated, for a number of kink angles. To find the maximum value of energy release rate a second degree polynomial is fitted to data obtained from analysis, and the derivative taken with respect to θ is equated to zero, gives the initiation angle according to G -criterion. Energy release rate can be calculated from displacement correlation technique, crack-closure integral, compliance change and J -integral [21]. Of these four techniques, J -integral, which can be obtained using the finite element program is used to determine energy release rate. J -integral, as defined by Rice [22] is given below.

$$J = \int_{\Gamma} U dy - \int_{\Gamma} \sigma_{ij} n_i \frac{\partial u_j}{\partial x} ds \quad (3.3)$$

In a two-dimensional body Γ is any path between the bottom crack face and the top

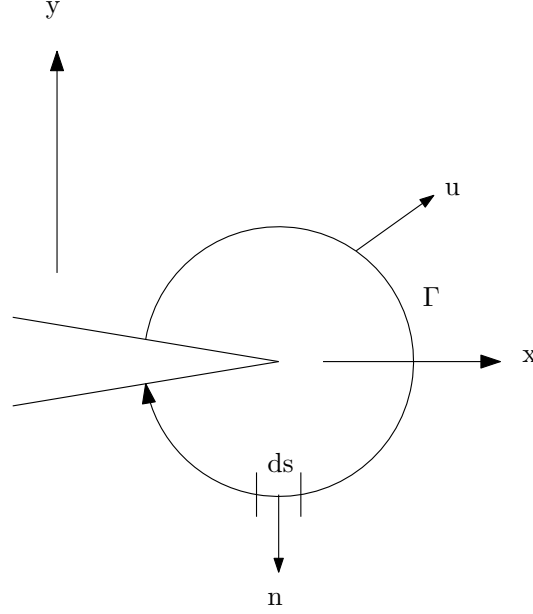


Figure 3.18. Closed contour path around the crack tip

crack face (Figure 3.18). In Equation 3.3, U is the strain energy density, u_j are the displacement components, σ_{ij} are stress components and n_i are the components of the outward unit vector normal to path Γ . For homogeneous materials, J-integral is path independent if there are no body forces, thermal strains and crack surface tractions (J-integral is constant in each contour) and, it is also known that for homogeneous materials:

$$J = G \quad (3.4)$$

But for nonhomogeneous material, in general J integral is not path independent. Gu [23] discussed a method for calculating stress intensity factors in FGMs and proposed that J-integral can provide accurate results if it is evaluated very close to the crack tip, using very fine mesh. It can be shown as,

$$J = G \quad \text{as} \quad \Gamma \rightarrow 0 \quad (3.5)$$

Anlas [24] introduced a new method to find J integral value as $\Gamma \rightarrow 0$ which does not require a very fine mesh. According to this method, J-integrals are calculated along

many contours and by fitting a fourth order polynomial to these data, a value for J as $\Gamma \rightarrow 0$, can be obtained.

In Figure 3.19, analytical and FEM results for crack initiation angles for center cracked homogeneous plate are plotted. The FEM and analytical results are close. In Figure 3.20, crack initiation angle results based on S and G criteria for center cracked homogeneous plate are plotted.

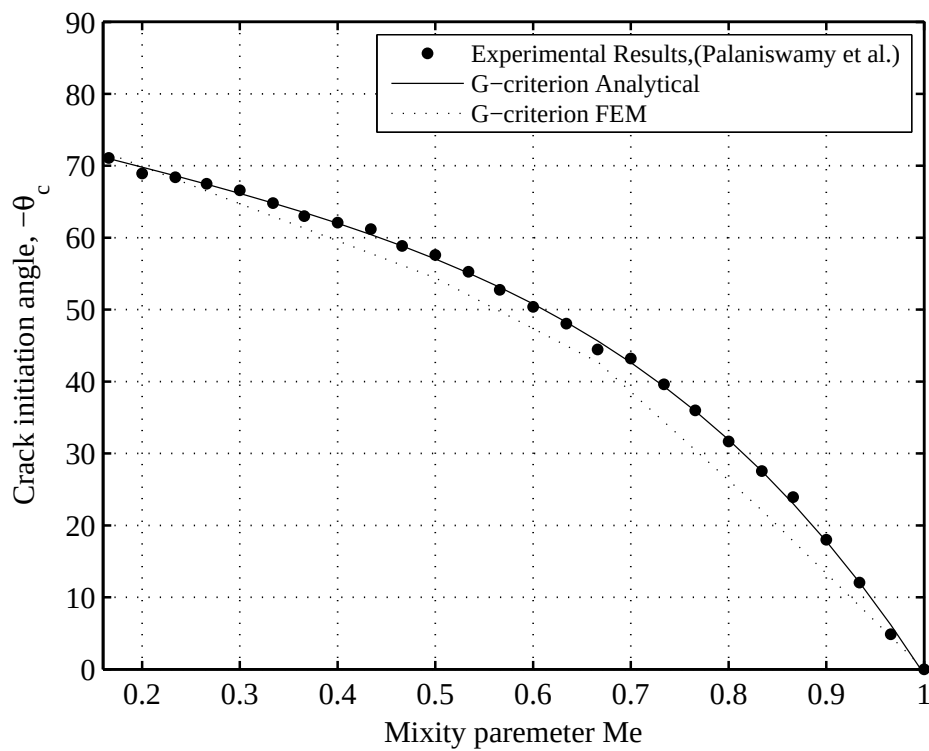


Figure 3.19. Comparison of crack initiation angle results based on G Criterion for center cracked homogeneous plate

Crack initiation angles determined by using G criterion are given in Table 3.9-3.14.

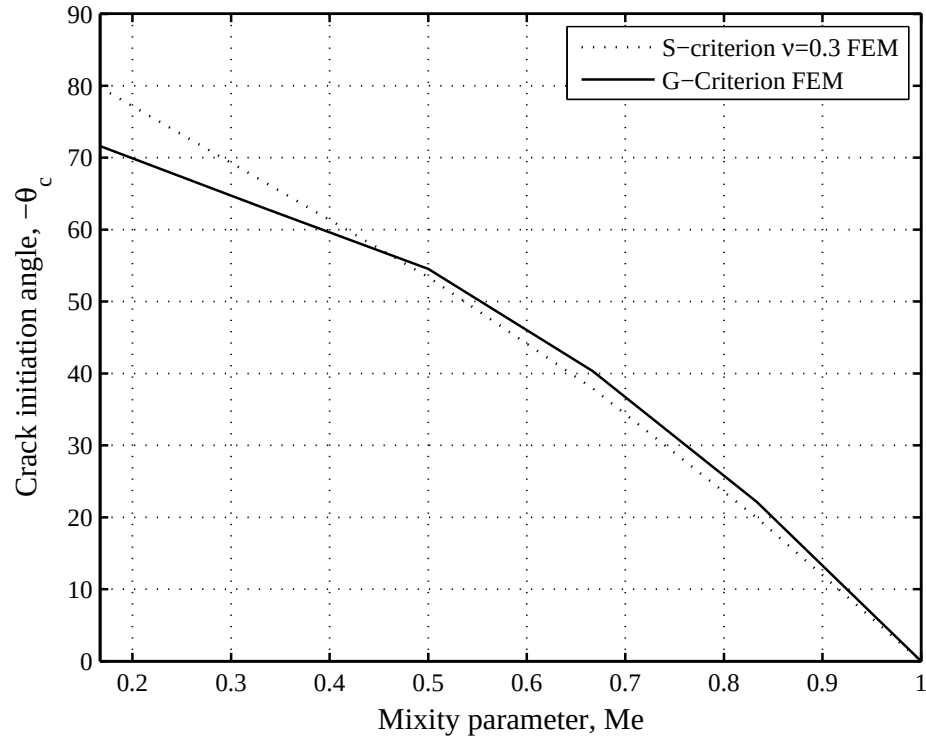


Figure 3.20. Comparison of crack initiation angle results based on S and G Criteria for center cracked homogeneous plate

Table 3.9. Crack initiation angles for edge cracked homogeneous plate using G criterion

Crack Angle, β	$-\theta_c$, $a/w=0.2$	$-\theta_c$, $a/w=0.3$	$-\theta_c$, $a/w=0.4$
[deg]	[deg]	[deg]	[deg]
30	53.67	54.41	50.95
45	44.49	42.19	42.64
60	29.99	29.87	29.48
75	15.08	15.30	15.11
90	0.00	0.00	0.00

Table 3.10. Crack initiation angles for edge cracked FGM plate ($E_2/E_1 = 0.1$) using
G criterion

Crack Angle, β	$-\theta_c$, a/w=0.2	$-\theta_c$, a/w=0.3	$-\theta_c$, a/w=0.4
[deg]	[deg]	[deg]	[deg]
30	56.76	56.10	54.45
45	44.45	45.52	43.16
60	30.81	31.10	31.05
75	15.82	16.31	16.35
90	0.00	0.00	0.00

Table 3.11. Crack initiation angles for edge cracked FGM plate ($E_2/E_1 = 0.2$) using
G criterion

Crack Angle, β	$-\theta_c$, a/w=0.2	$-\theta_c$, a/w=0.3	$-\theta_c$, a/w=0.4
[deg]	[deg]	[deg]	[deg]
30	54.81	54.47	53.12
45	43.98	43.60	42.84
60	30.28	30.89	30.65
75	15.72	16.13	16.17
90	0.00	0.00	0.00

Table 3.12. Crack initiation angles for edge cracked FGM plate ($E_2/E_1 = 10$) using G criterion

Crack Angle, β	$-\theta_c$, a/w=0.2	$-\theta_c$, a/w=0.3	$-\theta_c$, a/w=0.4
[deg]	[deg]	[deg]	[deg]
30	53.09	50.85	48.54
45	41.96	40.36	38.34
60	29.48	28.41	27.14
75	14.92	14.87	14.25
90	0.00	0.00	0.00

Table 3.13. Crack initiation angles for edge cracked FGM plate ($E_2/E_1 = 100$) using G criterion

Crack Angle, β	$-\theta_c$, a/w=0.2	$-\theta_c$, a/w=0.3	$-\theta_c$, a/w=0.4
[deg]	[deg]	[deg]	[deg]
30	51.25	47.94	43.82
45	40.30	37.22	33.54
60	27.93	25.67	23.02
75	14.43	13.21	11.98
90	0.00	0.00	0.00

Table 3.14. Crack initiation angles for center cracked homogeneous plate using G criterion

Crack Angle, β	Initiation angle, $-\theta_c$, $a/w=0.3$
[deg]	[deg]
15	71.64
30	63.00
45	54.53
60	41.17
75	22.16
90	0.00

The effect of material non-homogeneity and crack length for different crack lengths is represented in Figures 3.21-3.23. As E_2/E_1 increases, the crack initiation angle decreases, for constant crack angle β .

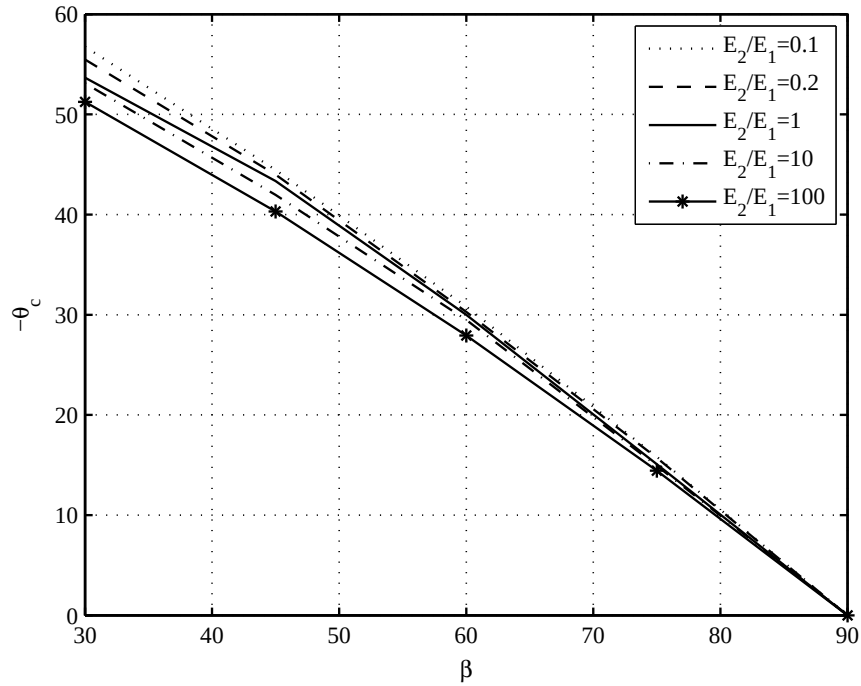


Figure 3.21. Crack initiation angles using G-criterion for different E_2/E_1 , for $a/w=0.2$

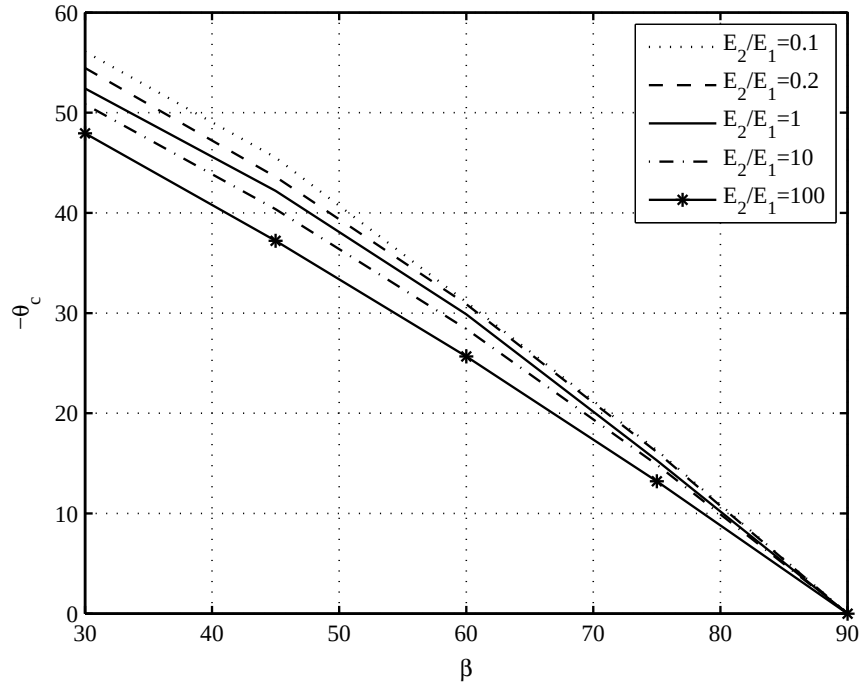


Figure 3.22. Crack initiation angles using G-criterion for different E_2/E_1 , for $a/w=0.3$

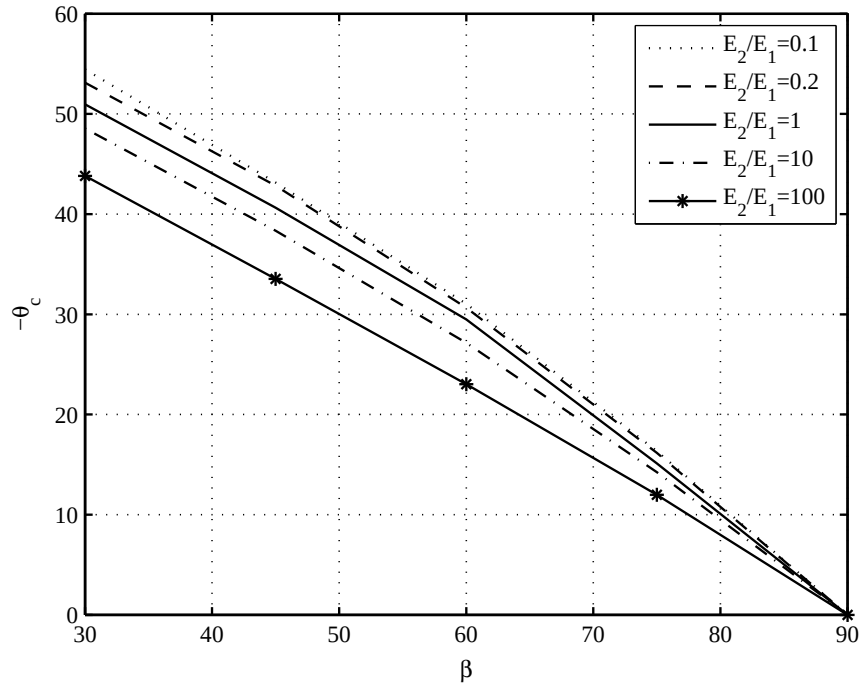


Figure 3.23. Crack initiation angles using G-criterion for different E_2/E_1 , for $a/w=0.4$

In Figures 3.10-3.14 effect of crack length on crack initiation angle can be observed. For homogeneous case, the effect of material is minor, as can be seen from

Figure 3.10, but for non-homogeneous cases, as non-homogeneity increases, the effect of crack length become more evident. For these cases, crack initiation angle decreases as the crack length increases, for the constant crack angle β .

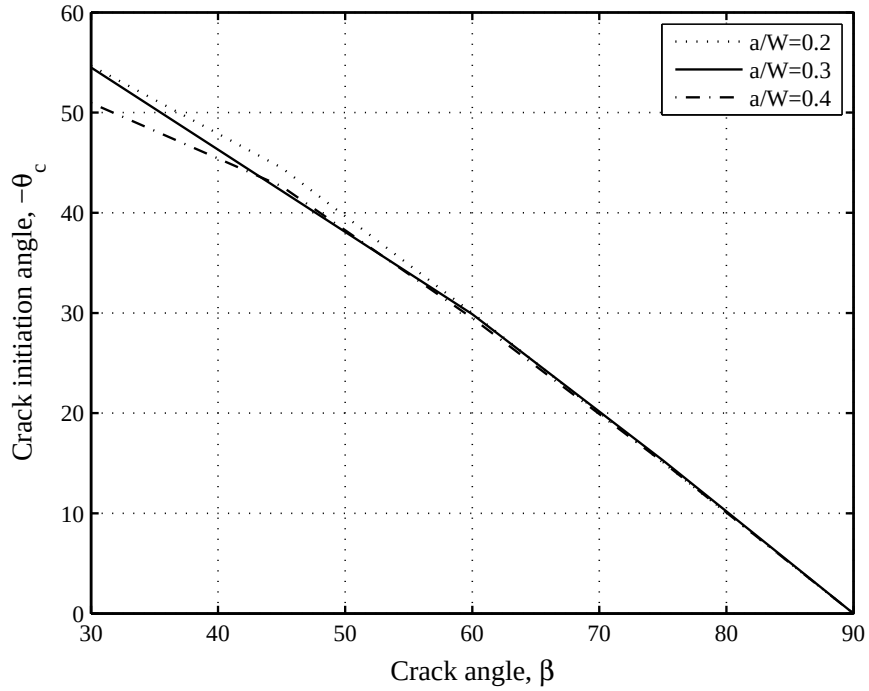


Figure 3.24. Crack initiation angles using G-criterion, for different a/w ratios, for homogeneous plate

Figures 3.29-3.30 represent the effect of crack length and the material non-homogeneity for specific crack angle, β . For the same geometry, as the material non-homogeneity increases, crack initiation angle decreases.

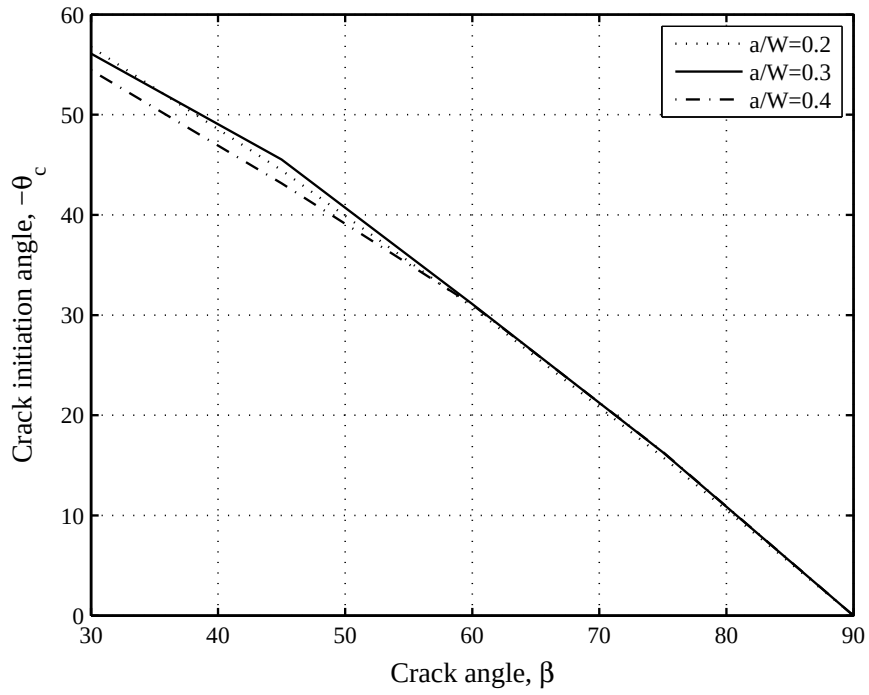


Figure 3.25. Crack initiation angles using G-criterion, for different a/w ratios, for FGM ($E_2/E_1 = 0.1$) plate

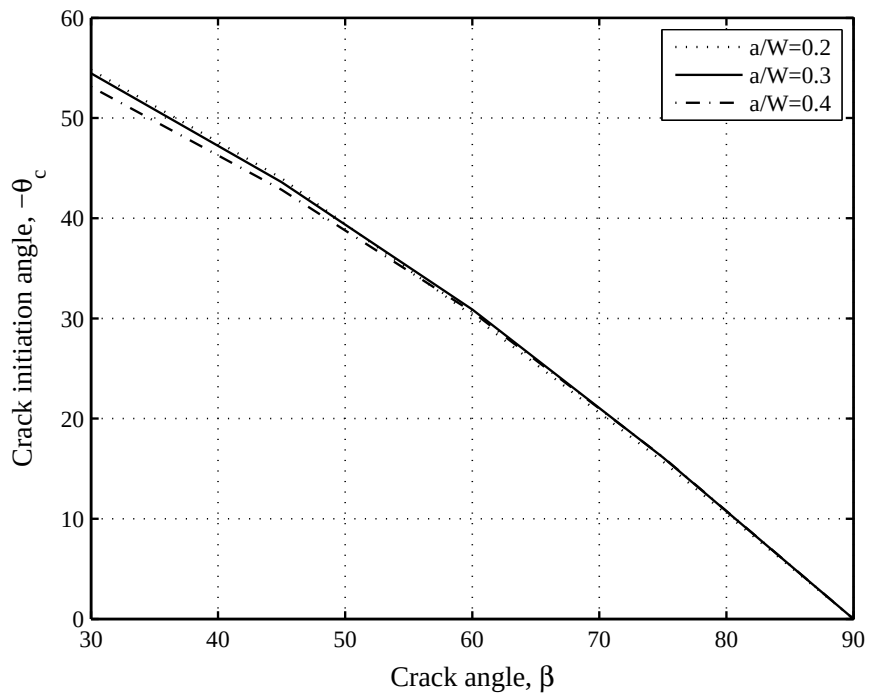


Figure 3.26. Crack initiation angles using G-criterion, for different a/w ratios, for FGM ($E_2/E_1 = 0.2$) plate

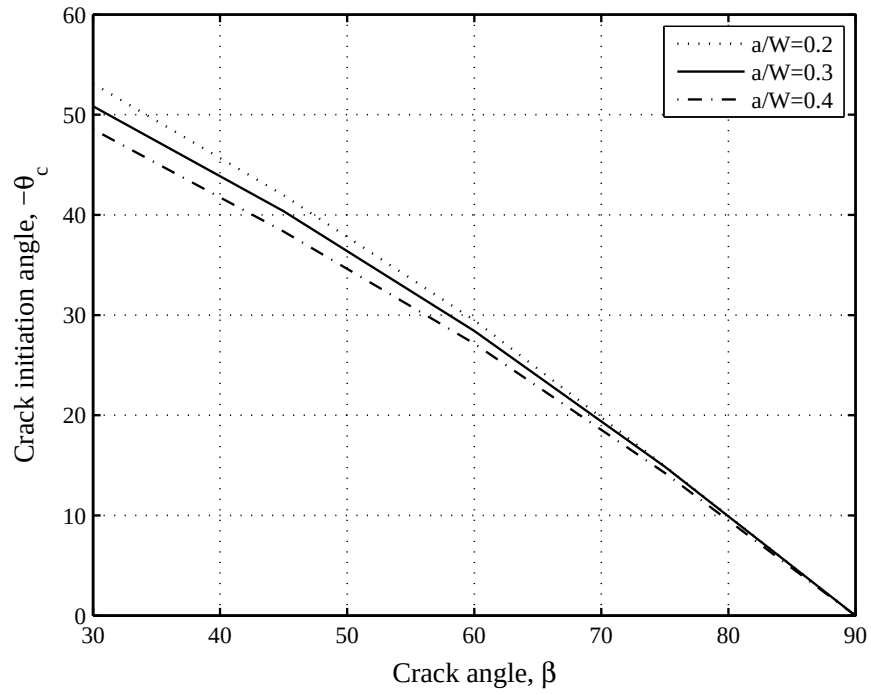


Figure 3.27. Crack initiation angles using G-criterion, for different a/w ratios, for FGM ($E_2/E_1 = 10$) plate

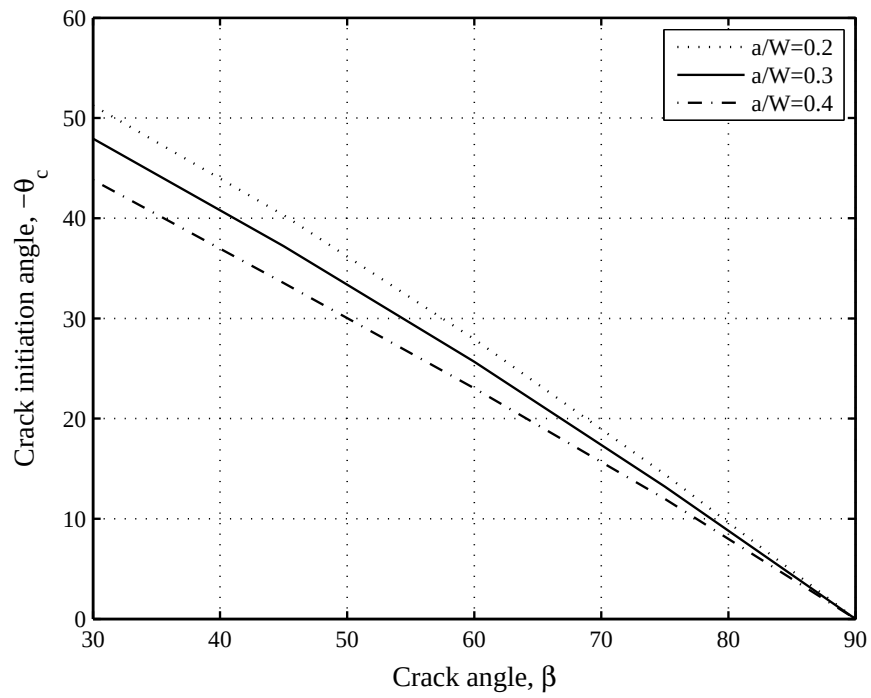


Figure 3.28. Crack initiation angles using G-criterion, for different a/w ratios, for FGM ($E_2/E_1 = 100$) plate

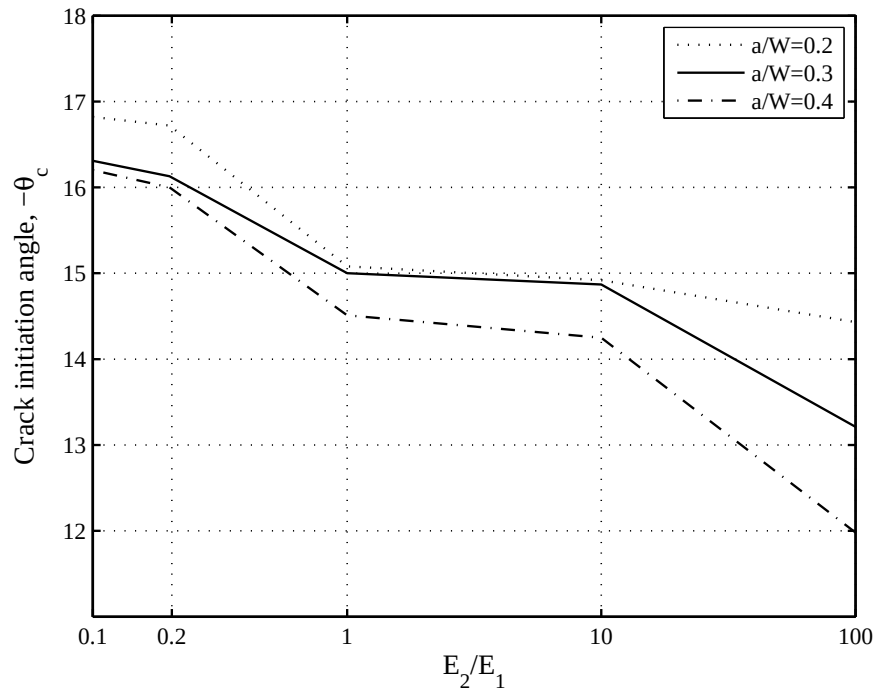


Figure 3.29. Crack initiation angles using G-criterion for different E_2/E_1 , for $\beta = 75^\circ$

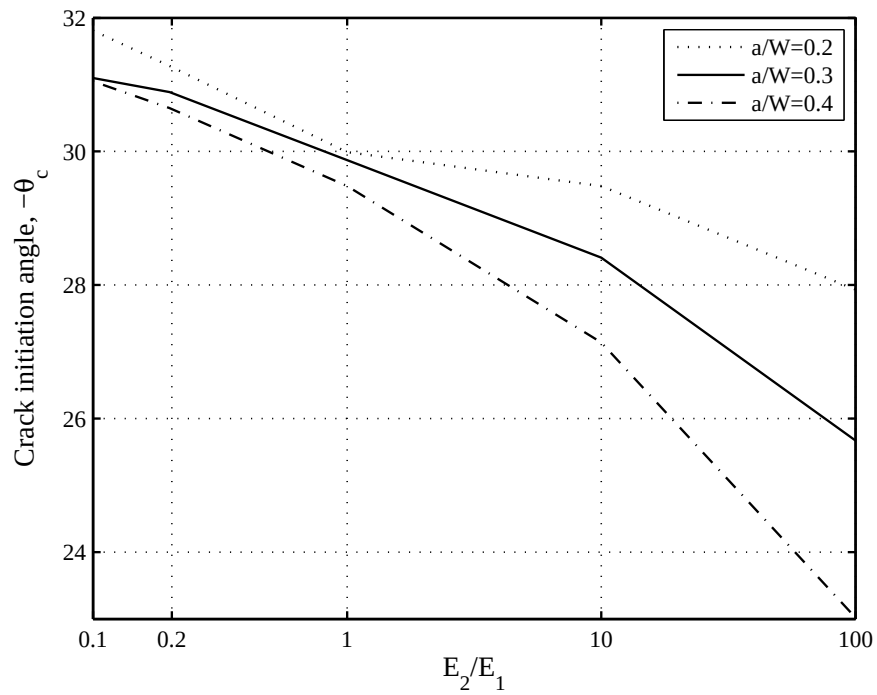


Figure 3.30. Crack initiation angles using G-criterion for different E_2/E_1 , for $\beta = 60^\circ$

3.2.3. Modified M-Criterion

Finally, M-criterion is used for numerical analysis. M-Criterion states that crack initiates where the triaxial stress is maximum at a constant radius around the crack tip. Triaxial stress is the ratio of hydrostatic stress to equivalent stress. By inserting asymptotic stress expressions into triaxial stress term and equating derivative of triaxial stress to zero gives the initiation angle analytically [16]. But while examining M-criterion in detail, it is realized that this criterion does not take effect of the T-stress into account. Therefore, we inserted the T-stress term into stress field equations (Equations 1.6-1.8) and derived the modified M-criterion. Hydrostatic and equivalent stress are:

$$M = \frac{\sigma_H}{\sigma_{eq}} \quad (3.6)$$

$$\sigma_H = \frac{1}{2\sqrt{2\pi r}} \left[2K_I \cos \frac{\theta}{2} - 2K_{II} \sin \frac{\theta}{2} + T\sqrt{2\pi r} \right] \quad (3.7)$$

$$\begin{aligned} \sigma_{eq} = & \frac{1}{\sqrt{2\pi r}} \left[K_I^2 \left(4 \cos^2 \frac{\theta}{2} 3 \cos^4 \frac{\theta}{2} \right) + K_{II}^2 \left(9 \cos^4 \frac{\theta}{2} - 10 \cos^2 \frac{\theta}{2} + 4 \right) \right. \\ & + K_I K_{II} \left(12 \sin \frac{\theta}{2} \cos^3 \frac{\theta}{2} - 8 \sin \frac{\theta}{2} \cos \frac{\theta}{2} \right) \\ & + T\sqrt{2\pi r} \left(K_I \left(4 \cos \frac{\theta}{2} + 12 \cos^5 \frac{\theta}{2} - 15 \cos^3 \frac{\theta}{2} \right) \right. \\ & \left. \left. + K_{II} \left(-12 \cos^4 \frac{\theta}{2} + 9 \sin \frac{\theta}{2} \cos^2 \frac{\theta}{2} \right) \right) + (T\sqrt{2\pi r})^2 \right]^{\frac{1}{2}} \quad (3.8) \end{aligned}$$

In order to understand the effect of T -stress and r_c , these terms are normalized. The non-dimensionless form of T-stress is represented as B .

$$B = \frac{T\sqrt{\pi a}}{K_{eff}} \quad (3.9)$$

where a is the crack length and K_{eff} is the effective stress intensity factor which can be defined as:

$$K_{eff} = \sqrt{(K_I^2 + K_{II}^2)} \quad (3.10)$$

And normalized r_c is represented as α .

$$\alpha = \sqrt{\frac{2r_c}{a}} \quad (3.11)$$

By inserting B , α into Equations 3.7-3.8 and applying M-criterion we get;

$$\frac{\partial M}{\partial \theta} = 0 \quad (3.12)$$

$$\frac{\frac{3}{4} \cos \frac{\theta}{2} \left((\mu^2 + 1)[M_1\mu + M_2](B\alpha)^2 + (\sqrt{\mu^2 + 1})[M_3\mu^2 + M_4\mu + M_5](Ba) \right)}{\left((\mu^2 + 1)(B\alpha)^2 + \sqrt{\mu^2 + 1}[M_{10}\mu + M_{11}](B\alpha) + [M_{12}\mu^2 + M_{13}\mu + M_{14}] \right)^{\frac{3}{2}}} + \frac{\frac{3}{4} \cos \frac{\theta}{2} \left([M_6\mu^3 + M_7\mu^2 + M_8\mu + M_9] \right)}{\left((\mu^2 + 1)(B\alpha)^2 + \sqrt{\mu^2 + 1}[M_{10}\mu + M_{11}](B\alpha) + [M_{12}\mu^2 + M_{13}\mu + M_{14}] \right)^{\frac{3}{2}}} = 0 \quad (3.13)$$

where $\mu = \frac{K_I}{K_{II}}$ and $M_1, M_2, M_3, M_4, M_5, M_6, M_7, M_8, M_9, M_{10}, M_{11}, M_{12}, M_{13}, M_{14}$ are angular functions given below.

$$M_1 = \left(\frac{-15}{2} \sin \frac{\theta_c}{2} \cos \frac{\theta_c}{2} + 10 \sin \frac{\theta_c}{2} \cos^3 \frac{\theta_c}{2} \right), \quad (3.14)$$

$$M_2 = \left(10 \cos^4 \frac{\theta_c}{2} - \frac{25}{2} \cos^2 \frac{\theta_c}{2} + 3 \right), \quad (3.15)$$

$$M_3 = \left(-7 \sin \frac{\theta_c}{2} \cos^2 \frac{\theta_c}{2} + 12 \sin \frac{\theta_c}{2} \cos^4 \frac{\theta_c}{2} \right), \quad (3.16)$$

$$M_4 = \left(-44 \cos^3 \frac{\theta_c}{2} + 24 \cos^5 \frac{\theta_c}{2} + 21 \cos \frac{\theta_c}{2} \right), \quad (3.17)$$

$$M_5 = \left(-8 \sin \frac{\theta_c}{2} + 25 \sin \frac{\theta_c}{2} \cos^2 \frac{\theta_c}{2} - 12 \sin \frac{\theta_c}{2} \cos^4 \frac{\theta_c}{2} \right), \quad (3.18)$$

$$M_6 = \left(-2 \sin \frac{\theta_c}{2} \cos^3 \frac{\theta_c}{2} \right), \quad (3.19)$$

$$M_7 = \left(-10 \cos^4 \frac{\theta_c}{2} + 8 \cos^2 \frac{\theta_c}{2} \right), \quad (3.20)$$

$$M_8 = \left(14 \sin \frac{\theta_c}{2} \cos^3 \frac{\theta_c}{2} - 12 \sin \frac{\theta_c}{2} \cos \frac{\theta_c}{2} \right), \quad (3.21)$$

$$M_9 = \left(6 \cos^4 \frac{\theta_c}{2} - 12 \cos^2 \frac{\theta_c}{2} + 4 \right), \quad (3.22)$$

$$M_{10} = \left(4 \cos \frac{\theta_c}{2} + 12 \cos^5 \frac{\theta_c}{2} - 15 \cos^3 \frac{\theta_c}{2} \right), \quad (3.23)$$

$$M_{11} = \left(-12 \sin \frac{\theta_c}{2} \cos^4 \frac{\theta_c}{2} + 9 \sin \frac{\theta_c}{2} \cos^2 \frac{\theta_c}{2} - 4 \sin \frac{\theta_c}{2} \right), \quad (3.24)$$

$$M_{12} = \left(4 \cos^2 \frac{\theta_c}{2} - 3 \cos^4 \frac{\theta_c}{2} \right), \quad (3.25)$$

$$M_{13} = \left(12 \sin \frac{\theta_c}{2} \cos^4 \frac{\theta_c}{2} - 8 \sin \frac{\theta_c}{2} \cos \frac{\theta_c}{2} \right), \quad (3.26)$$

$$M_{14} = \left(-10 \cos^2 \frac{\theta_c}{2} + 9 \cos^4 \frac{\theta_c}{2} + 4 \right). \quad (3.27)$$

Solutions of Equation 3.13 are illustrated in the Figure 3.31 for different $B\alpha$ values.

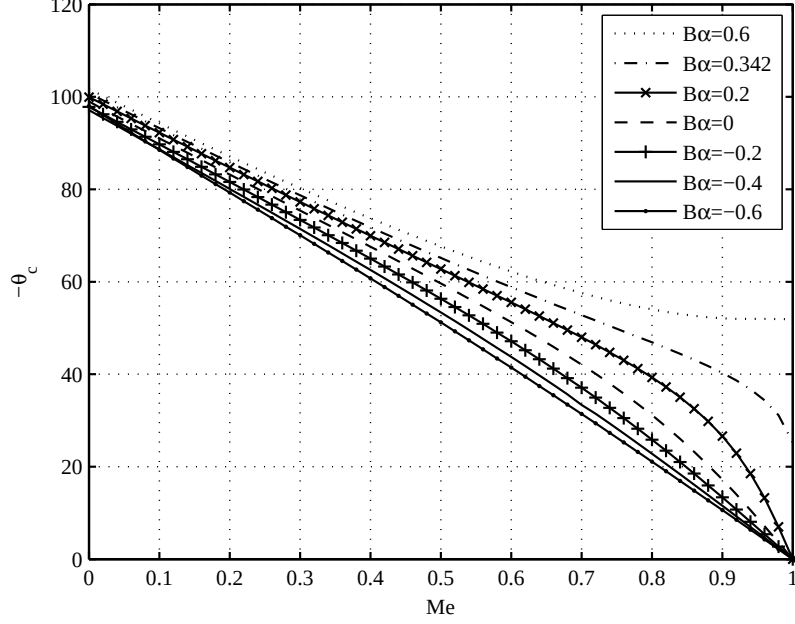


Figure 3.31. Crack initiation angles for different T-stress and r_c values

From Figure 3.31, it can be seen that as $B\alpha$ increases, the crack initiation angle increases. If we assume a constant r_c (α), for a constant Me (Mixity parameter), for positive values of T-stress (B), as T-stress increases, the crack initiation angle increases and for negative values of T-stress, as T-stress decreases, the crack initiation angle decreases. As can be seen from Figure 3.31, for constant r_c , The effect of T-stress is relatively small in Mode II dominant region. And T-stress has a drastic effect for the values of $0.4 < Me < 0.8$. In Mode I, it is also clear that, for constant r_c , T stress has no influence on crack initiation angle for $B\alpha < 0.342$. But for $B\alpha > 0.342$, it is found that the crack initiation angle is different from 0° . For example, for $B\alpha = 0.6$, the predicted fracture angle is $\pm 51^\circ$. These maxima points can be observed from Figure 3.32 where M vs θ is plotted.

For Mode I loading, $\theta_c = 0$, when the second derivative of the triaxial stress is solved, it is found that to provide the condition $\partial^2 \sigma_{\theta\theta} / \partial \theta^2 < 0$, $B\alpha$ must be less than 0.342. The fracture angles for Mode I loading, for different values of $B\alpha$, in the range of $0 < B\alpha < 2$ is shown in the Figure 3.33. From Figure 3.33 it is observed that the

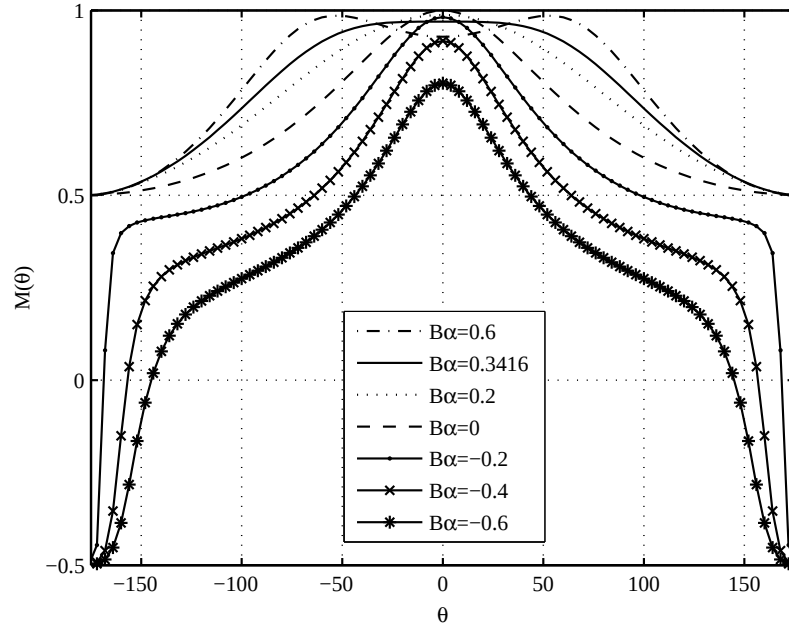


Figure 3.32. M distributions for different $B\alpha$ values under Mode I loading

analytical fracture angles has two values which are different than 0° for $B\alpha > 0.342$.

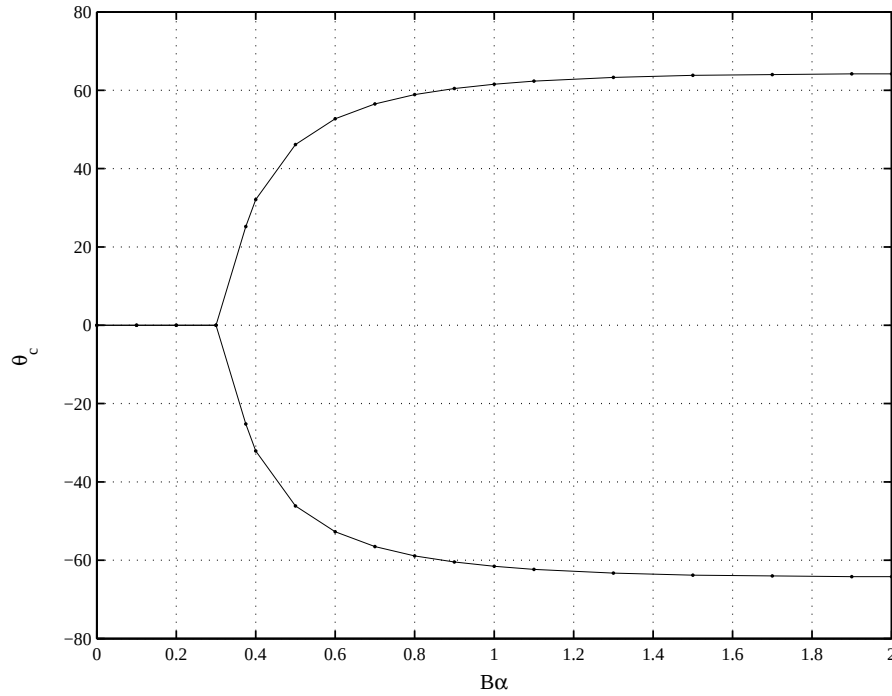


Figure 3.33. Variation of the fracture angle as a function of $B\alpha$ for Mode I loading

To understand the physical meaning of this case in Mode I loading, numerical and analytical values of T-stress are evaluated. Numerical $B\alpha$ and T-stress values are obtained from FE analysis and it is observed that T-stress is negative and $B\alpha$ never

becomes greater than 0.342 (Figures 3.34-3.35). Analytically, T stress is equal to $-\sigma$. It can be concluded that, for Mode I loading, discussing the crack initiation angle results for $B\alpha > 0.342$ is redundant because $B\alpha$ values are always negative in real applications.

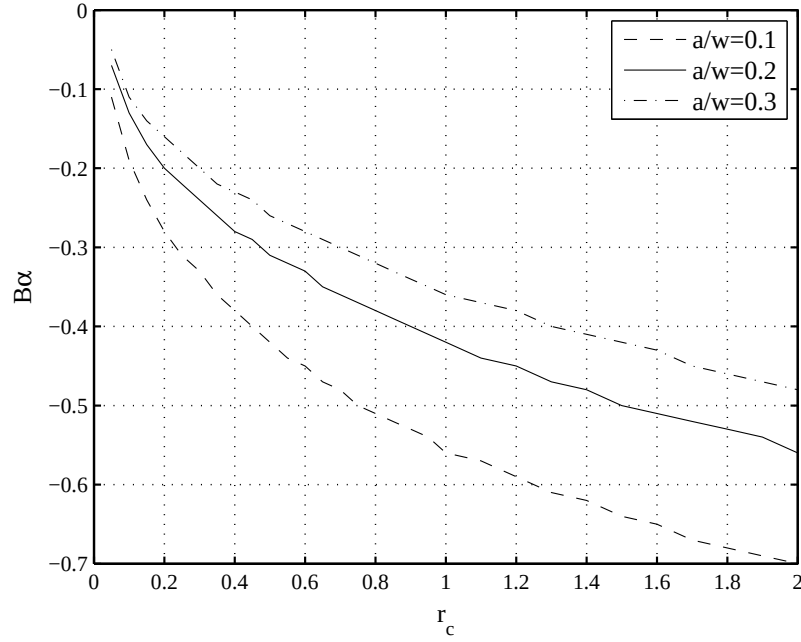


Figure 3.34. $B\alpha$ values for Mode I loading

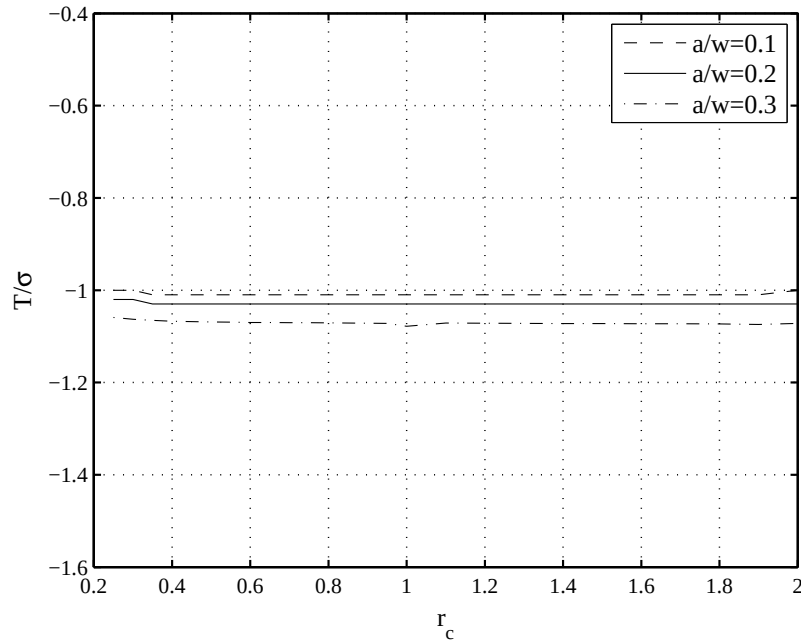


Figure 3.35. T-stress values for Mode I loading

To determine the crack initiation angle using M-criterion numerically, first triaxial stress is determined as a function of θ with different radii around the crack tip. For different radii triaxial stress can be determined in terms of σ_{xx} , σ_{yy} and σ_{eq} by using FE program for all nodes on the circular path (Figure 3.36). Then to find the maximum value of triaxial stress, a second degree polynomial is fitted to the triaxial stress plot its derivative is set to zero. The angle that corresponds to the maximum value of triaxial stress is obtained.

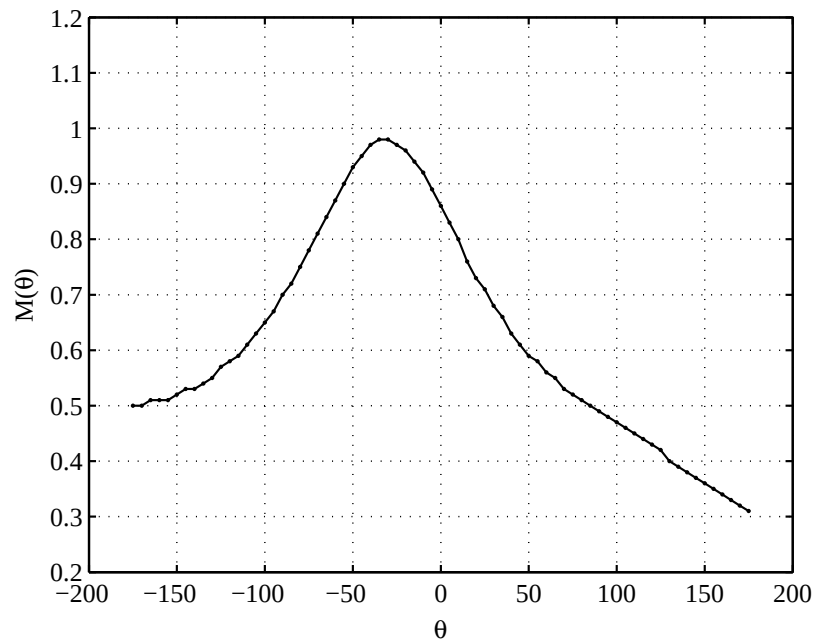


Figure 3.36. Triaxial stress around the crack tip

To verify numerical method, homogeneous center crack analysis is made and numerical results are compared with analytical ones. If we rewrite the modified M-criterion Equation 3.13 by combining Equations 1.18-1.22, Figure 3.37 can be obtained for different α values.

In Figure 3.37, the effect of r_c can be seen clearly in modified M-criterion. For α values different than zero gives dissimilar results compared to conventional M-criterion ($\alpha=0$) and this proves that the effect of T stress is important and is not negligible. In Figure 3.37 there is a intersection point at $Me=0.5$ ($\beta = 45^\circ$). For the case, $\beta < 45^\circ$ and $T > 0$, as r_c increases, crack initiation angle increases and for the case, $\beta > 45^\circ$ and

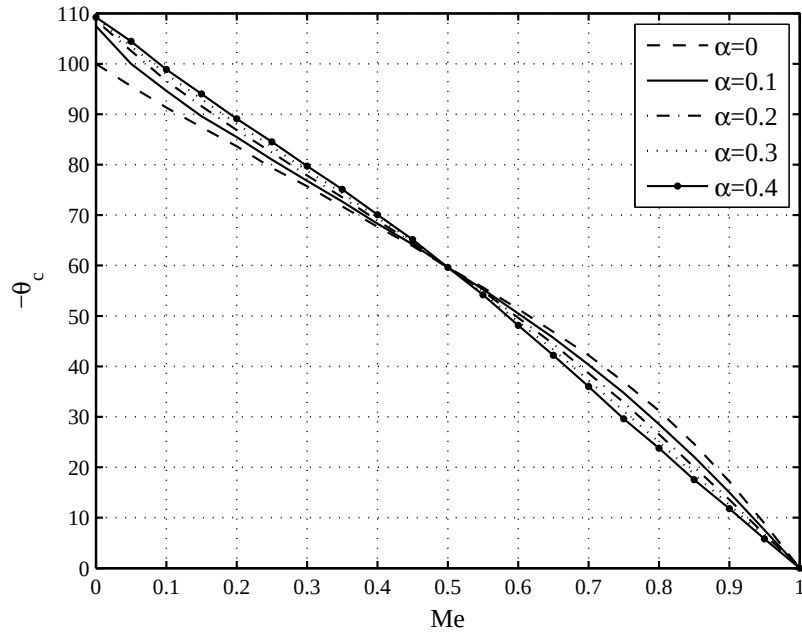


Figure 3.37. Analytical crack initiation angles for center cracked plate

$T < 0$, as r_c increases, crack initiation angle decreases. The comparison of analytical and numerical results for centered crack plate is given in Figures 3.38-3.40. FE analysis are in a good agreement with analytical results. FE results also support the opinion that the T-stress effects the crack initiation angle.

The crack initiation angles determined for $r_c/a = 0.05, 0.075$ and 0.1 for edge cracked homogeneous and FGM plates for different crack geometries are tabulated in the Tables 3.15-3.29.

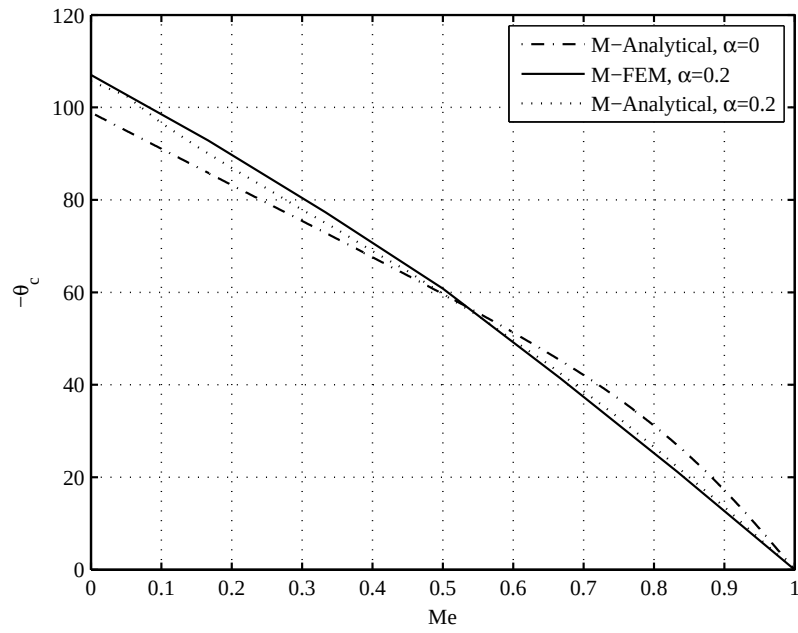


Figure 3.38. Comparison of analytical and numerical crack initiation angles for $\alpha = 0.2$, for center cracked homogeneous plate

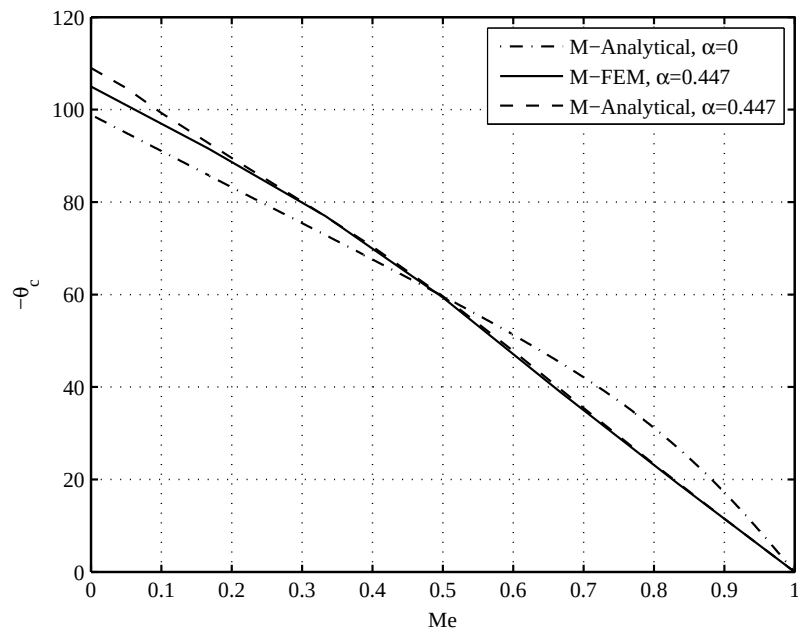


Figure 3.39. Comparison of analytical and numerical crack initiation angles for $\alpha = 0.447$, for center cracked homogeneous plate

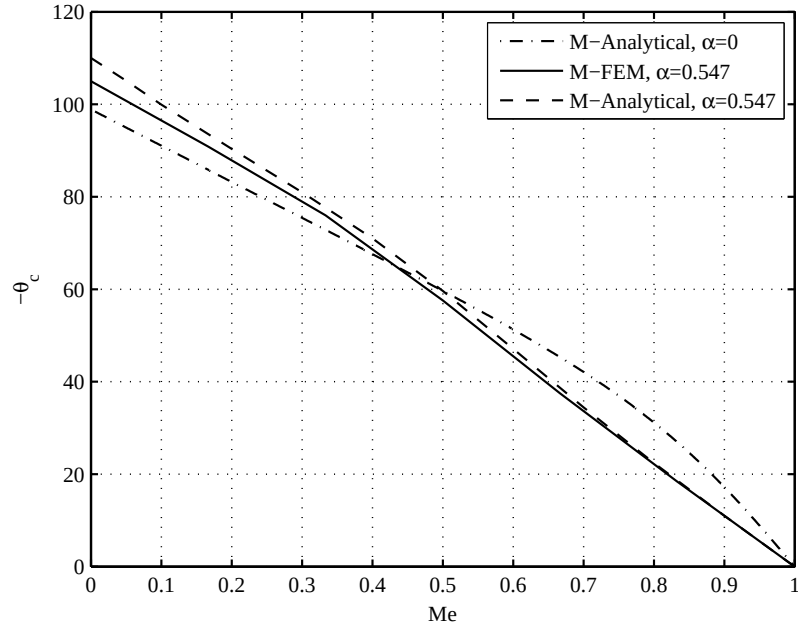


Figure 3.40. Comparison of analytical and numerical crack initiation angles for $\alpha = 0.547$, for center cracked homogeneous plate

Table 3.15. Crack initiation angles for edge cracked homogeneous plate for $a/w = 0.2$ using M criterion

Crack Angle, β	$-\theta_c, r_c/a = 0.05$	$-\theta_c, r_c/a = 0.075$	$-\theta_c, r_c/a = 0.1$
[deg]	[deg]	[deg]	[deg]
30	59.53	60.45	60.32
45	47.13	46.86	46.43
60	32.38	31.48	31.17
75	16.03	15.79	15.57
90	0.00	0.00	0.00

Table 3.16. Crack initiation angles for edge cracked FGM plate ($E_2/E_1 = 0.1$) for $a/w = 0.2$ using M criterion

Crack Angle, β	$-\theta_c, r_c/a = 0.05$	$-\theta_c, r_c/a = 0.075$	$-\theta_c, r_c/a = 0.1$
[deg]	[deg]	[deg]	[deg]
30	65.82	64.44	63.93
45	49.38	48.34	47.64
60	32.63	31.93	31.35
75	16.23	15.86	15.53
90	0.00	0.00	0.00

Table 3.17. Crack initiation angles for edge cracked FGM plate ($E_2/E_1 = 0.2$) for $a/w = 0.2$ using M criterion

Crack Angle, β	$-\theta_c, r_c/a = 0.05$	$-\theta_c, r_c/a = 0.075$	$-\theta_c, r_c/a = 0.1$
[deg]	[deg]	[deg]	[deg]
30	63.56	62.90	62.80
45	48.22	47.57	47.13
60	32.35	31.79	31.32
75	16.11	15.80	15.51
90	0.00	0.00	0.00

Table 3.18. Crack initiation angles for edge cracked FGM plate ($E_2/E_1 = 10$) for $a/w = 0.2$ using M criterion

Crack Angle, β	$-\theta_c, r_c/a = 0.05$	$-\theta_c, r_c/a = 0.075$	$-\theta_c, r_c/a = 0.1$
[deg]	[deg]	[deg]	[deg]
30	60.60	60.67	60.99
45	45.99	45.94	45.92
60	31.04	30.85	30.67
75	15.68	15.52	15.36
90	0.00	0.00	0.00

Table 3.19. Crack initiation angles for edge cracked FGM plate ($E_2/E_1 = 100$) for $a/w = 0.2$ using M criterion

Crack Angle, β	$-\theta_c, r_c/a = 0.05$	$-\theta_c, r_c/a = 0.075$	$-\theta_c, r_c/a = 0.1$
[deg]	[deg]	[deg]	[deg]
30	59.38	59.57	59.96
45	44.51	44.60	44.68
60	29.95	29.85	29.76
75	15.50	15.07	14.99
90	0.00	0.00	0.00

Table 3.20. Crack initiation angles for edge cracked homogeneous plate for $a/w = 0.3$
using M criterion

Crack Angle, β	$-\theta_c, r_c/a = 0.05$	$-\theta_c, r_c/a = 0.075$	$-\theta_c, r_c/a = 0.1$
[deg]	[deg]	[deg]	[deg]
30	59.09	59.75	60.01
45	45.52	45.46	45.52
60	30.96	30.66	30.47
75	15.67	15.38	15.21
90	0.00	0.00	0.00

Table 3.21. Crack initiation angles for edge cracked FGM plate ($E_2/E_1 = 0.1$) for
 $a/w = 0.3$ using M criterion

Crack Angle, β	$-\theta_c, r_c/a = 0.05$	$-\theta_c, r_c/a = 0.075$	$-\theta_c, r_c/a = 0.1$
[deg]	[deg]	[deg]	[deg]
30	62.67	63.14	62.70
45	48.07	47.47	46.64
60	32.15	31.59	30.80
75	16.09	15.74	15.26
90	0.00	0.00	0.00

Table 3.22. Crack initiation angles for edge cracked FGM plate ($E_2/E_1 = 0.2$) for $a/w = 0.3$ using M criterion

Crack Angle, β	$-\theta_c, r_c/a = 0.05$	$-\theta_c, r_c/a = 0.075$	$-\theta_c, r_c/a = 0.1$
[deg]	[deg]	[deg]	[deg]
30	62.07	61.97	61.94
45	47.36	46.97	46.45
60	30.90	31.45	30.82
75	16.01	15.73	16.04
90	0.00	0.00	0.00

Table 3.23. Crack initiation angles for edge cracked FGM plate ($E_2/E_1 = 10$) for $a/w = 0.3$ using M criterion

Crack Angle, β	$-\theta_c, r_c/a = 0.05$	$-\theta_c, r_c/a = 0.075$	$-\theta_c, r_c/a = 0.1$
[deg]	[deg]	[deg]	[deg]
30	58.38	58.73	59.49
45	44.38	44.47	44.67
60	30.17	30.07	29.97
75	15.32	15.21	15.05
90	0.00	0.00	0.00

Table 3.24. Crack initiation angles for edge cracked FGM plate ($E_2/E_1 = 100$) for $a/w = 0.3$ using M criterion

Crack Angle, β	$-\theta_c, r_c/a = 0.05$	$-\theta_c, r_c/a = 0.075$	$-\theta_c, r_c/a = 0.1$
[deg]	[deg]	[deg]	[deg]
30	55.94	56.39	57.35
45	41.75	41.98	42.40
60	28.16	28.20	28.33
75	14.28	14.24	14.25
90	0.00	0.00	0.00

Table 3.25. Crack initiation angles for edge cracked homogeneous plate for $a/w = 0.4$ using M criterion

Crack Angle, β	$-\theta_c, r_c/a = 0.05$	$-\theta_c, r_c/a = 0.075$	$-\theta_c, r_c/a = 0.1$
[deg]	[deg]	[deg]	[deg]
30	59.11	59.62	60.14
45	45.15	45.16	45.22
60	30.73	30.45	30.28
75	15.58	15.32	15.17
90	0.00	0.00	0.00

Table 3.26. Crack initiation angles for edge cracked FGM plate ($E_2/E_1 = 0.1$) for $a/w = 0.4$ using M criterion

Crack Angle, β	$-\theta_c, r_c/a = 0.05$	$-\theta_c, r_c/a = 0.075$	$-\theta_c, r_c/a = 0.1$
[deg]	[deg]	[deg]	[deg]
30	61.81	61.49	41.33
45	46.81	46.55	45.69
60	31.42	30.70	30.55
75	15.74	15.30	15.00
90	0.00	0.00	0.00

Table 3.27. Crack initiation angles for edge cracked FGM plate ($E_2/E_1 = 0.2$) for $a/w = 0.4$ using M criterion

Crack Angle, β	$-\theta_c, r_c/a = 0.05$	$-\theta_c, r_c/a = 0.075$	$-\theta_c, r_c/a = 0.1$
[deg]	[deg]	[deg]	[deg]
30	60.82	60.85	61.00
45	46.34	45.93	45.68
60	31.28	31.70	30.33
75	15.73	15.34	15.09
90	0.00	0.00	0.00

Table 3.28. Crack initiation angles for edge cracked FGM plate ($E_2/E_1 = 10$) for $a/w = 0.4$ using M criterion

Crack Angle, β	$-\theta_c, r_c/a = 0.05$	$-\theta_c, r_c/a = 0.075$	$-\theta_c, r_c/a = 0.1$
[deg]	[deg]	[deg]	[deg]
30	56.31	57.13	57.89
45	42.73	43.10	43.59
60	29.21	29.26	29.33
75	14.94	14.86	14.82
90	0.00	0.00	0.00

Table 3.29. Crack initiation angles for edge cracked FGM plate ($E_2/E_1 = 100$) for $a/w = 0.4$ using M criterion

Crack Angle, β	$-\theta_c, r_c/a = 0.05$	$-\theta_c, r_c/a = 0.075$	$-\theta_c, r_c/a = 0.1$
[deg]	[deg]	[deg]	[deg]
30	52.22	53.35	54.34
45	38.78	39.48	40.01
60	26.50	26.86	27.10
75	13.65	13.77	13.85
90	0.00	0.00	0.00

Crack initiation angles for $r_c/a=0.05, 0.75$ and 0.1 are illustrated in Figures 3.41-3.45. The results are very close to each other, therefore it is difficult to compare the results. The reason for that is the α values correspond to r_c/a are 0.32, 0.38 and 0.44 respectively. To see the effect of r_c on crack initiation angle, the α values should not be chosen so close.

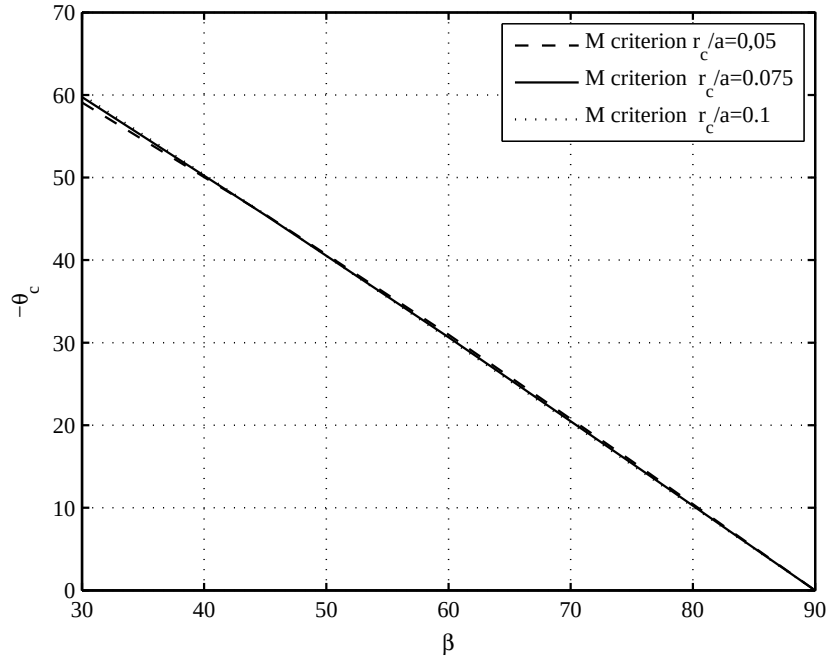


Figure 3.41. Comparison of crack initiation angle results based on M-criterion for $a/w=0.3$ edge cracked homogeneous plate for different r values

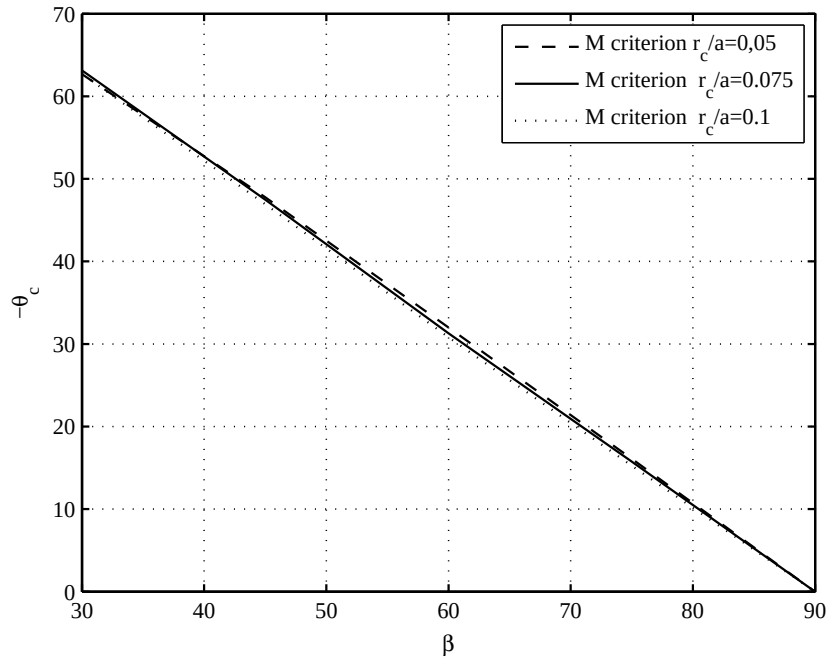


Figure 3.42. Comparison of crack initiation angle results based on M-criterion for $a/w=0.3$ edge cracked FGM($E_2/E_1 = 0.1$) plate for different r values

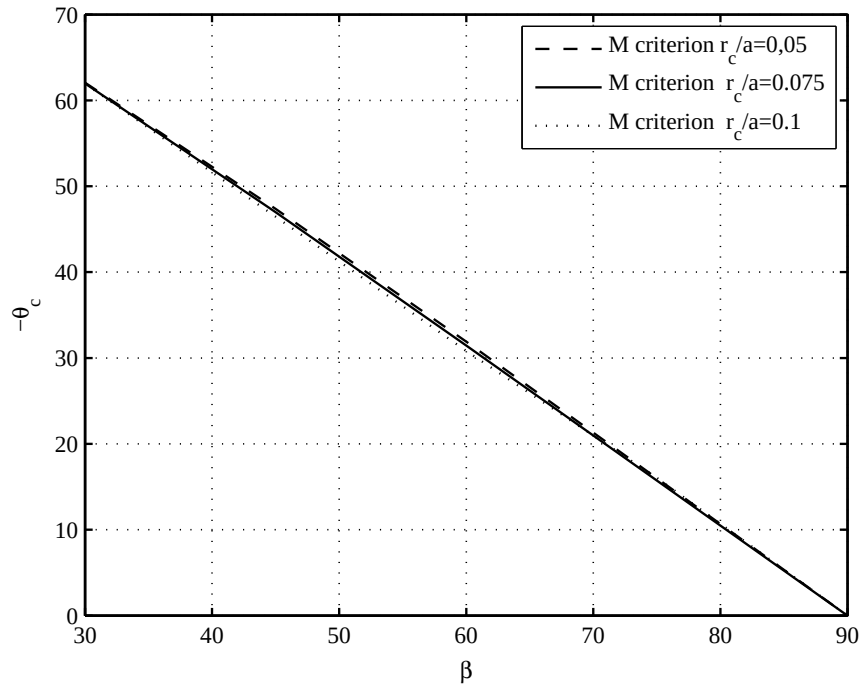


Figure 3.43. Comparison of crack initiation angle results based on M-criterion for $a/w=0.3$ edge cracked FGM($E_2/E_1 = 0.2$) plate for different r values

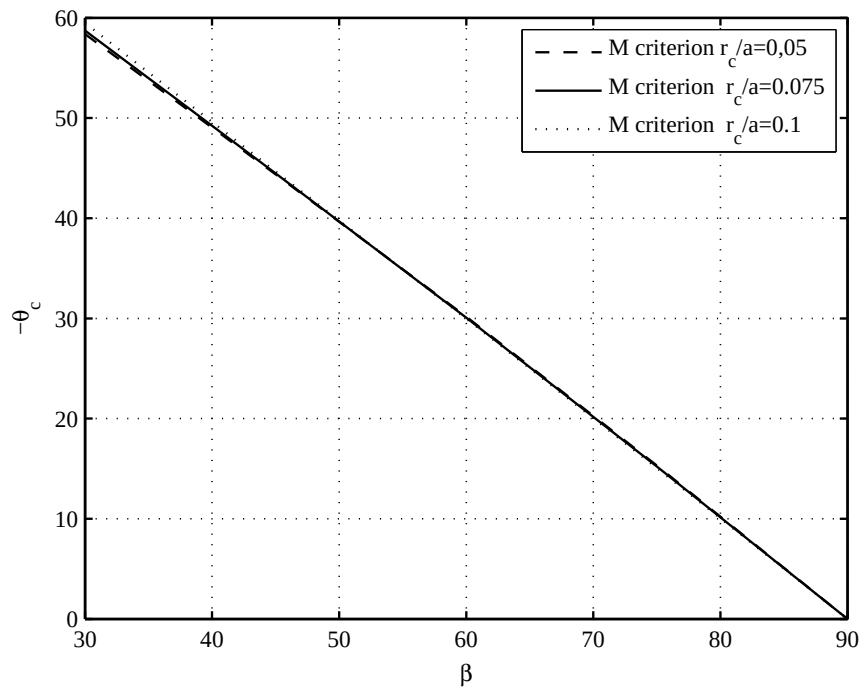


Figure 3.44. Comparison of crack initiation angle results based on M-criterion for $a/w=0.3$ edge cracked FGM($E_2/E_1 = 10$) plate for different r values

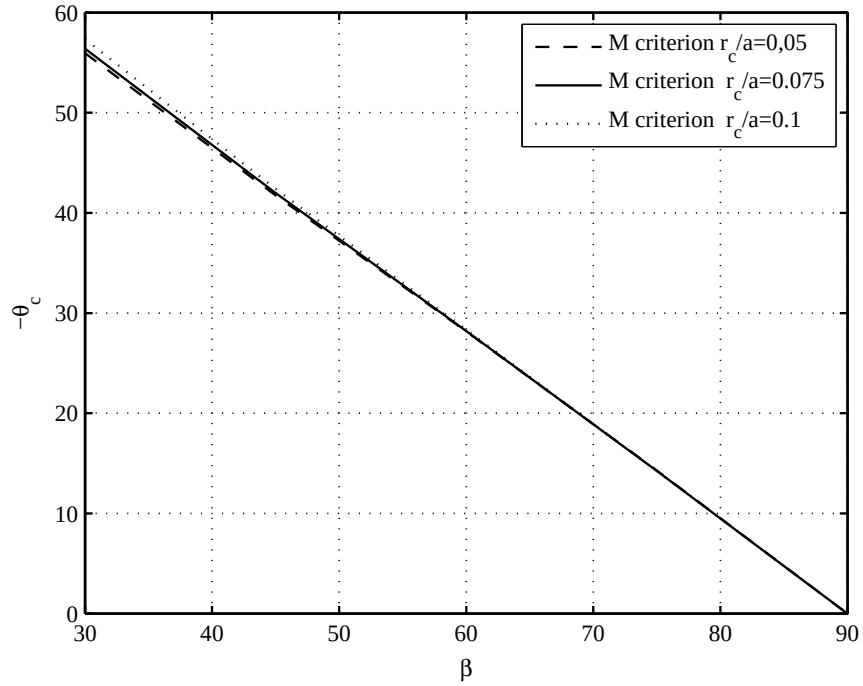


Figure 3.45. Comparison of crack initiation angle results based on M-criterion for $a/w=0.3$ edge cracked FGM($E_2/E_1 = 100$) plate for different r values

The effect of material non-homogeneity and crack length for different crack lengths is represented in Figures 3.46-3.48

Crack initiation angles for different crack length, a/w , are illustrated in the Figures 3.49-3.53.

To see the effect of material non-homogeneity on crack initiation angle, Figures 3.54-3.55 are plotted for a fixed crack angle β and for different a/w values. For the same geometry, as the material non-homogeneity increases, crack initiation angle decreases. But the change depends on crack length. From Figures 3.54-3.55, it can be seen that as the crack length increases, a/w , the change in crack initiation angle increases.

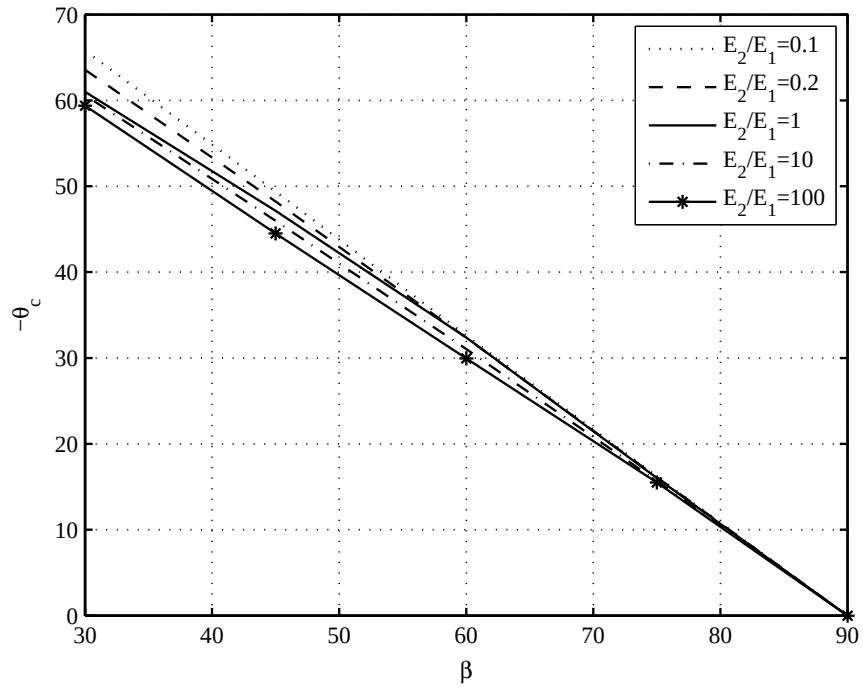


Figure 3.46. Crack initiation angles using M-criterion for different E_2/E_1 , for $r_c/a = 0.05, a/w = 0.2$

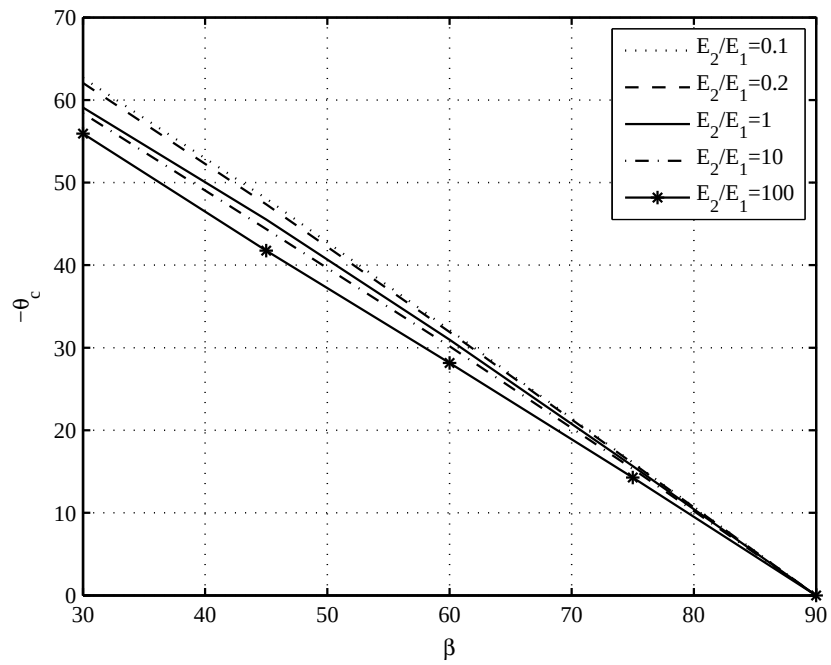


Figure 3.47. Crack initiation angles using M-criterion for different E_2/E_1 , for $r_c/a = 0.05, a/w = 0.3$

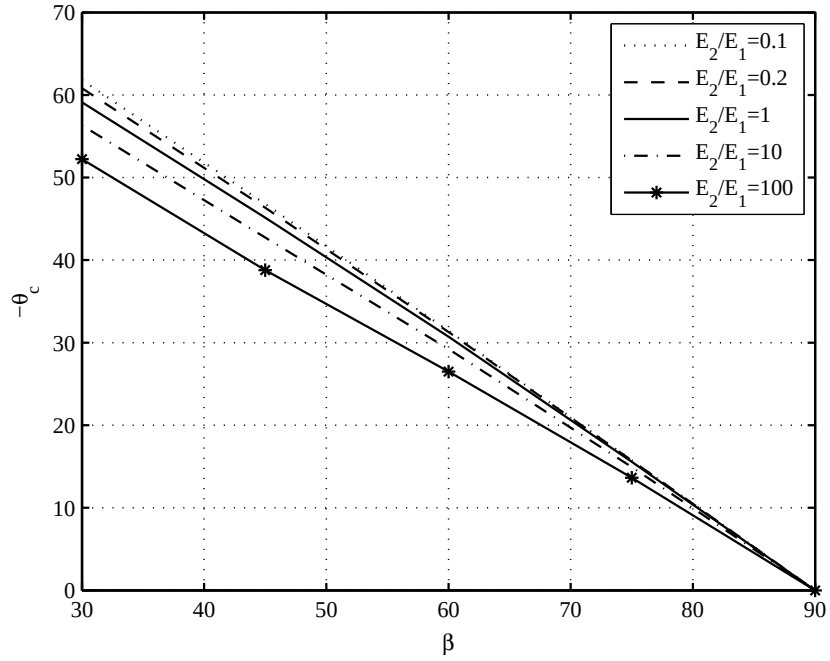


Figure 3.48. Crack initiation angles using M-criterion for different E_2/E_1 , for $r_c/a = 0.05$, $a/w=0.4$

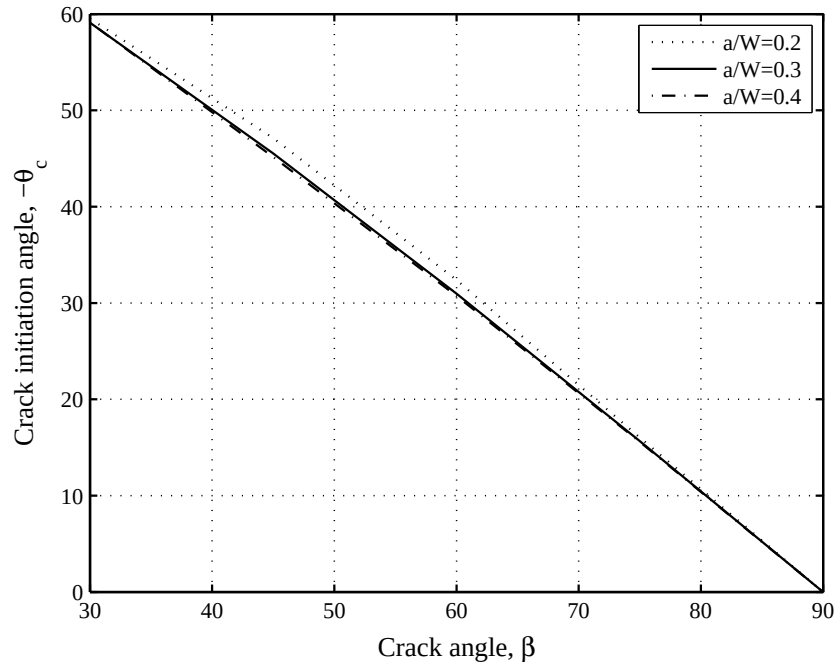


Figure 3.49. Crack initiation angles using M-criterion for different a/w ratios, for homogeneous plate, $r_c/a = 0.05$

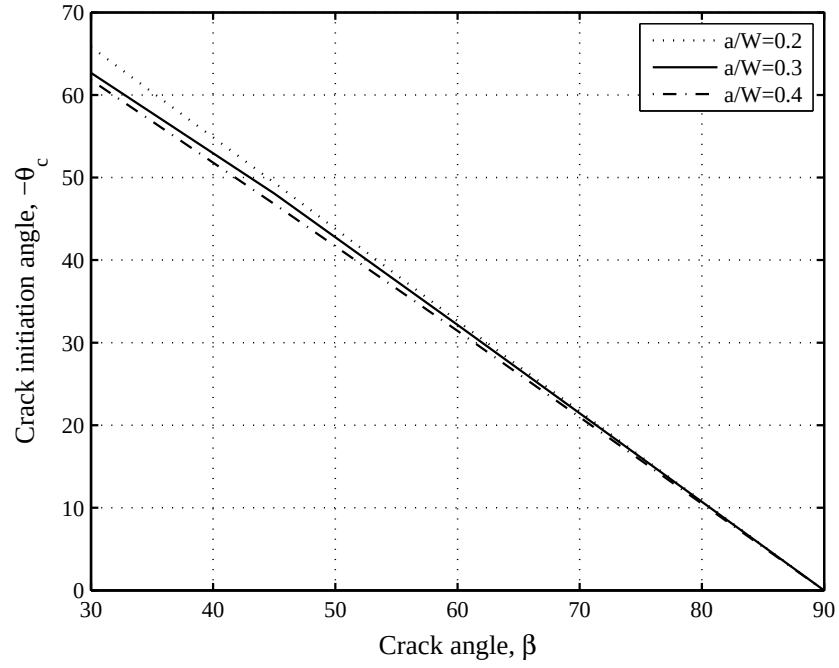


Figure 3.50. Crack initiation angles using M-criterion for different a/w ratios, for FGM ($E_2/E_1 = 0.1$) plate, $r_c/a = 0.05$

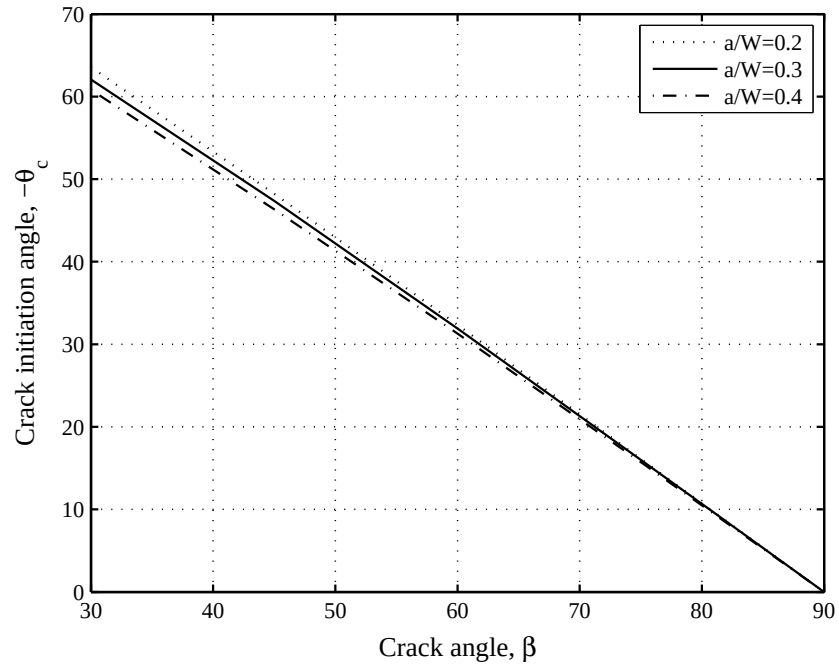


Figure 3.51. Crack initiation angles using M-criterion for different a/w ratios, for FGM ($E_2/E_1 = 0.2$) plate, $r_c/a = 0.05$

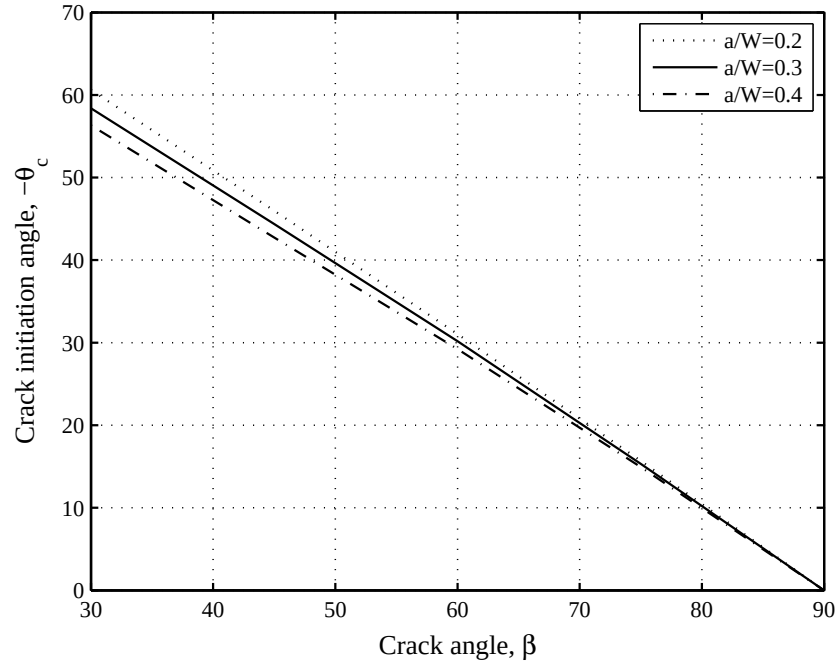


Figure 3.52. Crack initiation angles using M-criterion for different a/w ratios, for FGM ($E_2/E_1 = 10$) plate, $r_c/a = 0.05$

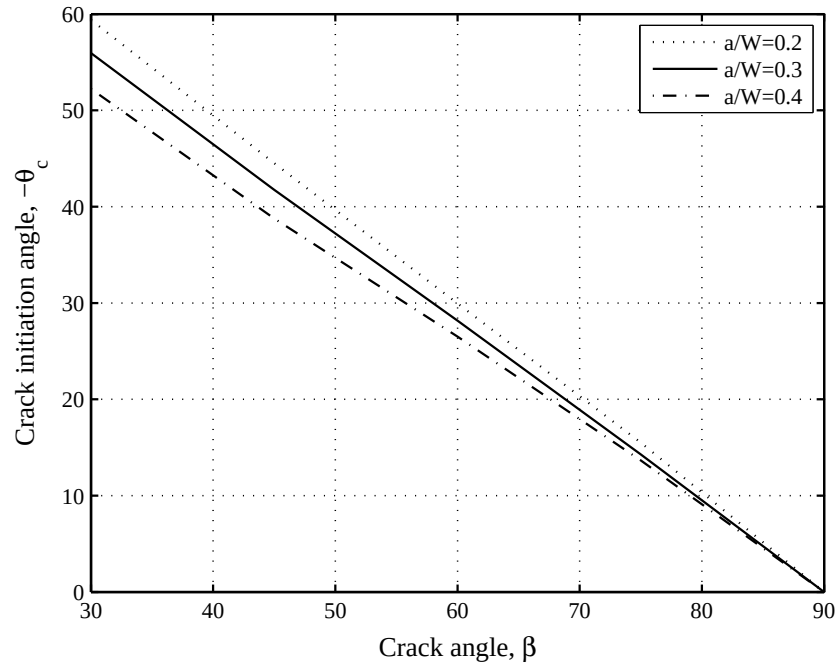


Figure 3.53. Crack initiation angles using M-criterion for different a/w ratios, for FGM ($E_2/E_1 = 100$) plate, $r_c/a = 0.05$

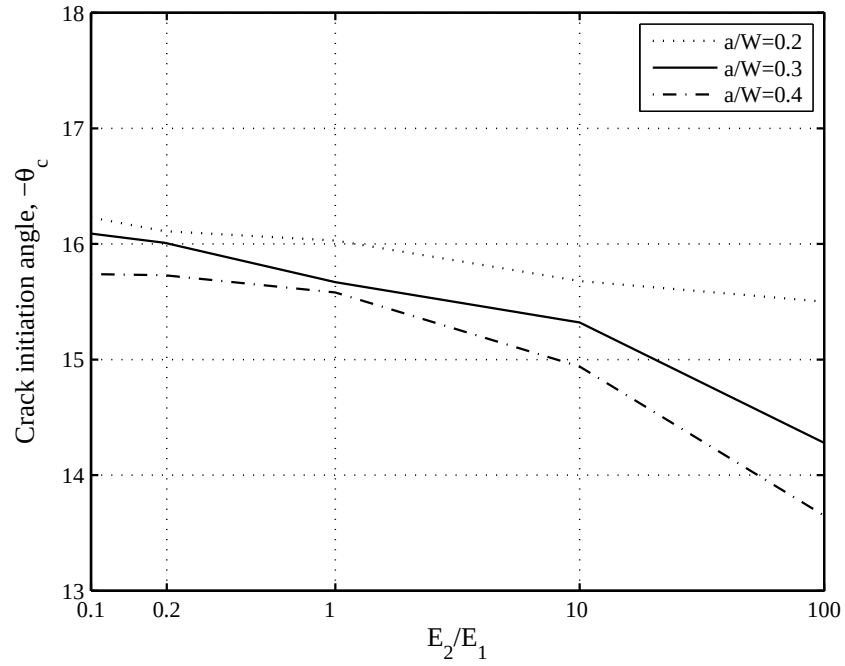


Figure 3.54. Crack initiation angles using M-criterion for different E_2/E_1 , for $\beta = 75^\circ$

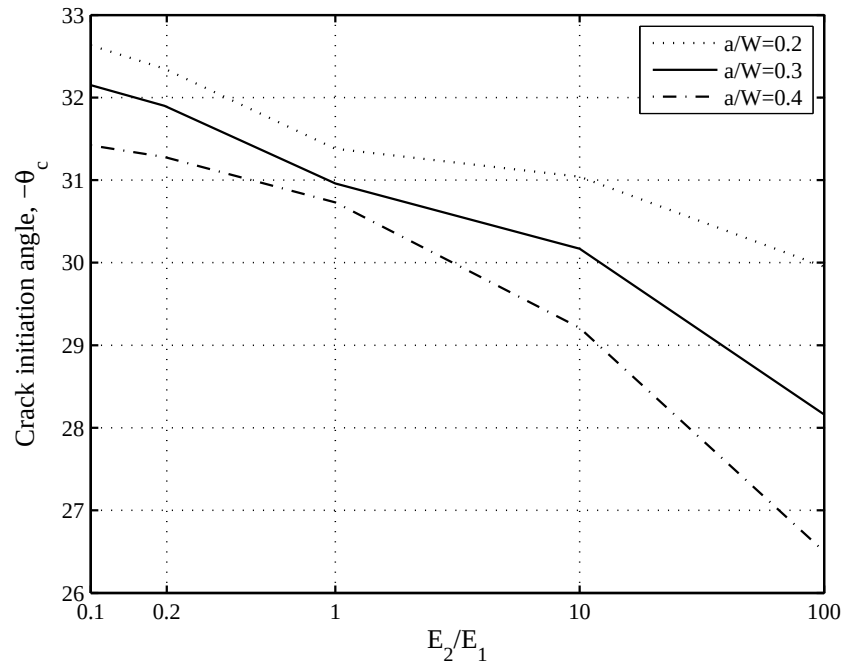


Figure 3.55. Crack initiation angles using M-criterion for different E_2/E_1 , for $\beta = 60^\circ$

3.2.4. Comparison of Numerical Results

To compare the crack initiation angle results obtained by S, G, M and MTS [5] criteria, Figures 3.56-3.70 are plotted for $a/w=0.2$, 0.3 and 0.4 and $E_2/E_1=0.1$, 0.2, 1, 10, 100. From Figures 3.58-3.70, it can be seen that MTS, S and G criteria give approximate results in Mode-I dominant region. For Mode II dominant region, MTS, S and G show better agreement than the M criterion. And also it can be concluded that S and G criteria results are close to MTS criterion $r_c/a = 0.01$ results. It can be easily seen that M-criterion results are distinct from others, for the values of $\beta > 50^\circ$. It is an expected situation because, analytical M-criterion gives different crack initiation angle results than others (Figure 4.1). When we compare FGM and homogeneous material plots, we can easily see that the form of the figures are the same.

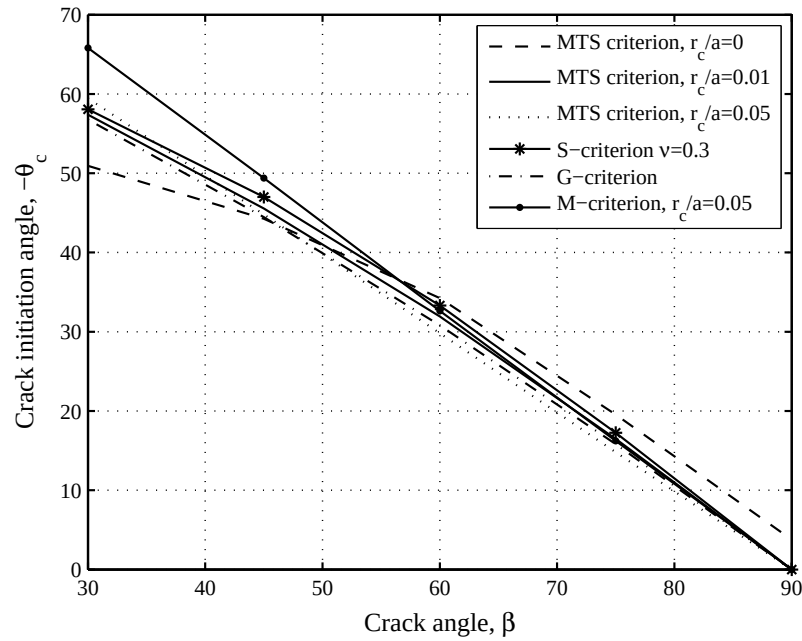


Figure 3.56. Comparison of crack initiation angle results based on S, G, M and MTS criteria for edge cracked FGM($E_2/E_1 = 0.1$) plate, for $a/w=0.2$

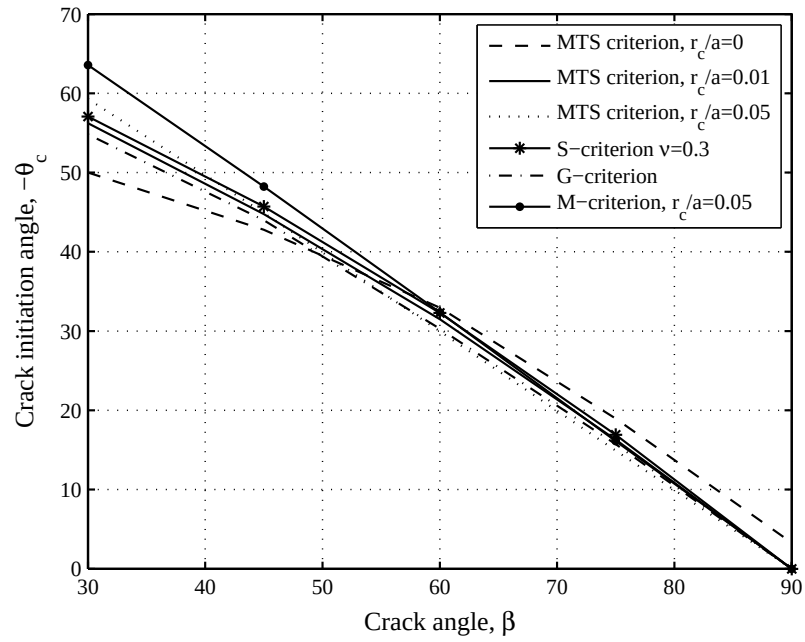


Figure 3.57. Comparison of crack initiation angle results based on S, G, M and MTS criteria for edge cracked FGM($E_2/E_1 = 0.2$) plate, for $a/w=0.2$

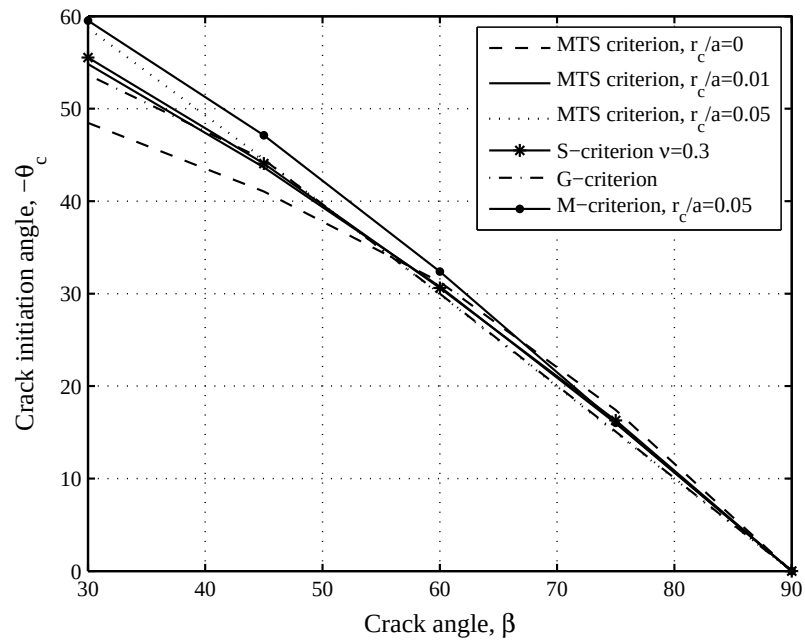


Figure 3.58. Comparison of crack initiation angle results based on S, G, M and MTS criteria for edge cracked homogeneous plate, for $a/w=0.2$

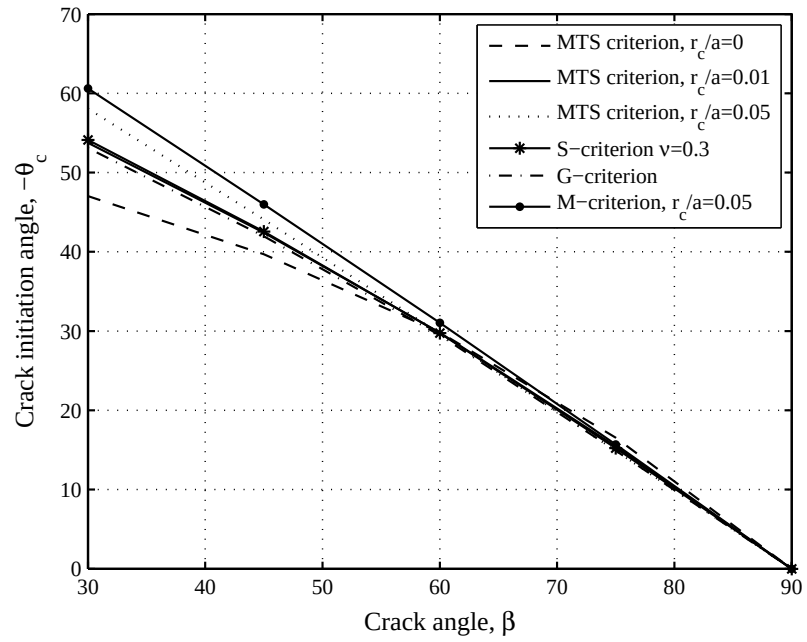


Figure 3.59. Comparison of crack initiation angle results based on S, G and MTS criteria for edge cracked FGM($E_2/E_1 = 10$) plate, for $a/w=0.2$

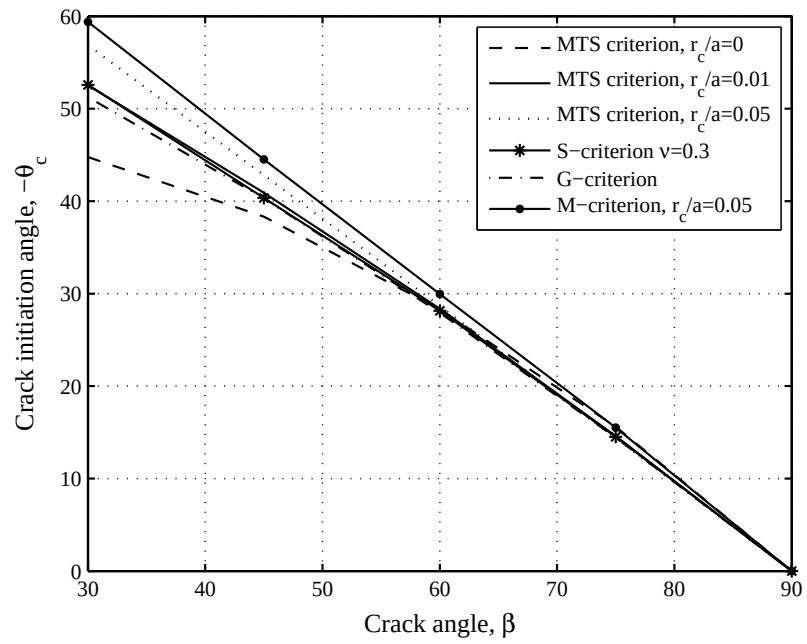


Figure 3.60. Comparison of crack initiation angle results based on S, G, M and MTS criteria for edge cracked FGM($E_2/E_1 = 100$) plate, for $a/w=0.2$

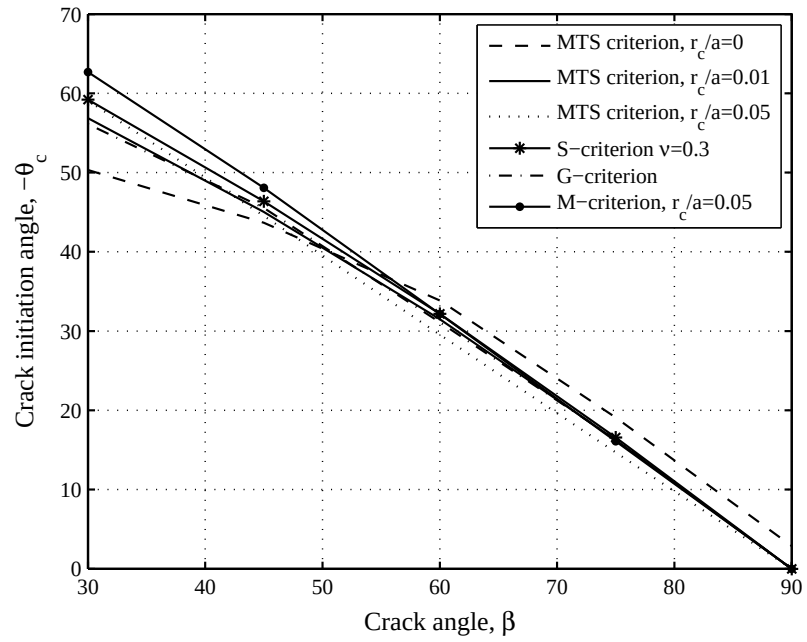


Figure 3.61. Comparison of crack initiation angle results based on S, G, M and MTS criteria for edge cracked FGM($E_2/E_1 = 0.1$) plate, for $a/w=0.3$

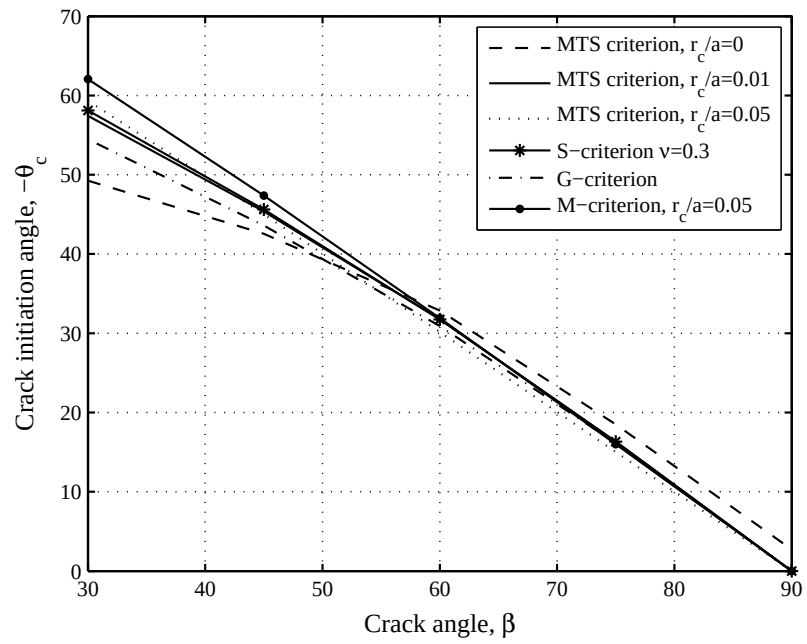


Figure 3.62. Comparison of crack initiation angle results based on S, G, M and MTS criteria for edge cracked FGM($E_2/E_1 = 0.2$) plate, for $a/w=0.3$

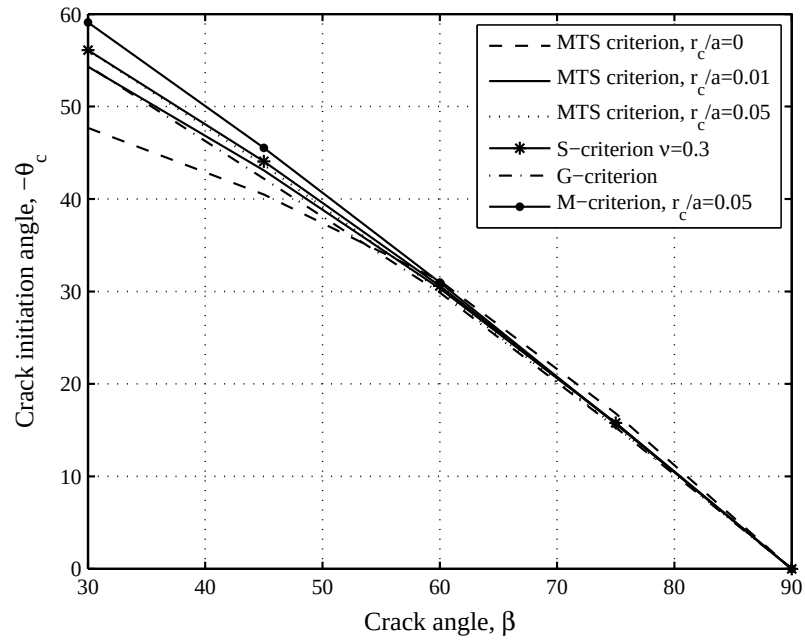


Figure 3.63. Comparison of crack initiation angle results based on S, G, M and MTS criteria for edge cracked homogeneous plate, for $a/w=0.3$

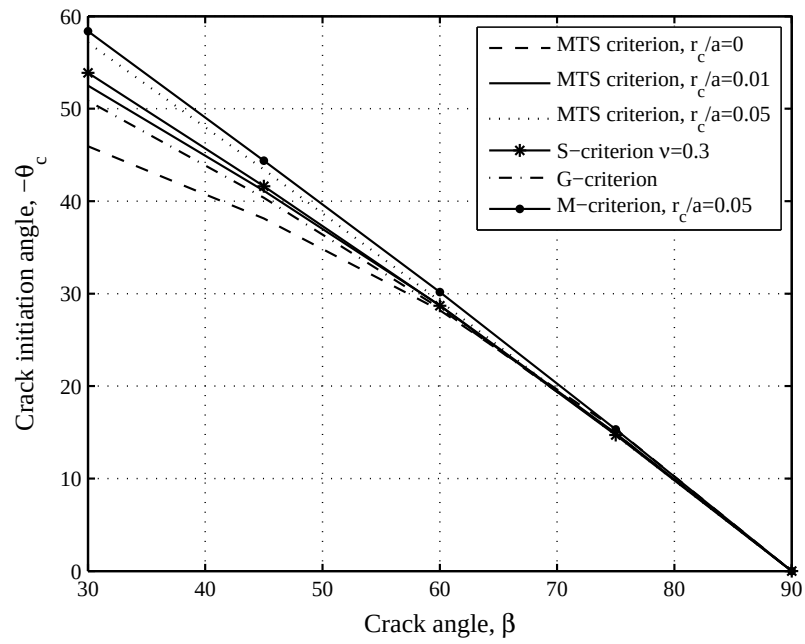


Figure 3.64. Comparison of crack initiation angle results based on S, G, M and MTS criteria for edge cracked FGM($E_2/E_1 = 10$) plate, for $a/w=0.3$

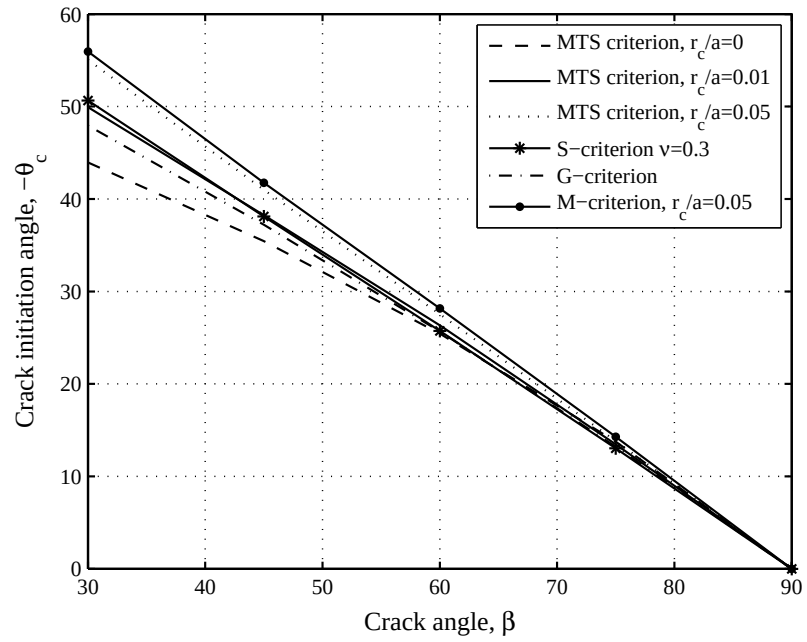


Figure 3.65. Comparison of crack initiation angle results based on S,G, M and MTS criteria for edge cracked FGM($E_2/E_1 = 100$) plate, for $a/w=0.3$

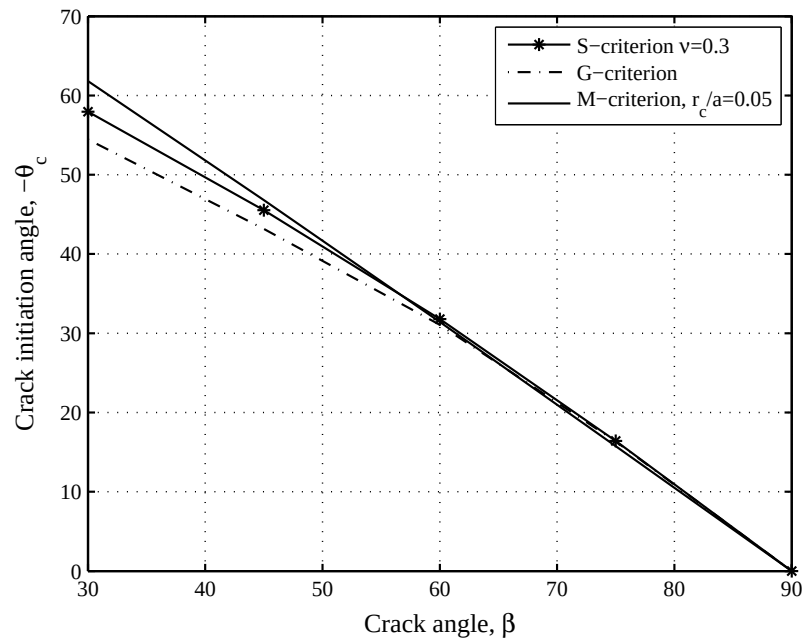


Figure 3.66. Comparison of crack initiation angle results based on S, G and M criteria for edge cracked FGM($E_2/E_1 = 0.1$) plate, for $a/w=0.4$

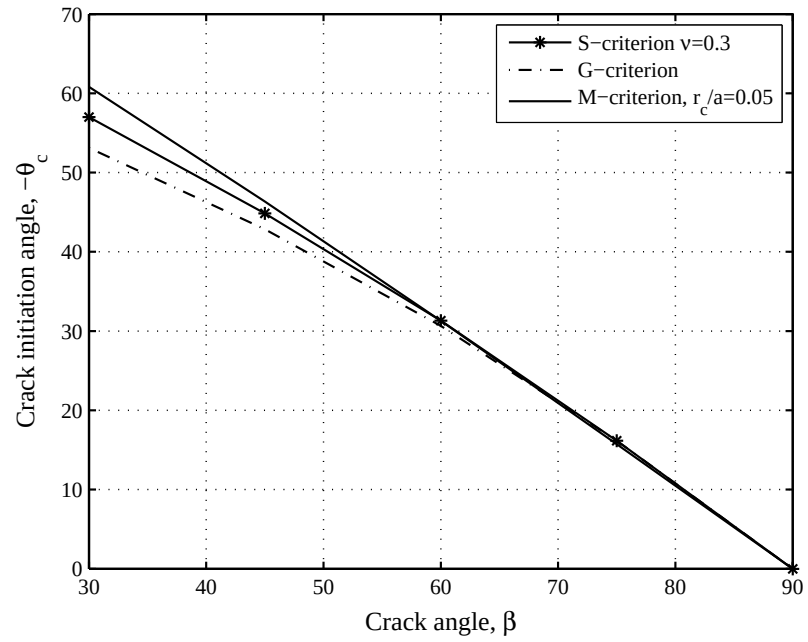


Figure 3.67. Comparison of crack initiation angle results based on S, G and M criteria for edge cracked FGM($E_2/E_1 = 0.2$) plate, for $a/w=0.4$

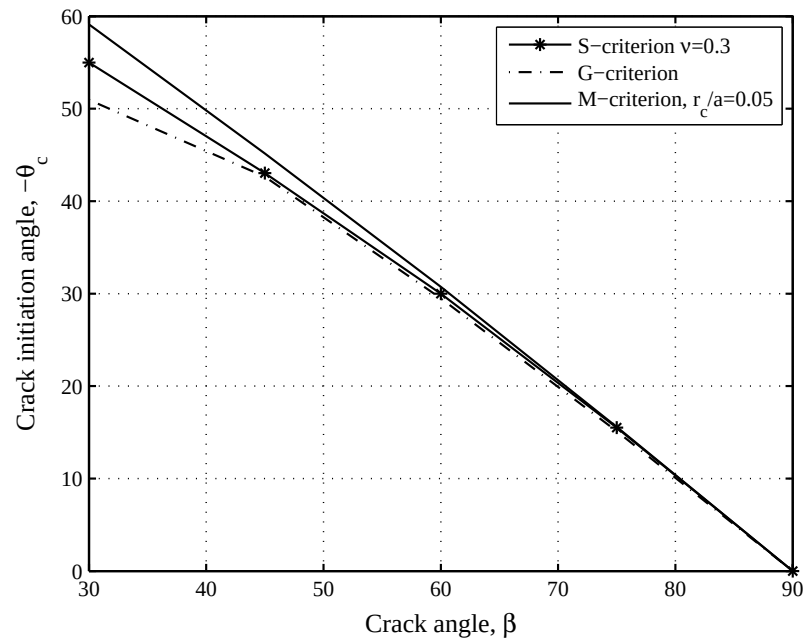


Figure 3.68. Comparison of crack initiation angle results based on S, G and M criteria for edge cracked homogeneous plate, for $a/w=0.4$

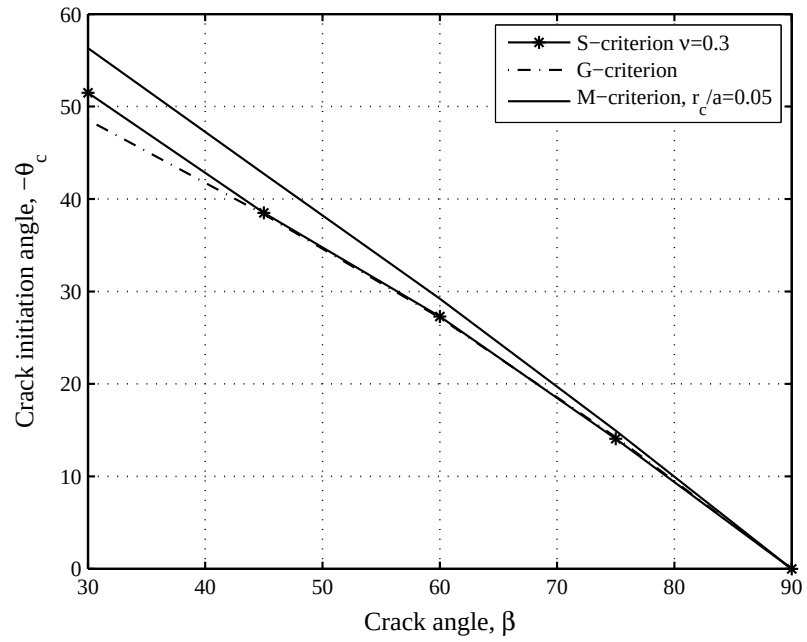


Figure 3.69. Comparison of crack initiation angle results based on S, G and M criteria for edge cracked FGM($E_2/E_1 = 10$) plate, for $a/w=0.4$

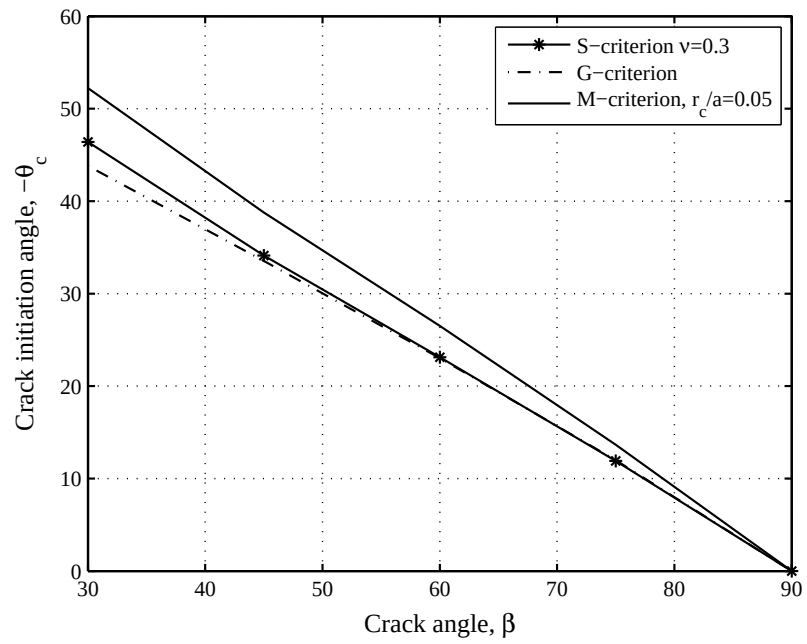


Figure 3.70. Comparison of crack initiation angle results based on S, G, M criteria for edge cracked FGM($E_2/E_1 = 100$) plate, for $a/w=0.4$

4. CONCLUSIONS

The purpose of this thesis was to determine computationally the crack initiation angles in functionally graded materials under mixed mode loading. Therefore, four criteria were discussed and compared to predict crack initiation angles in the case of an inclined edge crack. These criteria were maximum tangential stress (MTS), maximum energy release rate (G), minimum strain energy density (S) and maximum triaxial stress (M) criteria. Also, the deficiency of M criterion is realized and M criterion is modified. The effects of both the singular term and the T -term in the triaxial stress are included. Using this criterion, mixed mode fracture can be predicted for which the crack tip parameters K_I , K_{II} and T are known.

MTS, G, S and GMTS criteria are for brittle fracture and M is for ductile fracture under mixed mode loading. The observations for these criteria are:

- GMTS criterion is a modified MTS criterion which is additionally includes non-singular term, T stress, in the stress field equations.
- In applying GMTS criterion, we need to define another material characteristic r_c where to evaluate $\sigma_{\theta\theta max}$. Therefore, results are dependent of the choice for r_c .
- S criterion is the only criterion which depends on material property, ν , and state of stress (plane stress or plane strain).
- For all practical purposes, all three criteria give identical results for small ratios of $\frac{K_{II}}{K_I}$ and diverge slightly as this ratio increases (Figure 4.1).
- All models can be represented by a normalized fracture locus (Figure 4.2).
- In applying modified M criterion, we also need to define r_c where to evaluate M_{max} , therefore effect of T-stress is added to M criterion.

If the pair of SIF is inside the fracture loci (Figure 4.2), then that crack cannot initiate without sufficient increase in stress intensity factors. If outside, then the crack is locally unstable and will continue to propagate in either of the following ways:

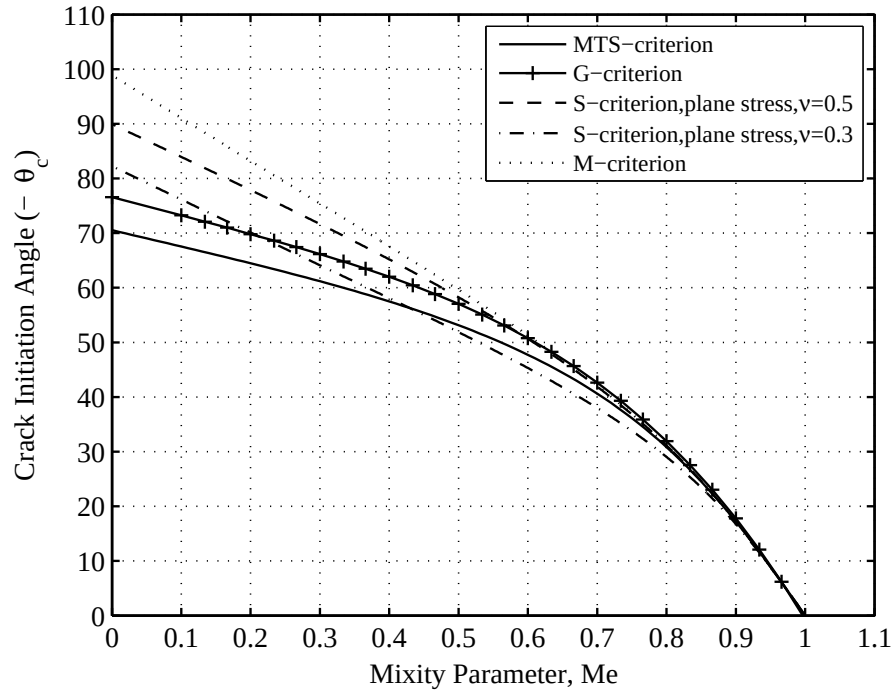


Figure 4.1. Comparison of crack initiation criteria under mixed mode loading

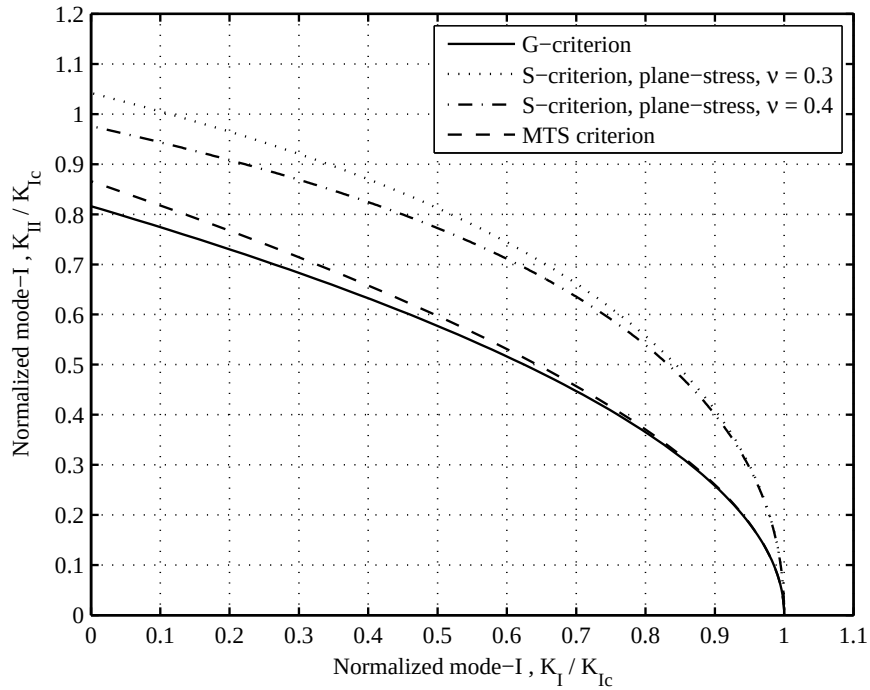


Figure 4.2. Locus of fracture diagram under mixed mode loading

- With an increase in the SIF (and the energy release rate G), thus resulting in a global instability, failure of the structure (crack reaching a free surface) will

occur.

- With a decrease in the SIF (and the energy release rate G), due to a stress redistribution, the SIF pair will return to within the locus [3].

To predict the crack initiation angles numerically, S, G and M criteria are chosen. As MTS criterion was studied previously by [5], the three of most common known criteria are compared in functionally graded materials. Before determining initiation angles, to check the accuracy of the models, SIFs are calculated and compared with analytical results for homogeneous and FGM case. Afterwards, to verify the method, crack initiation angles are determined by each criterion in inclined center crack for homogeneous material and results were good agreements with analytical ones. Then, crack initiation angles are determined by S and G criterion respectively for inclined edge crack for FGMs with varying E_2/E_1 ratios as: 0.1, 0.2, 1, 10, 100. Crack angles, β , were used as $90^\circ, 75^\circ, 60^\circ, 45^\circ, 30^\circ$. a/W ratios (crack length/plate width) were chosen as 0.2, 0.3, 0.4.

For functionally graded materials, crack initiation angles depend on: Material non-homogeneity, crack angle and crack length. Figure 4.3, illustrates the crack angles versus crack initiation angles obtained by MTS, S, G and M methods numerically. It can be concluded that, for small angles of β , all criteria give close results. Figure 4.4, represents the effect of the crack length and the material non-homogeneity for a specific crack angle, $\beta = 75^\circ$ and Figure 4.5, represents the effect of the crack angle and material non-homogeneity. For the same geometry, as the E_2/E_1 increases, crack initiation angle decreases. From Figures 4.4, 4.6 it can be seen that crack initiation angle decreases, while the ratio of crack length to the width of the plate, a/W , increases.

For future work, crack initiation angles can be determined for different material non-homogeneity types and for different geometries (i.e. 3D complex engineering applications.) various criteria. Also, a new criterion can be developed to determine crack initiation angles and experimental studies can be enhanced to support analytical and numerical results.

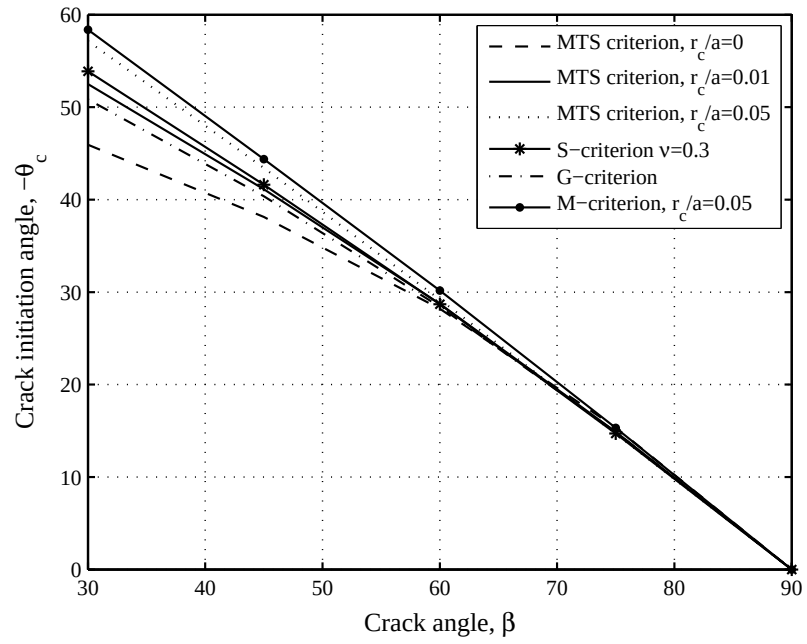


Figure 4.3. Comparison of crack initiation angle results for FGM ($E_2/E_1 = 10$) plate, for $a/w=0.3$

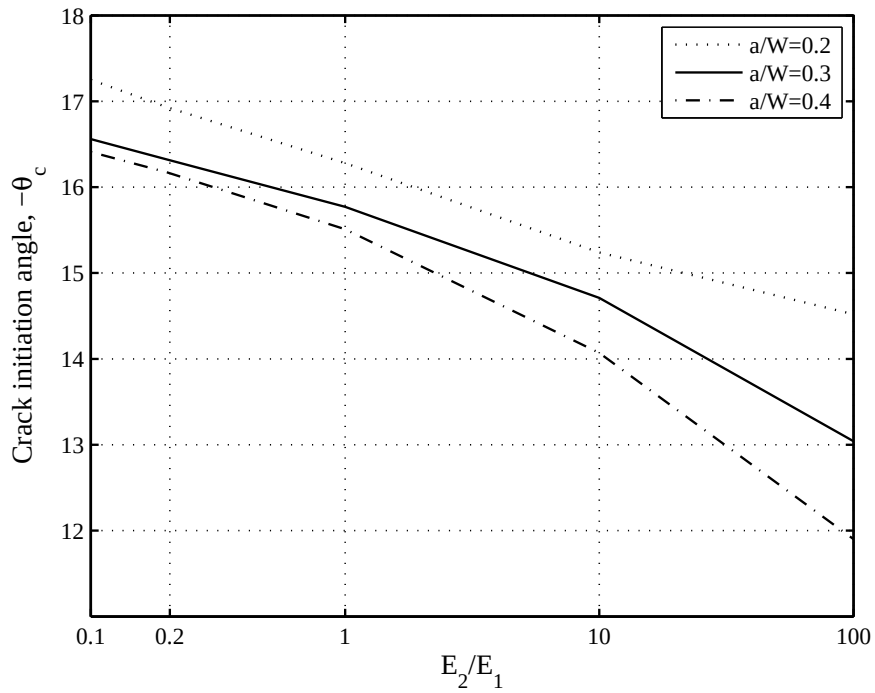


Figure 4.4. Effect of the crack length and the material non-homogeneity, for $\beta = 75^\circ$

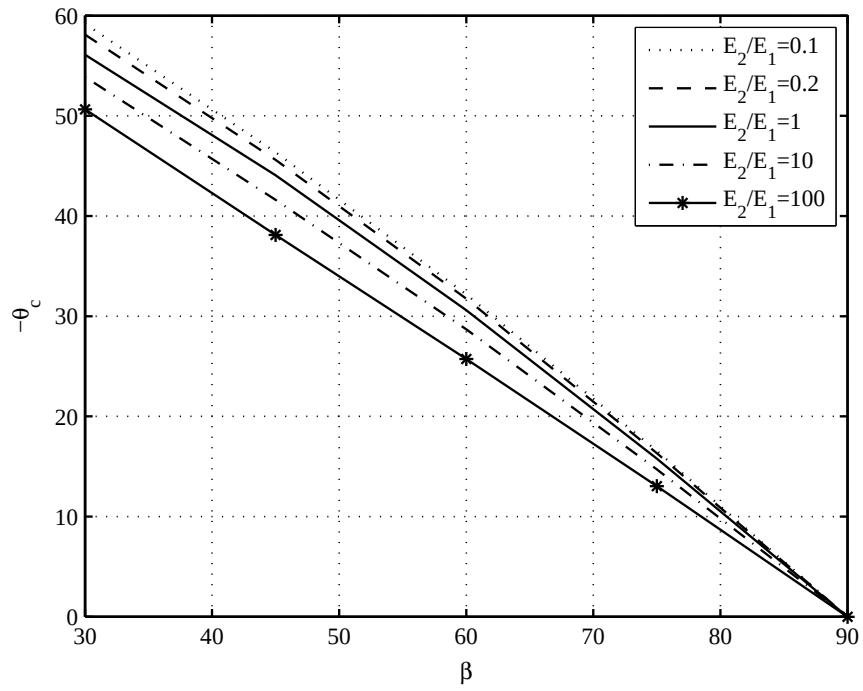


Figure 4.5. Effect of the crack angle and the material non-homogeneity, for $a/w=0.3$

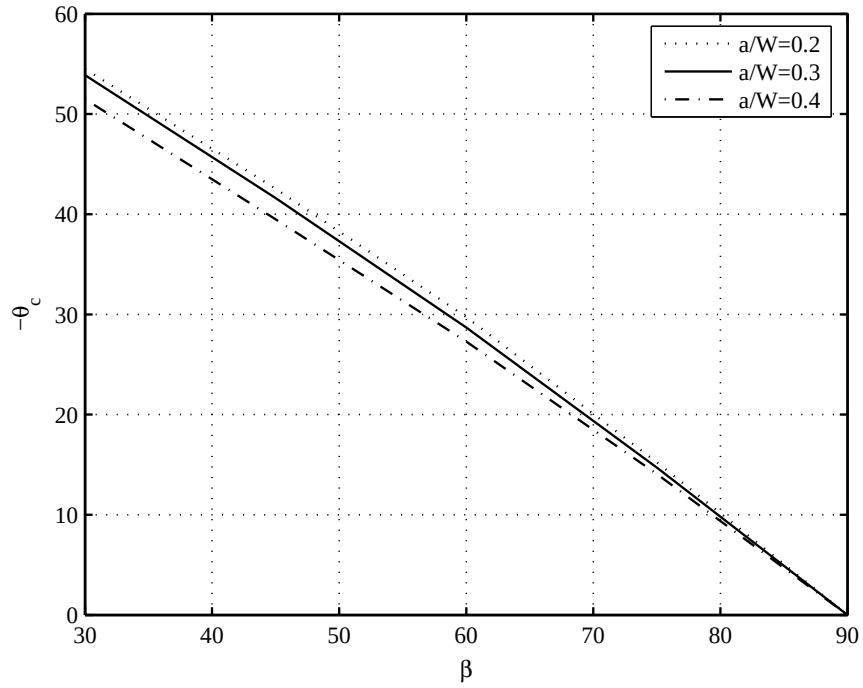


Figure 4.6. Effect of the crack length, for FGM ($E_2/E_1 = 10$)
plate

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